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LINEAR TOPOLOGICAL SPACES

D. H. HYERS

1. **Introduction.** In general analysis it is customary to study linear spaces for which there is defined a "norm," which takes the place of the absolute value of ordinary analysis in defining distance, limit point, continuity, and so on. Linear metric spaces more general than the normed spaces have also been studied by Fréchet, Banach, and several others. In line with the trend toward general topology, it seems natural to generalize still more by introducing *linear topological spaces*, that is, linear spaces which are at the same time topological spaces, in which the fundamental "linear" operations of addition and scalar multiplication are continuous. We shall always assume that the topology is subject to the axioms for a T_1-space.[1]

The topology of T_1-spaces may be introduced in various ways; by postulating a system of open sets or of neighborhoods with certain properties, and so on. We shall find it convenient to give a set of postulates for the topology of the linear space L in which "neighborhood" is the fundamental undefined notion. Since L is a topological group, it has a uniform topology and hence it is sufficient to consider neighborhoods of the origin. Moreover, the "uniform structure" implies that L is a completely regular Hausdorff space.[2]

The following notations will be used. The set of elements x having the property P will be denoted by $\{x; P\}$. If S and T are subsets of L, α is a fixed real number, and x a fixed point of L, $x+S$ denotes the set $\{x+y;\ y\in S\}$; $S+T$ denotes the set $\{y+z;\ y\in S,\ z\in T\}$; αS stands for the set $\{\alpha y;\ y\in S\}$. The notations $\cup$ and $\cap$ are used for union and intersection, respectively. The following definition of a linear topological space is equivalent to the one given above.[3]

Definition 1.1. A linear space L will be called a *linear topological space* (abbreviated l.t.s.) if and only if there is a system $\mathcal{V}$ of subsets

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[1] The definition given here seems to be due to Kolmogoroff [1], 1934. However, a linear space with a more general topology than that of a T_1-space was defined by Fréchet [2, 3] in 1926 under the name "topological affine space." The postulates for a T_1-space are given in Alexandroff and Hopf [1, p. 59]. The bracketed numbers refer to the bibliography.

[2] A. Weil [1, p. 13].

[3] For the proof see Hyers [4].

U of L called *neighborhoods of the origin* subject to the following postulates.[4]

(i) The intersection $\bigcap\{U;\ U\in\mathcal{V}\}$ of all the members of $\mathcal{V}$ is the one element set consisting of the origin θ.

(ii) Given $U\in\mathcal{V}$, $V\in\mathcal{V}$, there exists $W\in\mathcal{V}$ such that $W\subset U\cap V$.

(iii) If $U\in\mathcal{V}$ there exists $V\in\mathcal{V}$ such that $V+V\subset U$.

(iv) Given $U\in\mathcal{V}$ there exists $V\in\mathcal{V}$ such that $\alpha V\subset U$, for all real α satisfying $-1\leqq\alpha\leqq 1$.

(v) For every $x\in L$ and $U\in\mathcal{V}$, there exists a real number α such that $x\in\alpha U$.

Linear topological spaces may be regarded on the one hand as very special topological groups, which are abelian and have real "operators," or on the other hand as a generalization of Fréchet spaces, Banach spaces, and so on.

We now consider some examples of l.t.s.

EXAMPLE 1. *The space of all real functions $x(t)$ of a real variable t on a fixed interval (the interval may be finite or infinite).* The neighborhoods U of the origin are taken to be the sets $U(\sigma, \delta)=\{x;\ |x(t)|<\delta,\ t\in\sigma\}$, where σ ranges over all finite sets of real numbers on the interval, and δ ranges over all positive numbers.

EXAMPLE 2a. *Any linear normed space E with its usual "strong" topology.* Put $U(\delta)=\{x;\ \|x\|<\delta\}$, where $\delta>0$.

EXAMPLE 2b.[5] *A linear normed space E with its weak topology.* Corresponding to each finite set σ of linear continuous functionals $f(x)$ on E, and each $\delta>0$, define $U(\sigma, \delta)=\{x;\ |f(x)|<\delta,\ f\in\sigma\}$. A linear normed space E with its weak topology, considered as a l.t.s., will be denoted by $\mathcal{E}$.

EXAMPLE 3. *The space $l^{1/2}$ of sequences $x=(x_1, x_2, \cdots)$ such that* $|x|=(\sum_{i=1}^{\infty}|x_i|^{1/2})^2<\infty$. The neighborhoods U are defined to be the sets $U(\delta)=\{x;\ |x|<\delta\}$, where $\delta>0$.

The space $\mathcal{E}$ of Example 2b is of considerable importance in some recent developments of Banach space theory. In Example 1 the topology is simply that of "point-wise" convergence. This example illustrates the utility of the concept of a l.t.s. in situations where pointwise convergence is desired. The norm or "strong" topologies on the other hand (Example 2a) apply to function spaces in which the convergence of elements means the uniform convergence of functions, or convergence in the mean, and so on.

Example 1 may be generalized by considering the topological product of a family of l.t.s. Let A be any abstract set of arbitrary cardinal number, and let there be assigned to each $a \in A$, a l.t.s. L_a. The space $\prod L_a$ of functions $x(a)$, with $x(a) \in L_a$ for each $a \in A$, will be called the *linear topological product* of the family $\{L_a; a \in A\}$ of l.t.s. when addition and scalar multiplication are defined in the usual manner, and a typical neighborhood U of the origin in $\prod L_a$ is defined as

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[4] Except for (i), these postulates are identical to those given by von Neumann [1].

[5] For further examples of a similar nature related to Hilbert space see von Neumann [1, 2].

$$U = \{x(a);\, x(a_i) \in U_{a_i},\, i = 1, \cdots, n\},$$

where $a_1, \cdots, a_n$ is any finite set of indices and U_a is any neighborhood of the origin in the space L_a. It is easily shown that the linear topological product of a family of l.t.s. is itself a l.t.s. According to this definition, Example 1 may be thought of as the space of real numbers raised to the power of the continuum. The linear topological product of a finite number of normed spaces is itself a normed space, but this is not true for an infinite family of normed spaces.[6]

2. The properties of boundedness and convexity; normability and metrizability.

One of the first questions to come up in studying l.t.s. is to determine conditions under which the space under consideration is capable of being normed or metrized, and hence subsumed under more traditional systems.

We shall call two l.t.s. *isomorphic* if they are isomorphic as abstract linear spaces and if in addition the isomorphism is bicontinuous. A necessary and sufficient condition for a l.t.s. to be *normable*, that is, isomorphic to a normed space, was given by Kolmogoroff [1] in terms of two geometrical properties which will now be defined. The term "convex set" will be used in its usual sense: S is convex if x, $y \in S$, $0 < \alpha < 1$ imply $\alpha x + (1 - \alpha) y \in S$. A subset S of a l.t.s. L will be said to be *bounded* if for any neighborhood U of the origin there is a real number α such that $S \subset \alpha U$. This definition was given by von Neumann [1]. It can be shown[7] to be equivalent to an earlier definition due to Banach. Another equivalent definition of boundedness was given by Michal and Paxson [1].

THEOREM 1 (Kolmogoroff [1]). *A necessary and sufficient condition r a l.t.s. to be normable is that it contain an open set which is both nvex and bounded.*

The central idea of the proof is the Minkowski functional, which is ed to define the norm:

$$\|x\| = \inf\{\alpha;\, x \in \alpha U\}, \qquad)$$

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[6] See for example Bourgin [1, p. 639].

[7] Hyers [3].

where U is a bounded, open, symmetric, convex neighborhood of the origin. The triangular property of the norm is a consequence of the convexity of the fundamental neighborhood U. The property that $\|x\|=0$ implies $x=\theta$ follows from the fact that since U is bounded, the neighborhoods $U'=(1/n)U$, $n=1, 2, 3, \cdots$, form a complete neighborhood system of the origin.

It has been pointed out[8] that the next theorem can be easily deduced from a theorem of G. Birkhoff [1] on the metrization of topological groups.

THEOREM 2. *A l.t.s. L is metrizable as a space of type F (not necessarily complete) if and only if there exists a countable complete neighborhood system of the origin in L.*

The following definitions are suggested by the results of Theorem 1.

DEFINITION 2.1. A l.t.s. L is called *locally convex*[9] if there exists a complete neighborhood system of the origin, every neighborhood of which is convex.

DEFINITION 2.2. A l.t.s. L is called *locally bounded* if it contains a non-void bounded open set (Hyers [4]).

To return for a moment to the examples given in §1, we see that Examples 1, 2a and 2b are locally convex. The space $l^{1/2}$ of Example 3 is locally bounded but *not* locally convex (Tychonoff [1]). Examples 1 and 2b are, in general, spaces which do not have countable complete neighborhood systems of the origin and hence are not metrizable. This is easily demonstrated for Example 1. It was proved by von Neumann [2] for weak Hilbert space $\mathfrak{H}$ by constructing a set $S\subset\mathfrak{H}$ which had the origin as a limit point, and yet had the property that no sequence of the elements of S was convergent to the origin. Thus in non-metrizable l.t.s. the ordinary concept of sequential convergence may not be sufficient to give us all the limit points of the space. The space $\mathcal{E}$ of Example 2b is normable if and only if it is finite-dimensional (Wehausen [1]).

The property of local convexity plays an important role in many analytical theories, especially in the theories of linear functionals integrals, and functional equations. Fortunately, most of the mo important instances of l.t.s. do have the property of local convexit It is easily seen that the linear topological product of an arbitra family of locally convex spaces is itself locally convex.[10]

The property of local boundedness[11] puts a rather severe restricti on the l.t.s., as the following theorems show. It was mentioned in connection with Theorem 1 that the property of local boundedness implies that there exists a countable complete neighborhood system of the origin. Thus from Theorem 2 we have:

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[8] Hyers [3], Wehausen [1].

[9] The term "convex" is used by von Neumann [1] in place of "locally convex."

[10] This was pointed out by Bourgin [1].

[11] Defined by Hyers [4].

THEOREM 3. *Every locally bounded l.t.s. is metrizable.*

Again, by carrying through the same type of argument used in proving Theorem 1, one can prove[12] the following theorem (the functional $|x|$ is again defined by formula (1), but without requiring the convexity of U).

THEOREM 4. *If L is a locally bounded l.t.s. there exists a non-negative real valued functional x on L with the following properties:*

(i) $|x| = 0$ *implies that* $x = \theta$;

(ii) $|\alpha x| = |\alpha| \cdot |x|$ *for every x in L and every real* α;

(iii) *there exists a real number* $\beta \geqq 1$ *such that* $|x+y| \leqq \beta(|x| + |y|)$ *for all x and y in L;*

(iv) *the sets* $S(\delta) = \{x;\ x \in L,\ |x| < \delta\}$, *where* $\delta > 0$, *form a complete neighborhood system of the origin.*

Conversely, if L is a linear space and there exists a non-negative real valued functional $|x|$ defined on L and having the properties (i), (ii), (iii), *then L is a locally bounded l.t.s., when the topology is defined in accordance with* (iv).

The multiplier property (iii) is due to Bourgin [1]. The properties given by Hyers [4] were (i), (ii), (iv) and in place of (iii) an apparently weaker continuity property, that $|x+y| \to 0$ whenever $|x| \to 0$ and $|y| \to 0$. However, on the assumption of this continuity property and properties (i) and (ii) it is easily shown that (iii) holds.[13] For if $|x| \leqq \delta$, $|y| \leqq \delta$ imply that $|x+y| < 1$, then choose any x_1 and y_1 such that either x_1 or y_1 is not the zero element. Putting $x = \delta \cdot x_1/(|x_1| + |y_1|)$ and $y = \delta \cdot y_1/(|x_1| + |y_1|)$ gives $|x_1+y_1| < \beta(|x_1| + |y_1|)$, where $\beta = 1/\delta$. On the other hand, if $|x_1| = |y_1| = 0$, the inequality (iii) is obviously satisfied.

DEFINITION 2.3. The functional $|x|$ appearing in Theorem 4 will be called a *quasi-norm*, and β is called a *multiplier* of the *quasi-norm.*

One is tempted to look for the least multiplier for a given locally bounded space. However, Bourgin [1] has shown by an example that there may be no least multiplier. By Theorem 1 it is possible to take $\beta = 1$ if and only if the space is locally convex.

In normed spaces the triangular inequality implies that the norm is continuous at every point of the space. It is easily shown by an example that the properties (i), (ii), (iii), (iv) do not imply continuity everywhere. However, if the set U is bounded and open, the quasi-norm obtained by formula (1) is everywhere upper semi-continuous (Hyers [4]). The complete answer to this continuity question was recently given by Bourgin [1] who proved the following theorem.

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[12] For the proof see Hyers [3] and Bourgin [1].

[13] A different proof was given by Bourgin.

THEOREM 5. *In any locally bounded l.t.s. L it is possible to define a quasi-norm with the properties* (i), (ii), (iii), (iv) *of Theorem* 4, *and with the additional property*: (v) $|x|$ *is continuous in* x *at each point of* L.

Bourgin's proof, of course, involves a suitable selection of the fundamental neighborhood U occurring in the definition of the quasi-norm (see formula (1)). This neighborhood was chosen to be a sphere $\{x;\, \rho(x, \theta) < \alpha\}$, where $\rho(x, y)$ is a certain metric for the space (see Theorem 3). A particular case of a locally bounded l.t.s. with is not normable is the space $l^{1/2}$ (Example 3). Here the quasi-norm can be defined as $|x| = (\sum_{i=1}^{\infty} |x_i|^{1/2})^2$, and in this case the number 2 is a multiplier. Further examples are given by M. M. Day [1].

The following theorem shows that the results of requiring local compactness are the same for a l.t.s. as for a linear normed space. The term *compact* will be used in the "old fashioned" sense, that is, a set S is *compact* if every infinite subset of S has a limit point in S.

THEOREM 6. *A necessary and sufficient condition for a l.t.s. L to be isomorphic to a finite-dimensional Euclidean space is that L be locally compact, that is, that L contain an open set whose closure is compact.*

Theorem 6 may be proved by first showing that local compactness implies local boundedness, and hence the existence of a quasi-norm. The existence of a finite basis is then established by the method of F. Riesz [1], who proved the theorem for the space C of continuous functions.[14]

This result suggests the possibility of extending the Riesz theory of completely continuous linear functional equations to transformations of l.t.s. However, so far it has been found possible to do this only for locally bounded spaces (Hyers [4]).

3. **Fixed point theorems and functional equations.** The well known fixed point theorem of Brouwer for continuous transformations of an n-dimensional topological simplex into itself has been generalized to compact subsets of (possibly) infinitely-dimensional spaces by several

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[14] Theorem 6 may also be deduced from Theorem 43 of Pontrjagin [1, p. 170].

authors,[15] beginning with Birkhoff and Kellogg [1]. The numerous applications of such fixed point theorems to the proof of existence theorems for both ordinary and partial differential equations as well as to other functional equations are too well known to require extensive reiteration here. The Brouwer fixed point theorem was generalized to the case of bicompact convex subsets of a locally convex l.t.s. by Tychonoff [1], who also applied his results to obtain an existence theorem for infinite systems of differential equations. Tychonoff's theorem was also used by Paxson [1, 3] in the study of differential equations for functions whose values are in a l.t.s.

As beautiful as the concepts and methods are in this general approach to the solution of functional equations, it does have certain disadvantages. The method is "non-effective" in the sense that no method is given for constructing a set of approximations to the fixed point. Moreover, the method leads only to the existence of a fixed point, and tells us nothing about its uniqueness.

Another and older method for proving existence theorems for functional equations is the famous method of successive approximations of Picard. This method has the advantage of being "effective," and hence capable of being applied to obtain approximate solutions. It also provides a uniqueness proof. The "heavy" restriction made here is that the transformation involved satisfies a Lipschitz condition. An application of Picard's method to the proof of a fixed-point theorem for transformations of a subset of a complete metric space was given by Hildebrandt and Graves [1].[16] The author's Theorem 7 below[17] is a generalization of the result of Hildebrandt and Graves to transformations operating on l.t.s. In order to formulate a suitable analog of the Lipschitz condition in a l.t.s. we need the following definition.

DEFINITION 3.1. By a *Lipschitz system* for a l.t.s. L we shall understand a family of sets $K(x)$, with the following properties:

(1) To each x in L there corresponds a convex set $K(x)$.

(2) $x \in K(x)$ and $\theta \in K(x)$; if $x \neq \theta$ and $\alpha > 1$, then $\alpha x \notin K(x)$.

(3) $K(\alpha x) = \alpha K(x)$ for $\alpha \geqq 0$.

(4) $y \in K(x)$ implies $K(y) \subset K(x)$.

(5) Each $K(x)$ is bounded.

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[15] For example, Schauder [1]. Further references are given by Graves [1]. An application of a fixed point theorem to an existence theorem arising in a hydrodynamical problem was worked out by Weyl [1].

[16] For a discussion see Graves [1].

[17] Hyers [1, 2].

In the special case of a linear normed space E, the sets $K(x) = \{y; \|y\| \leqq \|x\|\}$ form a Lipschitz system. A sequence $x_n \in L$ will be called *fundamental* if, corresponding to each neighborhood U of the origin, there is a positive integer n_0 such that $x_m - x_n \in U$ for $m > n_0$ and $n > n_0$. A subset S of L will be called *sequentially complete* if every fundamental sequence $x_n \in S$ converges to a point of S. The closure of S, that is, the set consisting of S and its limit points, is denoted by $\bar{S}$.

THEOREM 7. *Let $z = f(y)$ be a continuous transformation of Y into part of itself, where Y is a sequentially complete subset of a linear topological space L. Let $\{K(y)\}$ be any Lipschitz system with respect to which the function $f(y)$ satisfies the Lipschitz condition*

$$f(y) - f(z) \in \mu K(y - z)$$

for all pairs $y, z \in Y$, where μ is a positive constant less than one. Then the transformation $z = f(y)$ of Y into part of itself has a unique fixed point.

PROOF. For any chosen $y_0 \in Y$, define y_n recurrently by $y_{n+1} = f(y_n)$, $n = 0, 1, 2, \cdots$, so that $y_n \in Y$ for all n. Using properties (3) and (4) of Lipschitz systems and an induction we obtain

$$y_{n+1} - y_n \in \mu^n K(y_1 - y_0), \qquad n = 1, 2, \cdots.$$

Hence we have

$$\begin{aligned} y_{m+p} - y_m &= y_{m+p} - y_{m+p-1} + \cdots + y_{m+1} - y_m \\ &\in \mu^{m+p-1} K(y_1 - y_0) + \cdots + \mu^m K(y_1 - y_0) \\ &\in (\mu^{m+p-1} + \cdots + \mu^m) K(y_1 - y_0), \end{aligned}$$

where the last step is justified since by (1), $K(y_1 - y_0)$ is a convex set. By (5) the set $K(y_1 - y_0)$ is bounded, so that there is a $\delta > 0$ corresponding to each neighborhood U of the origin θ such that $0 < \beta < \delta$ implies $\beta K(y_1 - y_0) \subset U$. Now $0 < \mu < 1$ so that $\sum \mu^n$ converges, and for sufficiently large m and any p we have $y_{m+p} - y_m \in U$, whence y_n is a fundamental sequence. The existence of a fixed point now follows immediately, while the uniqueness is readily proved by using property (2) together with the inequality $0 < \mu < 1$.

THEOREM 8. *In any locally convex linear topological space there always exist complete neighborhood systems $\mathcal{V}$: $(U, V, W, \cdots)$ of the origin with the properties:*

(i) *if $U \in \mathcal{V}$, then U is convex;*

(ii) *if $U \in \mathcal{V}$ then $\alpha U \in \mathcal{V}$, for each real α.*

For any such neighborhood system U *put* $K(x)=\bigcap\{U;\ U\in\mathcal{V},\ x\in U\}$. *Then the sets* $K(x)$ *form a Lipschitz system for the space* L *and we have* $\bar{K}(x)=\bigcap\{\bar{U};\ U\in\mathcal{V},\ x\in\bar{U}\}$.

This theorem gives us a useful way of finding Lipschitz systems in any particular locally convex space. For instance, in weakly topologized Hilbert space $\mathfrak{H}$ the theorem tells us that the sets $K(x)$ of all z for which $|(z, y)| \leqq |(x, y)|$, as y runs over $\mathfrak{H}$, form a Lipschitz system.

Theorem 7 in conjunction with Theorem 8 may be used to prove existence theorems for differential or integral equations (Hyers [1, 2]). By interpreting the space L in different ways a variety of existence theorems can be proved. For instance, on taking L to be the space (s) of all real sequences $x=(x_1, x_2, \cdots)$, topologized by component-wise convergence, we can obtain an existence theorem for an infinite system of ordinary differential equations. Again if L is taken to be the weakly topologized Hilbert space $\mathfrak{H}$, one may establish an existence theorem for "weak" differential equations for functions whose values lie in Hilbert space.

4. **Pseudo-norms.** The notion of a pseudo-norm, introduced by J. von Neumann [1] for the case where the l.t.s. is locally convex, is both useful and suggestive in theories of linear transformations, linear functionals, differentials, and so on. It springs from the same idea already used in proving Kolmogoroff's normability theorem, and in connection with locally bounded spaces, namely, the Minkowski functional. It differs from a norm (or the quasi-norm introduced in §2) in that the vanishing of the pseudo-norm of an element x does not imply that $x=\theta$. The following treatment is based upon the author's generalization[18] of von Neumann's pseudo-norm, so that it may be applied to any l.t.s.

DEFINITION 4.1. An arbitrary set D with elements $a, b, c, \cdots$ together with a binary relation $>$ is called a *directed system*[19] if the following postulates are satisfied:

(i) either $a>b$ or $a \not> b$ for every pair a, b in D;

(ii) if $a>b$ and $b>c$ then $a>c$;

(iii) given $a\in D$ and $b\in D$ there exists $c\in D$ such that $c>a$ and $c>b$.

Properties (i) and (ii) state that D is partially ordered, and (iii) is the "composition" property of Moore and Smith.

DEFINITION 4.2. A linear space L will be said to be *pseudo-normed* with respect to a directed system D if there exists a real-valued functional $|x|_d$ defined for all $x \in L$ and $d \in D$ which satisfies the following postulates.

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[18] Hyers [5]. See also LaSalle [2].

[19] Moore and Smith [1]; G. Birkhoff [2]; Tukey [1]. In our notation $a>b$ includes the statement $a=b$.

(1) $|x|_d \geqq 0$; $|x|_d = 0$ for all $d \in D$ implies $x = \theta$.

(2) $|\alpha x|_d = |\alpha| \cdot |x|_d$ for all real α and every $x \in L$.

(3) Given $e \in D$ there exists $d \in D$ such that $|x+y|_e \leqq |x|_d + |y|_d$ for all x and y in L.

(4) $|x|_d \geqq |x|_e$ whenever $d > e$, and for all x in L.

Remark. The property (3) in Definition 4.2 is due to LaSalle [2]. A weaker property was used in the author's original definition.

As a simple example of a pseudo-norm, consider the space (s) of all real sequences $x = (x_1, x_2, \cdots)$ and take D to be the ordered set of positive integers. Then the functional $|x|_d = \max\{|x_i|; 1 \leqq i \leqq d\}$ is a pseudo-norm for (s) with respect to the set of positive integers.

Having defined a pseudo-norm for a space, the topology, obtained by defining x to be a limit point of a set S if, given $\delta > 0$ and $d \in D$, there exists a point $y \neq x$ such that $|x-y|_d < \delta$, is called the *topology generated by the pseudo-norm.* In the example just given the pseudo-norm generates a topology equivalent to the usual metric topology for (s).

Theorem 9. *Every pseudo-normed linear space L is a l.t.s. in which the pseudo-norm generates the topology of L. Conversely, given any l.t.s. L, there exists a directed system D with respect to which L may be pseudo-normed in such a way that the pseudo-norm generates the topology of L.*[20]

Corollary. *The directed system D for a l.t.s. may be identified with a certain complete neighborhood system $\{U\}$ of the origin, in which $U_1 > U_2$ means $U_1 \subset U_2$.*

The directed system D for a l.t.s. L will be called *a directed system associated with L.*

Theorem 9 affords an alternative definition of a l.t.s. In some cases it seems simpler to define a pseudo-norm directly instead of defining an equivalent neighborhood topology.

A pseudo-norm is called *triangular* if it satisfies the postulate:

(3a) $|x+y|_d \leqq |x|_d + |y|_d$ for all x, $y \in L$ and $d \in D$; that is, if $d = e$ in property (3) of Definition 4.2.

The "pseudo-metric" of von Neumann [1] is a triangular pseudo-norm for which D is a suitably chosen complete neighborhood system of θ. It is possible to define a triangular pseudo-norm for a l.t.s. L if and only if L is locally convex.[21] A good many properties of l.t.s. and their subsets can be stated conveniently in terms of the pseudo-norm. Thus, it is easily shown on the basis of Theorem 9 that a subset S of a l.t.s. L is bounded if and only if, given $d \in D$, there exists $\mu > 0$ such that $|x|_d < \mu$ for all $x \in S$.

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[20] Hyers [5]; LaSalle [2].

[21] von Neumann [1].

The notion of pseudo-norm has been generalized still more by J. P. LaSalle [1, 3] to apply to linear spaces in which the "scalars" are elements of a normed ring instead of real numbers, and also to topological spaces which need not be linear spaces.

5. **Linear transformation and functionals.** In this section we shall consider additive transformations $T(x)$ on a l.t.s. L_1 to a l.t.s. L_2. The following theorem was proved by Wehausen [1] for the case in which L_1 and L_2 are both locally convex. However, the proof remains unaltered for the present more general statement.

THEOREM 10. *Let L_1 and L_2 be two l.t.s. and let D_1 and D_2 be associated directed systems for L_1 and L_2, respectively. An additive transformation T on L_1 to L_2 is continuous if and only if to each e in D_2 there corresponds a d in D_1 and a real number $\mu \geqq 0$ such that $|T(x)|_e \leqq \mu |x|_d$, for all x in L_1.*

COROLLARY. *A linear continuous transformation on L_1 to L_2 takes bounded sets into bounded sets.*

In the study of linear operators on L_1 to L_2, one of the natural questions which arises is that of finding a suitable topology for the set $\mathfrak{T}$ of all continuous linear transformations T on L_1 to L_2 which will make $\mathfrak{T}$ a linear topological space. A satisfactory topology was given by LaSalle [2] in terms of a pseudo-norm. The focus of attention is the set $\mathfrak{B}_1$ of all bounded sets B of L_1 which contain the origin, rather than a neighborhood system of the origin in L_1. $\mathfrak{B}_1$ is a directed system if we define $B > B'$ to mean $B \subset B'$. Consider the set D of all pairs $e = (d, B)$, where $d \in D_2$, $B \in \mathfrak{B}_1$, and D_2 is a directed system associated with L_2. Then E is also a directed system, if we define $e > e'$ to mean that $d > d'$ and $B > B'$.

THEOREM 11. *For each $T \in \mathfrak{T}$ and each $e \in E$, put $|T|_e = \sup \{ |T(x)|_d ; x \in B \}$. Then $|T|_e$ is a pseudo-norm for the linear space $\mathfrak{T}$, and E is an associated directed system. In case L_2 is locally convex, then $|T|_e$ is a triangular pseudo-norm.*[22]

In particular, Theorem 11 provides a topology for the space L^* of all linear functionals on a l.t.s. L. An equivalent topology in this special case was used by Bourgin [1].

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[22] LaSalle [2, Theorems 5, 6].

We now turn to the special case of functionals on L. The following existence theorems, due to Wehausen [1] and LaSalle [2] respectively, may be proved on the basis of the Hahn-Banach theorem (Banach [1, p. 27, Theorem 1]). These two theorems once again emphasize the importance of convexity properties in connection with linear functionals.

THEOREM 12. *If L is a locally convex l.t.s. and D is an associated directed system, then for any $x_0 \in L$ and any $d \in D$ there exists a linear continuous functional $f(x)$ defined on L with the property that $f(x_0) = |x_0|_d$.*

It is easily shown by example (for example, the sequence space $l^{1/2}$) that this condition of local convexity, while sufficient, is not necessary for the existence of non-null linear continuous functionals. This remark adds interest to LaSalle's result:

THEOREM 13. *A necessary and sufficient condition that there exist a non-null continuous linear functional on a l.t.s. L is that there exist a convex open set which is properly contained in L.*

The question of the existence of linear continuous functionals in function spaces of the type L^p and l^p, where $0<p<1$, has been studied by M. M. Day [1]. In particular it was shown that every continuous linear functional defined on the space L^p of real functions $f(x)$ on the interval $(0, 1)$ is identically zero when $0<p<1$.

In the light of Theorems 12 and 13 the desirability of requiring local convexity in order to establish a satisfactory theory of linear functionals on a l.t.s. becomes apparent. In the case of a locally convex l.t.s., a good many generalizations of theorems on linear functionals which are well known for Banach spaces can be carried through. In particular, theorems of the Helly type (Banach [1, pp. 55–57]) on the extension of functionals, which are important for their applications to the problem of moments and the solution of infinite systems of linear equations, have been generalized by Wehausen [1] and Bourgin [1].

The relationship between a locally convex l.t.s. L and the space L^* of linear functionals on L has been studied from a new viewpoint by G. W. Mackey [1, 2]. He starts with an abstract linear space X, singles out a certain subset of the set of all linear (additive and homogeneous) functionals on X, and considers the problem of what topologies in X will make the distinguished set of functionals the *continuous* functionals. In the case of *normed* spaces, there is a theorem of Fichtenholz [1] to the effect that two norms in the same linear space give the same topology if and only if the same linear functionals are continuous with respect to both norms. Such is not the case, however, in the case of locally convex l.t.s., as shown by the example of a Banach space with its weak and strong topologies. This leads to the concepts of *relatively weak* and *relatively strong* l.t.s., and to several theorems involving these notions. (In the example just referred to, the norm topology t_1 is relatively strong, and the weak topology t_2 obtained from the set of all linear continuous functionals is relatively weak. Any other locally convex topology for the same space and leading to the same set of continuous linear functionals will be stronger than t_2 and weaker than t_1.) In particular it is shown that a locally convex l.t.s. of the second category is relatively strong. This theorem generalizes Wehausen's result [1] that an infinitely-dimensional Banach space with its weak topology is of the first category.

6. **Differentials.** In functional analysis a number of definitions of a differential have been used.[23] Perhaps the simplest is the "weak" or Gateaux differential $df(x+\alpha\delta x)/d\alpha|_{\alpha=0}$ which is simply the "variation" used in the calculus of variations. However, the Gateaux differential does not have many of the usual properties associated with the total differential for functions of two or more real variables, and we shall be interested in the generalization of the *total* differential. At this point it might be well to list the properties which a differentiable function should have. Denoting the differential of $f(x)$ with increment δx by $df(x; \delta x)$ we shall require that (I) if a function is differentiable at a point it is continuous at this point; (II) the composite function rule holds, that is, if $g(x)=f(\phi(x))$ and $d\phi$ and df exist then dg exists and $dg(x; \delta x)=df(y; dy)$, where $y=\phi(x)$, $dy=d\phi(x; \delta x)$; (III) the differential is linear and continuous in the increment; (IV) $df(x; \delta x)$ is a first order approximation to the difference $f(x+\delta x)-f(x)$ when δx is "close" to θ.

Property (IV) is purposely left vague. The various definitions which will be discussed differ principally in the exact interpretation of this property.

The standard definition of a differential of a function on E_1 to E_2, where E_1 and E_2 are linear normed spaces, is due to Fréchet [1]. A function on E_1 to E_2 is Fréchet differentiable at the point x with the differential $df(x; \delta x)$ if $df(x; \delta x)$ is a linear continuous function of δx on E_1 to E_2 such that $\lim_{h\to\theta}\|f(x+h)-f(x)-df(x; h)\|/\|h\|=0$. It is well known that the Fréchet differential has the four properties listed above. The definition shows the sense in which it satisfies (IV).

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[23] For example, see Fréchet [1, 2]. For a discussion see Graves [1].

The differential calculus has been extended to include functions whose arguments and values lie in l.t.s. and also in topological groups by A. D. Michal and his students. The first step in this direction was made by Michal and Paxson [1, 2] in 1936. However, their definition did not prove entirely satisfactory, since the domain and range of the functon had to lie in the same space, and the rule (II) for the differentiation of composite functions was not demonstrated. Since then a number of different definitions have been proposed having the properties (I), (II), (III), (IV). We consider first Michal's M-differential.[24]

Definition 6.1. Let L_1, L_2 be l.t.s., and let $f(x)$ be defined on a neighborhood N of a point $x_0 \in L_1$ to L_2. The function $f(x)$ will be said to be *M-differentiable* at $x = x_0$, and $df(x_0;\ \delta x)$ will be called an *M-differential* of $f(x)$ at $x = x_0$ with increment δx if and only if

(1) $df(x_0;\ \delta x)$ is a linear continuous function on L_1 to L_2;

(2) there exists a function $\epsilon(x_0, x_1, x_2)$ with arguments in L_1 and values in L_2 such that

(a) $\epsilon(x_0, \theta, x) = \theta$ for all $x \in L_1$;

(b) $\epsilon(x_0, x_1, \lambda x_2) = \lambda\epsilon(x_0, x_1, x_2)$; for all $\lambda > 0$, for all x_1 in some neighborhood of $\theta \in L_1$ and for all $x_2 \in L_1$;

(3) there exists a neighborhood U of $\theta \in L_1$ such that $f(x_0 + \delta x) - f(x_0) - df(x_0;\ \delta x) = \epsilon(x_0, \delta x, \delta x)$ for all $\delta x \in U$.

The M-differential can be shown to have the properties I, II, III, and the Definition 6.1 shows the sense in which it satisfies IV. Every M-differential is a Gateaux differential, so that the function $df(x_0;\ \delta x)$ is unique, for a given x_0 and a given $f(x)$. When L_1 and L_2 are normable spaces, *every Fréchet differential is also an M-differential*, but the converse statement is not true, as is shown by the following example.

Let L_1 be the linear normed space of all sequences $x = (x_1, x_2, \cdots)$ such that only a finite number of the x_i's are different from zero, where $\|x\| = \sup\{|x_i|;\ 0 < i < \infty\}$. Let L_2 be the linear normed space of all bounded sequences, with the norm defined in the same way. Define the function $f(x) = (f_1(x), f_2(x), \cdots)$ on L_1 to L_2 by means of the formula $f_r(x) = (r^{1/2} + r^{3/2}x_r^2)^{-1}$, $r = 1, 2, 3, \cdots$. It can be shown that $f(x)$ is M-differentiable at the origin with $df(\theta;\ \delta x) = \theta$, by defining $\epsilon(\theta, y, z) = -r^{1/2}y_r z_r/(1 + r y_r^2)$ and verifying that the conditions (1), (2), (3) of Definition 6.1 are satisfied.

Now if we assume that $f(x)$ is Fréchet differentiable, then there is a $\delta > 0$ such that $\|\epsilon(\theta, y, y)\| \leqq \|y\|/4$ for $\|y\| < \delta$. Choose n so large that $n^{1/2} > \delta^{-1}$, and choose y so that $y_r = 0$ for $r \neq n$, and $y_n = n^{-1/2}$. Clearly

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[24] Michal [1, 2]. A number of other definitions of differentials have been given by the same author. See Michal [1–5].

$\|y\| < \delta$. However, $\|\epsilon(\theta, y, y)\|/\|y\| = 1/2$. Hence $f(x)$ is *not* Fréchet differentiable at the origin. This example shows that the M-differential satisfies the condition (IV) in an essentially weaker sense than does the Fréchet differential.

Since Fréchet's original definition leans so heavily on the concept of a norm, it is natural to try to phrase a definition in terms of pseudo-norms (Hyers [6]).

DEFINITION 6.2. Let L_1 and L_2 be l.t.s., and let D_1, D_2 be associated directed systems. The function $f(x)$ on an open set $G \subset L_1$ to L_2 will be said to be *F-differentiable* at the point x of G, and $df(x; \delta x)$ will denote its F-differential if

(a) $df(x; h)$ is linear and continuous in h on L_1 to L_2;

(b) for every $e \in D_2$ there corresponds $d \in D_1$ such that, given $\eta > 0$, there exists $\delta > 0$ with $|f(x+h) - f(x) - df(x; h)|_e \leqq \eta |h|_d$ whenever $|h|_d < \delta$.

Note that d depends only on e, while δ depends on e, d and η. It can be shown that every F-differentiable function is also M-differentiable. The F-differential has the properties I–IV. In addition, when L_1 and L_2 are normable, the F-differential reduces to the Fréchet differential. It follows that the M-differential is more general than the F-differential.

The notion of a differential for functions whose arguments and values are in certain abelian topological groups was also introduced by Michal [3, 4]. The definition of the differential was of the same general character as Definition 6.1, except that λ was, of course, restricted to be an integer, and a stronger uniformity condition was imposed on the ϵ-function. This uniformity property was strong enough to make the definition reduce to that of Fréchet when the topological abelian groups were taken to be linear normed spaces and when the homogeneity property of the ϵ-function was assumed to hold for all real scalars. However, for l.t.s. this modified M-differential is *less* general than the F-differential. Higher order differentials and their properties were investigated by the same author.[25]

Later LaSalle (1) introduced differentials for transformations of a still more general type of space—a vector space in which the scalar multipliers were replaced by elements of a valued ring. His definition makes use of a suitably generalized pseudo-norm. Differentials for functions on noncommutative topological groups have been studied by K. Millsaps [1].

E. J. Pinney [1] has studied the calculus of variations for functionals on a l.t.s., making use of the m-differential.

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[25] Michal [5].

Although a good many definitions of differentials have been given and many of their more elementary properties have been studied, much remains to be done in the development of the differential calculus for transformations of one l.t.s. into another. In particular it would be interesting to obtain a generalization of the implicit function theorems proved by Hildebrandt and Graves [1] for the case of linear normed spaces. These theorems have many applications in functional analysis and it is desirable to extend their range of application still farther.

7. **Convergence and completeness.** We have seen how the topology of a l.t.s. could be generated by means of neighborhoods or by means of pseudo-norms. Another equivalent way is to use a suitably generalized notion of *convergence*. In non-metric topologies it is clearly not sufficient to consider merely the convergence of a sequence in the ordinary sense, since, by von Neumann's example already referred to, a point p may be a limit point of a set and yet no sequence of elements of the set may converge to the point. However, by using some ideas of Moore and Smith [1] on the convergence of integrals, Garrett Birkhoff [2] succeeded in generalizing the notion of convergence so that the convergence topology is equivalent to the neighborhood topology, even in very general spaces.[26] Given an abstract directed system D (Definition 4.1) and a topological space T, a *directed set*[27] x_α is defined to be a function on D to T. A directed set x_α is said to *converge to the limit point* x if, for every neighborhood V_x of x, there exists $\alpha_0 \in D$ such that $\alpha > \alpha_0$ implies that $x_\alpha \in V_x$. Thus the generalization consists in replacing the ordered set of positive integers coming into the definition of a sequence by directed systems D.

In terms of this concept of convergence Birkhoff defined "completeness" of a l.t.s. as follows. First, a directed set x_α will be called *fundamental* if, for every neighborhood U of the origin, there exists α_0 such that $\alpha > \alpha_0$, $\beta > \alpha_0$ imply that $x_\alpha - x_\beta \in U$. A l.t.s. L will be said to be *complete* if every fundamental directed set converges to some $x \in L$. Completeness is an extensionally obtainable property,[28] although most of the "natural" instances of l.t.s. are not complete. For example, weak Hilbert space is not complete, in Birkhoff's sense.

Several other types of completeness have been defined for l.t.s., all of which coincide in metric spaces. The most obvious definition is that of "sequential completeness" already used in connection with the fixed point theorem of a previous section. This is the weakest type of completeness.

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[26] For a discussion see Tukey [1].

[27] The term "directed function" might have been better.

[28] Birkhoff [2].

J. von Neumann [1] has defined "topological completeness" by generalizing another criterion for completeness in metric spaces. A subset S of a l.t.s. is called *totally bounded* if for every neighborhood U of the origin there is a finite set $x_1, x_2, \cdots, x_n$ of points of L such that the family of sets x_i+U, $i=1, \cdots, n$, covers S. Then L is said to be *topologically complete* if every totally bounded set is compact. It can be shown[29] that every topologically complete l.t.s. is also sequentially complete. In particular weak Hilbert space $\mathcal{H}$ is topologically and hence sequentially complete.

Still other definitions of completeness have been given by Šmulian [1] and Taylor [1, 2]. Šmulian calls a l.t.s. *quasi-complete* if every totally bounded closed set is complete in Birkhoff's sense, and Taylor calls a l.t.s. *boundedly complete* if every bounded fundamental directed set is convergent, that is, if every bounded closed set is complete.

If we denote by C the class of complete l.t.s., by B the class of those which are boundedly complete, by Q those which are quasi-complete, by T the topologically complete and by S the sequentially complete l.t.s., then we have the following inclusion relations:

$$C \subset B \subset Q \subset T \subset S.$$

An important theorem on metric spaces states that every complete metric space is of the second category. However, this is not true[30] for all l.t.s. belonging to the classes S, T, Q or B.

The notions of convergence of directed sets and the various notions of completeness and compactness arising therefrom have been applied to the study of Banach spaces by a large and still growing number of authors, including Alaoglu, Bourgin, Day, Dunford, Goldstine, Milman, Pettis, Šmulian, Taylor and several others. The ideas centering around the notion of weak (sequential) convergence as developed by Banach [1] are generalized by the use of the convergence of directed sets instead of sequences. This often makes possible the elimination of the hypothesis of separability which usually occurs in Banach's results. To quote only one example, a well known theorem[31] states that if a Banach space E is separable and has a weakly compact unit sphere then E is reflexive. An equivalent statement of this result may be obtained[32] by replacing the weak compactness condition by the condition that $\mathcal{E}$ be topologically complete. Goldstine [1] generalized this result by the use of convergence of directed sets, and obtained a

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[29] von Neumann [1].

[30] Wehausen [1]; Mackey [2].

[31] Banach [1, p. 189].

[32] Taylor [2, §6].

necessary and sufficient condition for reflexivity. Goldstine's result may be stated in either of the two forms:

A Banach space E is reflexive if and only if

(a) $\mathcal{E}$ is boundedly complete,[33]

(b) $\mathcal{E}$ is quasi-complete.[34]

Several other conditions for reflexivity have been given by Milman, Day, Bourgin and others in terms of various types of weak compactness of the unit sphere. However, for lack of space we are unable to pursue this interesting topic further.

8. **Conclusion.** There are several topics which we must leave with but the briefest mention. The subject of integration of functions whose values are in l.t.s. has been considered by a number of authors. In particular, integrals of the Lebesgue type have been defined by G. Birkhoff [2], R. S. Phillips [1], and the present author.[35] The work has been carried the farthest by Phillips, who showed that by properly choosing the neighborhood topology for the special case of a Banach space, his integral can be made to include the integrals of Birkhoff, Dunford, Gelfand, and Pettis, depending on the "strength" of the topology used. This is an example of how the concept of l.t.s. may be used to unify various theories which appear quite diverse from the standpoint of Banach space theory.

The theory of almost periodic functions has been studied by S. Bochner and J. von Neumann [1] in a very general setting, in which the arguments of the functions lie in groups, and the values in a l.t.s.

Another interesting topic is the relation between l.t.s. and abelian topological groups. The question of characterizing the l.t.s. among all abelian topological groups has been answered by E. W. Paxson [4] and R. C. James [1]. The question of characterizing the subgroups if a l.t.s. has not yet been satisfactorily solved in general. Partial answers have been given by James [2] and Hyers [5].

In conclusion we mention another unsolved problem, namely, under what conditions does a linear transformation of one l.t.s. into another have an inverse? The general type of existence proof used for metric spaces seems unavailable here, since non-metric l.t.s. are usually of the first category, even when they satisfy completeness conditions. Apparently little has been done on this problem even for special types of transformations such as $T(x)=x+G(x)$ where $G(x)$ is completely continuous.

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[33] Taylor [1, 2]. $\mathcal{E}$, as usual, denotes the Banach space with its weak topology.

[34] Šmulian [1].

[35] Hyers [1].

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The University of Southern California and
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THE OCTOBER MEETING IN NEW YORK

The four hundred seventh meeting of the American Mathematical Society was held at Columbia University on Saturday, October 28, 1944. The attendance included the following one hundred fifty-five members of the Society:

R. L. Anderson, R. G. Archibald, H. E. Arnold, L. A. Aroian, F. E. Baker, P. T. Bateman, E. G. Begle, Felix Bernstein, Lipman Bers, Garrett Birkhoff, Gertrude Blanch, H. W. Bode, C. B. Boyer, A. B. Brown, G. E. Bulloch, Hobart Bushey, J. H. Bushey, S. S. Chern, Edmund Churchill, B. H. Colvin, T. F. Cope, Richard Courant, M. D. Darkow, J. A. Daum, M. M. Day, John DeCicco, Jesse Douglas, Jacques Dutka, Samuel Eilenberg, Paul Erdös, J. M. Feld, A. D. Fialkow, L. A. Fine, M. C. Foster, R. M. Foster, R. H. Fox, G. A. Foyle, Hans Fried, Bernard Friedman, K. O. Friedrichs, G. N. Garrison, Abe Gelbart, David Gilbarg, B. P. Gill, R. O. Goodman, W. H. Gottschalk, H. J. Greenberg, Bernard Greenspan, H. M. Griffin, C. C. Grove, F. C. Hall, P. R. Halmos, G. A. Hedlund, Olaf Helmer, Erik Hemmingsen, M. R. Hestenes, Edwin Hewitt, Einar Hille, Abraham Hillman, G. P. Hochschild, T. R. Hollcroft, E. M. Hull, W. A. Hurwitz, (L.) C. Hutchinson, B. M. Ingersoll, S. A. Joffe, Irving Kaplansky, Edward Kasner, M. E. Kellar, J. W. Kitchens, J. R. Kline, E. G. Kogbetliantz, Horace Komm, D. M. Krabill, M. E. Ladue, Solomon Lefschetz, Max LeLeiko, Howard Levene, W. J. LeVeque, Howard Levi, Norman Levinson, Marie Litzinger, W. R. Longley, L. H. Loomis, Simon Lopata, E. R. Lorch, A. N. Lowan, Eugene Lukacs, Janet McDonald, L. A. MacColl, G. W. Mackey, Saunders MacLane, H. F. MacNeish, A. J. Maria, D. M. H. Maria, W. T. Martin, A. E. Meder, L. W. Miller, Deane Montgomery, G. W. Mullins, C. A. Nelson, A. V. Newton, P. B. Norman, J. C. Oxtoby, Gordon Pall, F. W. Perkins, W. A. Pierce, Harry Pollard, E. L. Post, M. H. Protter, Hans Rademacher, Tibor Radó, J. F. Ritt, H. A. Robinson, P. C. Rosenbloom, Raphael Salem, H. E. Salzer, Arthur Sard, L. J. Savage, I. J. Schoenberg, E. J. Scott, I. E. Segal, C. E. Shannon, L. L. Silverman, M. H. Slud, P. A. Smith, Ernst Snapper, I. S. Sokolnikoff, Herbert Solomon, J. J. Stoker, E. G. Straus, W. C. Strodt, Fred Supnick, Otto Szász, Feodor Theilheimer, C. J. Thorne, Leonard Tornheim, J. W. Tukey, Annita Tuller, Andrew Vazsonyi, R. M. Walter, S. E. Warschawski, W. R. Wasow, Alan Wayne, G. C. Webber, Louis Weisner, A. P. Wheeler, J. E. Wilkins, John Williamson, Y. C. Wong, S. C. Wu, Z. S. Wurtele, Daniel Zelinsky, M. F. Zucker, Antoni Zygmund.

There was a session for contributed papers in the morning at which Professor W. T. Martin presided. At the general session in the afternoon Professor Gordon Pall of McGill University gave an address entitled *The arithmetical invariants of quadratic forms*. Professor Garrett Birkhoff presided at the afternoon session.

Titles and cross references to the abstracts of papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1–9 were read in the morning session and papers 10–29 were presented by title. Paper 1 was read by Dr. Pollard. Mr. Hsiung was introduced by Professor T. R. Hollcroft and Professor Hua and Mr. Min were introduced by Professor A. T. Brauer.

1. R. P. Boas and Harry Pollard: *Completeness of sets of functions. I. The Legendre functions on* $(-1, 1)$. Preliminary report. (Abstract 50-11-271.)

2. J. E. Wilkins: *A special class of surfaces in projective differential geometry.* II. (Abstract 50-11-281.)

3. Otto Szász: *On Lebesgue summability and its generalization to integrals.* (Abstract 50-11-275.)

4. L. A. Aroian: *The frequency function of the product of two normally distributed variables.* (Abstract 50-11-283.)

5. J. J. Stoker: *Nonlinear theory of curved elastic sheets.* (Abstract 51-1-35.)

6. Edward Kasner: *Multi-valued symmetries.* (Abstract 51-1-41.)

7. John DeCicco: *Survey of polygenic functions.* (Abstract 51-1-38.)

8. Einar Hille: *The commutative n-parameter case of semi-groups of linear transformations.* (Abstract 51-1-15.)

9. R. H. Fox: *An application of the complete homotopy group.* (Abstract 51-1-49.)

10. R. F. Arens: *Topologies for the class of continuous functionals on a completely regular space.* (Abstract 50-11-284-*t*.)

11. Stefan Bergman: *Representation of potentials of electric charge distributions.* (Abstract 50-11-276-*t*.)

12. Paul Erdös: *Converse of Fabry's gap theorem.* (Abstract 50-9-216-*t*.)

13. H. W. Eves: *Calculating machine computation forms for the three-point problem on a rectangular coordinate system.* (Abstract 50-11-277-*t*.)

14. A. L. Foster: *The idempotent elements of a commutative ring form a Boolean algebra; application to principal ideal rings; ring duality and general transformation theory.* (Abstract 50-11-263-*t*.)

15. C. C. Hsiung: *New geometrical characterizations of some conjugate nets.* (Abstract 50-9-241-*t*.)

16. L. K. Hua: *On the distribution of quadratic non-residues and the Euclidean algorithm in real quadratic fields.* I. (Abstract 50-11-264-*t*.)

17. L. K. Hua and S. H. Min: *On the distribution of quadratic non-residues and the Euclidean algorithm in real quadratic fields.* II. (Abstract 50-11-265-*t*.)

18. B. M. Ingersoll: *On singularities of solutions of linear partial differential equations.* III. (Abstract 50-11-272-*t*.)

19. Nathan Jacobson: *A definition of the radical for an arbitrary ring.* (Abstract 50-11-266-*t*.)

20. Nathan Jacobson: *A problem on algebraic algebras.* (Abstract 51-1-8-*t*.)

21. R. L. Jeffery and D. S. Miller: *Convergence factors for generalized integrals.* (Abstract 50-9-219-*t*.)

22. Jakob Levitzki: *Chain-conditions and nilpotency.* (Abstract 50-11-267-*t*.)

23. A. E. Meder and Elizabeth Hallett: *The equipollence of two systems of axioms.* (Abstract 50-11-282-*t*.)

24. Raphael Salem: *On power series with integral coefficients.* (Abstract 50-11-274-*t*.)

25. I. J. Schoenberg: *An interpolation formula derived from heat-flow.* I. (Abstract 50-11-278-*t*).

26. I. J. Schoenberg: *On smoothing and subtabulation of empirical functions by means of heat-flow.* II. (Abstract 50-11-279-*t*.)

27. A. H. Stone: *On the normality of product spaces.* Preliminary report. (Abstract 50-11-286-*t*.)

28. Alexander Weinstein and J. R. Pounder: *On two elementary problems of mechanics and electromagnetic theory.* (Abstract 50-11-280-*t*.)

29. G. W. Whitehead: *On products in homotopy groups.* (Abstract 50-11-288-*t*.)

T. R. HOLLCROFT,
Associate Secretary

THE ANNUAL MEETING OF THE SOCIETY

The fifty-first Annual Meeting of the American Mathematical Society was held at the Museum of Science and Industry, Chicago, Illinois, Friday and Saturday, November 24 and 25, 1944, in conjunction with meetings of the Mathematical Association of America. Over two hundred persons registered, including the following one hundred seventy-seven members of the Society:

V. W. Adkisson, A. A. Albert, Warren Ambrose, B. M. Armstrong, W. A. Asprey, Max Astrachan, Reinhold Baer, R. M. Ballard, R. H. Bardell, Walter Bartky, Felix Bernstein, H. R. Beveridge, S. F. Bibb, H. L. Black, G. A. Bliss, Henry Blumberg, D. G. Bourgin, M. G. Boyce, A. J. Brandt, R. W. Brink, R. H. Bruck, E. L. Buell, R. S. Burington, Herbert Busemann, W. H. Bussey, Albert Cahn, W. D. Cairns, W. B. Carver, W. B. Caton, Subrahmanyan Chandrasekhar, J. H. Chanler, E. W. Chittenden, R. V. Churchill, M. D. Clement, R. H. Cole, J. J. Corliss, D. R. Curtiss, Paul D'Arco, J. Edgar Davis, John DeCicco, M. W. Dehn, Douglas Derry, Paul Eberhart, Paul Erdös, H. P. Evans, H. S. Everett, Will Feller, J. V. Finch, N. J. Fine, L. R. Ford, R. H. Fox, J. S. Frame, Evelyn Frank, R. E. Fullerton, H. M. Gehman, J. W. Givens, G. D. Gore, L. M. Graves, L. W. Griffiths, V. G. Grove, P. E. Guenther, D. W. Hall, P. R. Halmos, R. W. Hamming, W. L. Hart, J. O. Hassler, E. D. Hellinger, Fritz Herzog, E. H. C. Hildebrandt, T. H. Hildebrandt, J. D. Hill, D. L. Holl, T. R. Hollcroft, S. P. Hughart, H. K. Hughes, C. C. Hurd, M. H. Ingraham, R. D. James, Karl Johannes, A. W. Jones, B. W. Jones, William Karush, D. E. Kearney, I. F. Keeler, H. R. Kingston, J. R. Kline, L. A. Knowler, J. C. Koken, W. C. Krathwohl, Joseph Landin, E. P. Lane, R. E. Langer, E. H. Larguier, D. H. Leavens, B. A. Lengyel, A. L. Lewis, A. T. Lonseth, C. C. MacDuffee, H. F. MacNeish, Murray Mannos, Morris Marden, A. V. Martin, R. M. Martin, W. T. Martin, J. R. Mayor, Karl Menger, A. N. Milgram, H. J. Miser, W. L. Miser, A. C. Moeller, J. T. Moore, C. W. Moran, E. M. Morenus, E. J. Moulton, F. H. Murray, J. R. Musselman, A. L. Nelson, John von Neumann, C. V. Newsom, Ivan Niven, E. P. Northrop, F. S. Nowlan, E. B. Ogden, Rufus Oldenburger, J. M. H. Olmsted, F. W. Owens, Gordon Pall, P. M. Pepper, George Piranian, J. C. Polley, J. F. Randolph, R. B. Rasmusen, W. T. Reid, Haim Reingold, P. R. Rider, L. V. Robinson, W. H. Roever, A. E. Ross, E. H. Rothe, Arthur Saastad, Hans Samelson, R. G. Sanger, O. F. G. Schilling, K. C. Schraut, G. E. Schweigert, I. E. Segal, M. E. Shanks, H. A. Simmons, R. C. Simpson, E. R. Smith, G. W. Smith, W. S. Snyder, R. C. Stephens, B. M. Stewart, E. B. Stouffer, E. G. Swafford, H. P. Thielman, T. Y. Thomas, C. J. Thorne, W. J. Trjitzinsky, Henry Van Engen, J. I. Vass, W. A. Vezeau, H. S. Wall, M. J. Weiss, E. L. Welker, P. A. White, G. W. Whitehead, W. M. Whyburn, L. R. Wilcox, J. E. Wilkins, K. P. Williams, R. S. Wolfe, M. A. Wurster, Oscar Zariski, M. C. Zeller, H. J. Zimmerberg.

Sessions for the reading of contributed papers were held Friday at 2:00 P.M., Saturday at 10:45 A.M. and 3:15 P.M. The presiding officers for these sessions were Dean E. B. Stouffer, Professor E. W. Chittenden and Professor Karl Menger.

The eighteenth Josiah Willard Gibbs Lecture was given Friday evening by Professor John von Neumann on *The ergodic theorem and statistical mechanics.* Vice President L. M. Graves presided. The attendance was about three hundred.

The annual business meeting and election of officers was held on Saturday morning, Vice President Graves presiding. The proceedings are included later in the report. At the close of the business session, Vice President Graves announced the award of the Cole Prize in Algebra to Professor Oscar Zariski of the Johns Hopkins University for the following series of memoirs: (a) *Some results in the arithmetic theory of algebraic varieties,* Amer. J. Math. vol. 61 (1939) pp. 249–294; (b) *Algebraic varieties over ground fields of characteristic zero,* Amer. J. Math. vol. 62 (1940) pp. 187–221; (c) *The reduction of the singularities of an algebraic surface,* Ann. of Math. vol. 40 (1939) pp. 639–689; (d) *Local uniformization on algebraic varieties,* Ann. of Math. vol. 41 (1940) pp. 852–896. Professor Zariski gave a brief résumé of these memoirs.

Saturday afternoon with Professor John von Neumann presiding, Professor Will Feller of Brown University gave an address entitled *Limit theorems in the theory of probability.*

Sessions of the Mathematical Association of America were held on Saturday evening and Sunday morning.

There was a dinner Saturday evening at Hotel Windermere West for mathematicians and their guests. The toastmaster was President W. M. Whyburn of Texas Technological Institute. Professor R. E. Langer of the University of Wisconsin spoke on some problems which may be expected to confront mathematicians in the postwar period, and dwelt briefly upon the great loss sustained by mathematics and by the members of this Society in the recent death of Professor G. D. Birkhoff. Professor W. T. Martin of Syracuse University presented resolutions expressing the thanks and appreciation of the members of the Society and the Association to the local committee, the staffs of the Museum of Science and Industry and of the Hotel Windermere for the excellent arrangements.

At the meeting of the Board of Trustees at 6:00 P.M., November 24, 1944, in the Hotel Windermere East, there was no quorum present and the Board adjourned to January 6, 1945. The Council met at 9:00 P.M. on November 24, 1944, in the Hotel Windermere East.

The Secretary announced the election of the following nineteen persons to membership in the Society (the address given is that of the member at the time of his application):

Mr. Harold William Becker, Mare Island Navy Yard, Calif.;
Dr. Donald Lee Benedict, Sylvania Electric Products, Inc., Flushing, N. Y.;

Miss Grace L. Bolton, New Jersey College for Women;
Lieutenant Paul Francis Byrd, Army Air Corps;
Mr. Frederick Merle Carpenter, Missouri School of Mines;
Mr. Jasper Brown Jeffries, Metallurgical Laboratory, University of Chicago;
Dr. Herbert Jehle, Harvard University;
Mr. Richard Eugene Krug, Milwaukee Public Library;
Dr. Vincent Owen McBrien, College of the Holy Cross;
Dr. Donald Curtis May, Jr., Bureau of Ordnance, Navy Department;
Lieutenant Edwin Evariste Moise, U.S.N.R., Naval Communications Annex, Washington, D. C.;
Mr. Philip Peak, University School, Indiana University;
Dr. Robert Collom Rand, U.S.N.R.;
Mr. Shien-Siu Shü, School of Mechanics, Brown University;
Professor Allen Simmons, Bethany College, Bethany, W. Va.;
Dr. Karl H. Stahl, State Teachers College, California, Pa.;
Dr. Paul Porter Sutton, Geophysical Laboratory, Carnegie Institution of Washington;
Mr. Henry Zatzkis, University of North Carolina.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

Columbia University: Mr. Aaron Galuten, Professors Sigfrido Carlos Mazza and Milton da Silva Rodrigues, Mr. Hing Tong;
University of Pennsylvania: Mr. Charles W. Saalfrank;
University of Washington: Mr. Marvin Clifford Stippes.

The following appointments by President M. H. Stone were reported: as representative of the Society at the inauguration of W. M. Whyburn as President of Texas Technological College on September 30, 1944, Professor H. J. Ettlinger; as representative of the Society at the inauguration of Homer Levi Dodge as President of Norwich University on October 9, 1944, Dean Elijah Swift; as representative of the Society at the inauguration of Howard Foster Lowry as President of The College of Wooster on October 21, 1944, Professor C. O. Williamson; as tellers for the election at the 1944 Annual Meeting, Professor W. C. Krathwohl, Drs. Mary E. Ladue and W. C. Strodt; as a member of the committee to study the role of the Society in the publication of mathematical research (replacing Professor A. E. Taylor), Professor G. C. Evans; as a member of the Committee on Printing Contracts for the period 1945–1947, Professor R. M. Foster (committee now consists of Professor R. M. Foster, Dean M. H. Ingraham, and Professor J. D. Tamarkin); as a member of the Committee on Places of Meetings for the period 1945–1947, Professor W. T. Reid (committee now consists of Professors W. B. Carver, Chairman, C. R. Adams, and W. T. Reid); as new members of the Committees to Select Hour Speakers: for Annual and Summer Meetings, Professor C. C. MacDuffee (committee now consists of Professors J. R. Kline, Chairman, C. C. MacDuffee, and J. D. Tamarkin); for Eastern Sectional Meetings, Professor Oystein Ore (committee now consists of Professors T. R. Hollcroft, Chairman, E. J. McShane, and Oystein Ore); for Western Sectional Meetings, Professor Tibor Radó (committee now consists of Professors R. H. Bruck, Chairman, A. A. Albert, and Tibor Radó); for Far Western Sectional Meetings, Professor D. H. Lehmer (the complete membership of this committee will be announced later).

Certain invitations to give hour addresses were announced: Professor S. S. Chern for the February, 1945, meeting in New York City; Professors A. P. Morse and Witold Hurewicz for the April, 1945, meeting in New York City.

The Council voted to invite Professor Enrico Fermi to deliver the nineteenth Josiah Willard Gibbs Lecture at the 1945 Annual Meeting.

The Secretary reported that the ordinary membership in the Society is now 2,754, including 210 nominees of institutional members and 64 life members. There are also 86 institutional members. The total attendance at all meetings in 1944 was 1,126; the number of papers read was 284; there were 14 hour addresses, 2 symposium addresses, 1 Gibbs Lecture, and four Colloquium Lectures; the number of members attending at least one meeting was 714. The Secretary also reported that an effort had been made to increase the membership through the cooperation of chairmen of departments of mathematics.

At the annual election which closed on November 25, and at which 578 votes were cast, the following officers were elected:

President, Professor T. H. Hildebrandt.

Vice President, Professor J. M. Thomas.

Secretary, Professor J. R. Kline.

Associate Secretary, Professor T. R. Hollcroft.

Treasurer, Professor B. P. Gill.

Librarian, Professor Arnold Dresden.

Members of the Editorial Committee of the Bulletin, Professor Saunders MacLane and Dean E. B. Stouffer.

Member of the Editorial Committee of the Transactions, Professor Oscar Zariski.

Member of the Editorial Committee of the Colloquium Publications, Professor C. C. MacDuffee.

Member of the Editorial Committee of Mathematical Reviews, Professor Oswald Veblen.

Members of the Editorial Committee of Mathematical Surveys, Professors A. A. Albert, Nelson Dunford, and J. D. Tamarkin.

Member of the Editorial Committee of the American Journal of Mathematics, Professor Richard Brauer.

Members at large of the Council, Professors H. F. Bohnenblust, S. S. Cairns, H. B. Curry, M. H. Ingraham, I. S. Sokolnikoff.

Members of the Board of Trustees, Professors W. R. Longley, Marston Morse, G. W. Mullins, Dean R. G. D. Richardson, and Dr. Warren Weaver.

Professor Hermann Weyl was appointed as a member of the Editorial Committee of the American Journal of Mathematics, to fill the unexpired term of Professor G. D. Birkhoff.

The Council adopted the following resolution on the death of Professor G. D. Birkhoff:

> On November 12, 1944, this Society lost through death an illustrious and distinguished leader—a widely known and esteemed member—George David Birkhoff. Professor Birkhoff was in the sixty-first year of his life. He was in possession of his full mental vigor, and to outward appearances of his whole physical strength as well. The news of his death, therefore, came to many of his friends unexpectedly and unmitigated in its suddenness.
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> From the time of his early manhood to that of his end Professor Birkhoff was a force in American mathematics, and only less directly so in the wider realm of general American scholarship. Intellectually he was alert, active and versatile; as a mathematician he was peculiarly original and profound. His researches, which were numerous and prolific of important results, in many instances blazed the ways in which other investigators have found it fruitful to follow. At their best they sufficed at times to encompass difficulties the resolution of which served to open, or to reopen afresh, important channels of mathematical development. In their broader aspects the works of Professor Birkhoff have become widely known. In their details an enumeration of them would be lengthy and impressive. The whole measure of the man, however, would reach beyond the bounds of any such listing, complete though the catalogue might be.
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> Professor Birkhoff's personality was vigorous and dynamic. Over a period of almost forty years he graced the meetings and journals of this Society with the presentations and publications of his results. He entered actively into the discussions of this Council, officiated as editor of the Transactions, and participated in the work and deliberations of many committees. He was the Society's president in 1925 and 1926.
>
> Professor Birkhoff was a distinguished teacher, extraordinarily gifted for the guidance of advanced students and particularly of those who came to him for training for mathematical research. Such students found him admirably accessible intellectually, and profited greatly from his faculty for the sensing and definition of problems which were at once significant and yet within a modest technical reach. He gave willingly of his time and counsel, and seemed able to impart without effort his own enthusiasms and his own high critical standards. Many members of this Society have had such contacts with him. Many in a sense look upon him, therefore, as the master, with an almost filial regard.
>
> Professor Birkhoff's quality as a scholar was widely recognized. A number of important prizes were won by him, among them the Bôcher Prize of this Society. His name appears on the rolls of the National and American Academies and upon that of the American Philosophical Society. The American Association for the Advancement

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of Science bestowed its annual prize upon him in 1926 and elevated him to its presidency in 1937. More than a dozen universities, over half of them in foreign lands, conferred the honorary doctorate upon him, and no less than half a score of foreign mathematical societies extended their fellowship or honorary membership to him. It seems almost fair to say that Professor Birkhoff had come to be looked upon in this country and abroad as the symbol of the stature of American mathematics.

With Mrs. Birkhoff, who by her graciousness and charm has endeared herself widely among its membership, this Society grieves the departure of Professor Birkhoff. In the richness of his influence and in the wealth of his accomplishments, however, it takes pride and finds the deepest satisfaction.

This resolution was read at the annual business meeting and the members of the Society stood for a minute in respect to the memory of their distinguished colleague. The Council requested the incoming president to appoint a committee to study the problem of a suitable memorial for Professor Birkhoff.

The Council voted to hold the 1945 Summer Meeting June 24–25 in Montreal, Canada, at the conclusion of the Canadian Mathematical Congress. The invitation to meet at Macdonald College of McGill University was accepted with the appreciation of the Council.

Since the 1944 Annual Meeting was held before the close of the Society's fiscal year, it was impossible to present a full report of the Treasurer. (Excerpts of the final report of the Treasurer, however, appear on pages 35–38 of this Bulletin.) It was reported: (1) That total receipts would exceed estimates by considerable amounts. This is due to the fact that more than 150 ordinary members have been added during the year, the amount from contributing memberships has increased, all institutional memberships have been continued, and subscription lists for the Society's periodicals have grown. (2) That total disbursements would be within expectations and the actual excess of receipts over expenditures would be a considerable amount. The surplus which has been thus created will be needed to meet the difficult financial situation which will probably confront the Society in the years immediately following the war.

The Librarian reported the following additions to the Library: 28 books, 58 bound volumes of periodicals, 306 printed dissertations, 136 dissertations on microfilm. He also reported that 90 members of the Society have used the library during the academic year 1943–1944.

The Bulletin Editorial Committee reported that it would publish approximately 1,000 pages in 1944. The Committee is able to publish a research paper with reasonable promptness after it is received from the author whereas a few years ago a year or more elapsed between the receipt of the paper and the time of publication. The editors regard the present accelerated publication as highly desirable and are making plans to maintain it. The Editorial Committee is no longer adhering to the limitation of ten or fewer pages for papers to be published in the Bulletin. It was reported that Professors R. H. Bruck, R. H. Cameron, Irving Kaplansky, J. C. Oxtoby, and Leo Zippin have been appointed as Assistant Editors, to replace Professors P. O. Bell, O. F. G. Schilling, Garrett Birkhoff, W. B. Carver, Morris Marden, and Virgil Snyder.

The Transactions Editorial Committee reported that it would publish approximately 1,100 pages during 1944. The backlog of papers continues low and manuscripts are published within six or eight months after receipt.

The Mathematical Reviews Editorial Committee reported that abstracts of mathematical literature have been published in the usual form despite the continued difficulties ensuing from the war. The situation regarding foreign literature has been somewhat alleviated and the coverage of the mathematical literature of the world is practically complete with the exception of certain journals from Italy. The editors feel that this situation will be relieved shortly. During the year the Edinburgh Mathematical Society has been added to the list of sponsors of Mathematical Reviews. The total number of subscriptions is at present 1,332, while the financial situation continues to develop favorably.

On recommendation of the Colloquium Editorial Committee, the Council voted to invite Professor Hassler Whitney to deliver the Colloquium Lectures at the 1946 Summer Meeting.

The War Policy Committee reported concerning its activities since the last meeting of the Council: (1) The subcommittee appointed to advise the Examinations Staff of the Armed Forces Institute (Professors W. T. Reid, Chairman, Ralph Beatley, L. L. Dines, W. L. Hart, C. C. MacDuffee) has completed examinations in trigonometry, analysis, geometry, college algebra, differential and integral calculus, which will be submitted to the Institute for accreditation purposes. The work of this subcommittee should prove of aid to college authorities in assigning college credit for work taken under the auspices of the Institute. (2) Chairman Stone has appointed a subcommittee on historical records of mathematicians in war activities consisting of the following: Professors J. R. Kline (temporary chairman), R. C. Archibald, Marston Morse. This committee is empowered to add to its membership. (3) The Office of Scientific Personnel is continuing its activities in the matter of placement of scientists in various phases of the war effort. This Office is now in a transition period and in the near future its work will be mostly concerned with the problem of postwar placement. The War Policy Committee has voted to continue to support the Office of which Dr. M. H. Trytten is now the Director. (4) The War Policy Committee is also studying the tremendous problem which will be created by the shortage of scientific personnel which is the result of the decrease in the number of graduate students during the war.

The resignations of Professors G. B. Price and A. D. Michal as Associate Secretaries were accepted. (Professor R. H. Bruck was subsequently elected by the Council, with the Board of Trustees approval, to fill the unexpired term of Professor Price.)

Titles and cross references to papers read at the meeting follow below. Papers 1 to 6 were read Friday afternoon, papers 7 to 11 Saturday morning, papers 12 to 16 Saturday afternoon, and papers 17 to 38, whose abstract numbers are followed by the letter *t*, were read by title. Mr. C. C. Hsiung was introduced by Professor T. R. Hollcroft, Mr. Walter Kohn by Professor Alexander Weinstein, and Mr. C. T. Loo by Professor E. J. McShane.

1. W. H. Roever: *Second derivatives of the potential function of the earth's weight field of force.* (Abstract 51-1-33.)

2. V. G. Grove: *Quadrics associated with a curve on a surface.* (Abstract 51-1-39.)

3. W. J. Trjitzinsky: *Singular integral equations with Cauchy kernels.* (Abstract 51-1-22.)

4. W. S. Snyder: *Derivatives of set functions.* (Abstract 51-1-21.)

5. P. R. Halmos: *On an incompressible transformation.* (Abstract 51-1-36.)

6. A. T. Lonseth: *Error-limitation for the method of least squares.* (Abstract 51-1-46.)

7. R. H. Bruck: *Quasigroup theory.* II. *The lower central series.* Preliminary report. (Abstract 50-11-261.)

8. R. E. Fullerton: *Linear operators with range in a space of differentiable functions.* (Abstract 51-1-14.)

9. H. S. Wall: *The convergence of a positive definite J-fraction in the limit-circle case.* (Abstract 50-9-228.)

10. R. H. Fox: *On topologies for function spaces.* (Abstract 51-1-50.)

11. Felix Bernstein: *The swastika and the Sicilian triskelon from the standpoint of "higher geometry."* (Abstract 51-1-37.)

12. H. K. Hughes: *The asymptotic developments of a class of entire functions.* (Abstract 51-1-16.)

13. T. R. Hollcroft: *The probability of repetitions.* (Abstract 51-1-43.)

14. W. T. Reid: *A matrix differential equation of Riccati type.* (Abstract 51-1-19.)

15. P. A. White: *On regular transformations.* (Abstract 50-11-287.)

16. N. J. Fine: *Congruence properties of the elementary symmetric functions.* (Abstract 51-1-6.)

17. R. P. Agnew: *A simple and natural notation for the theory of summability of series and sequences.* (Abstract 50-11-268-*t*.)

18. R. P. Agnew: *Spans in Lebesgue and uniform spaces of translations of step functions.* (Abstract 50-11-269-*t*.)

19. E. F. Beckenbach: *Concerning the definition of harmonic functions.* (Abstract 50-11-270-*t*.)

20. L. M. Blumenthal: *Generalized elliptic spaces and quadratic forms.* (Abstract 51-1-48-*t*.)

21. R. H. Bruck: *Quasigroup theory.* III. *Finite p-loops.* Preliminary report. (Abstract 50-11-262-*t*.)

22. R. H. Bruck: *Quasigroup theory.* IV. *Associatral series.* Preliminary report. (Abstract 51-1-3-*t*.)

23. C. J. Everett: *The basis theorem for vector spaces over rings.* (Abstract 51-1-4-*t*.)

24. C. J. Everett and S. M. Ulam: *Projective algebra.* I. (Abstract 51-1-5-*t*.)

25. H. W. Eves: *A geometrical note on the isocenter.* (Abstract 51-1-25-*t*.)

26. C. C. Hsiung: *A ternary of plane curvilinear elements with a common singular point.* (Abstract 51-1-40-*t*.)

27. Fred Kiokemeister: *The Asano postulates for orders in a linear algebra.* (Abstract 51-1-10-*t*.)

28. Fred Kiokemeister and G. W. Whitehead: *A coset theory for left loops.* (Abstract 51-1-11-*t*.)

29. Walter Kohn: *Contour integration in the theory of the spherical pendulum and the heavy symmetrical top.* (Abstract 51-1-27-*t*.)

30. A. T. Lonseth: *A note on relative errors in systems of linear equations.* (Abstract 51-1-45-*t*.)

31. C. T. Loo: *Note on the strong summability of Fourier series.* (Abstract 51-1-17-*t*.)

32. C. T. Loo: *Two Tauberian theorems in the theory of Fourier series.* (Abstract 51-1-18-*t*.)

33. A. N. Lowan and H. E. Salzer: *Formulas for inverse interpolation within a square grid in the complex plane.* (Abstract 51-1-28-*t*.)

34. Isaac Opatowski: *A botanical application of Jensen's inequality.* (Abstract 51-1-31-*t*.)

35. Ida Roettinger: *An operational approach to the solution of boundary value problems by generalized Fourier series.* (Abstract 51-1-32-*t*.)

36. A. R. Schweitzer: *Functional relations valid in the domains of abstract groups and Grassmann's space analysis.* II. (Abstract 51-1-20-*t*.)

37. Seymour Sherman: *Complex polynomials and polygonal domains.* (Abstract 51-1-12-*t*.)

38. Antoni Zygmund: *On smooth functions.* (Abstract 51-1-23-*t*.)

T. R. HOLLCROFT,
Associate Secretary

APPENDIX

EXCERPTS FROM REPORT OF TREASURER

December 8, 1944

TO THE BOARD OF TRUSTEES OF THE
AMERICAN MATHEMATICAL SOCIETY

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1944. The following comments may be of interest.

Investment Portfolio

On November 30, 1944, the market value of securities held for Invested Funds exceeded book value by $3,566, and the market value of securities held for Current Funds exceeded book value by $482. In view of these figures and of the conservative character of many of the securities, the reserves held in accounts "Reserve for Investment Losses" and "Profit on Sales of Securities" may be considered sufficient protection against contingent depreciation in market value.

The investment portfolio now includes government bonds 29 per cent, other bonds 20 per cent, preferred stock 14 per cent, common stock 30 per cent, cash in savings banks 7 per cent.

Income from Investments

Income received during the year from investments of Current Funds amounted to $962 exclusive of $87 earmarked for International Congress. This represents a return of 2.1 per cent computed on average book value of investments. Income on Invested Funds amounted to $6,851 representing a return of approximately 3$\frac{2}{3}$ per cent. Total investment income from all sources was thus $7,900 representing a return somewhat under 3$\frac{1}{3}$ per cent. These rates of return are slightly under the corresponding rates for the fiscal year 1943 as the result of purchases of additional government bonds.

Income from the Henderson Estate was $4,511, which is slightly more than was received in 1943 from this source.

Increase in Surplus and Assets

Surplus account shows an increase of $9,212 after transferring $5,463 from Surplus to Endowment Fund Principal. This excellent outcome of the operations for the year is to be attributed among other things to a gain of some 150 ordinary memberships, a gain of $100 in contributing memberships, larger subscription lists for all of the Society's periodicals supported from general receipts, and unexpectedly heavy sales of back volumes. It is to be discounted a little by the fact that payment for the List of Members now in the course of publication will be deferred to the next fiscal year.

Further light on the results of the year's financial operations is given by noting that total assets increased during the year by $23,637 so that their amount is now $258,209. This increase reflects, in addition to the above mentioned increase in surplus, a gain in subscriptions to and in back volume sales of Mathematical Reviews, the fact that no provision had to be made for printing new volumes either of the Colloquium Series or of Mathematical Surveys, and the accumulation of income in the special funds held by the Society. The assets include $1,719 of a grant from the Rockefeller Foundation held for the use of the War Policy Committee.

Respectfully submitted,
BENNINGTON P. GILL,
Treasurer

BALANCE SHEET

Assets

	November 30, 1944	November 30, 1943
CURRENT FUNDS:		
Cash	$ 27,973.79	$ 19,846.11
Investments	41,839.37	36,731.06
	$ 69,813.16	$ 56,577.17
INVESTED FUNDS:		
Cash	$ 336.09	$ 1,273.67
Investments	188,059.77	176,721.29
	$188,395.86	$177,994.96
TOTAL ASSETS	$258,209.02	$234,572.13

Liabilities

CURRENT FUNDS:		
Mathematical Reviews	$ 11,936.09	$ 14,252.62
Colloquium	7,368.15	5,761.69
Mathematical Surveys	1,236.10	840.03
Reprinting Funds	3,765.26	2,239.10
Prize Funds and Other Special Funds Accumulated Income	3,332.75	2,296.18
International Congress	6,233.39	6,146.06
War Policy Committee	1,719.41	
Miscellaneous	72.56	80.99
Sinking Fund	768.96	592.34
Profit on Sales of Securities	413.36	612.86
Surplus	32,967.13	23,755.30
	$ 69,813.16	$ 56,577.17
INVESTED FUNDS:		
Endowment Fund Principal	$ 65,573.18	$ 60,110.68
Prize Funds and Other Special Funds	31,033.22	31,033.22
Life Membership and Subscription Reserve	4,032.68	4,434.81
Colloquium	5,000.00	5,000.00
Mathematical Reviews	65,000.00	60,000.00
Reserve for Investment Losses	4,385.89	4,385.89
Profit on Sales of Securities	13,370.89	13,030.36
	$188,395.86	$177,994.96
TOTAL LIABILITIES	$258,209.02	$234,572.13

SUMMARY STATEMENT OF INCOME AND EXPENDITURES

1943–1944

	1944		1943	
	Receipts	*Disbursements*	*Receipts*	*Disbursements*
GENERAL RECEIPTS:				
Dues—Ordinary Memberships	$18,060.64		$16,514.71	
Dues—Contributing Memberships	948.06		859.63	
Dues—Institutional Memberships	7,510.02		6,616.52	
Initiation Fees	793.95		378.84	
Investment Income	8,836.16		8,612.83	
Miscellaneous	353.84		39.96	
Grant from Rockefeller Foundation for War Policy Committee	3,750.00		1,250.00	
GENERAL DISBURSEMENTS:				
Secretaries		$ 5,642.71		$ 5,590.02
Treasurer		1,070.77		1,093.58
Officers' Traveling		600.00		581.23
Library		984.46		893.15
War Policy Committee Expenses		2,030.59		1,584.17
Emergency		158.38		4.90
Total	$40,252.67	$10,486.91	$34,272.49	$ 9,747.05
Excess of General Receipts	$29,765.76		$24,525.44	
PUBLICATION:				
Bulletin	$ 2,462.10	$11,189.68	$ 1,710.80	$11,536.88
Bulletin Reprinting	856.40		626.71	
Transactions	7,097.63	8,644.20	5,886.83	8,309.42
Transactions Reprinting	669.76		1,172.40	889.16
Colloquium	3,210.08	1,603.62	3,951.38	994.73
Mathematical Reviews	15,453.49	12,770.02	10,235.88	12,167.95
Mathematical Surveys	713.82	317.75	3,600.89	2,760.86
Semicentennial Publications	180.65	4.03	76.35	
American Journal		2,500.00		2,500.00
Total	$30,643.93	$37,029.30	$27,261.24	$39,159.00
Excess Cost of Publication		$ 6,385.37		$11,897.76
OTHER:				
Special Funds (including Congress)	$ 1,223.90	$ 100.00	$ 1,273.41	$ 3,600.00
Profit on Sales of Securities	141.03		9,160.49	
Gift from R. C. Archibald				80.97
Miscellaneous		8.43		12.03
From Surplus:				
Transactions Reprinting				750.00
Mathematical Reviews		1,000.00		300.00
Mathematical Surveys				1,000.00
Total	$ 1,364.93	$ 1,108.43	$10,433.90	$ 5,743.00
Difference	$ 256.50		$ 4,690.90	
Net Change in Assets	$23,636.89		$17,318.58	
ASSETS BEGINNING OF YEAR	$234,572.13		$217,253.55	
ASSETS END OF YEAR	$258,209.02		$234,572.13	

THE NOVEMBER MEETING IN LOS ANGELES

The four hundred ninth meeting of the American Mathematical Society was held at the University of California, Los Angeles, on Saturday, November 25, 1944. The attendance was about forty, including the following twenty-seven members of the Society:

Harry Bateman, Clifford Bell, Myrtie Collier, J. M. Danskin, P. H. Daus, C. H. Dix, W. H. Glenn, H. J. Hamilton, O. G. Harrold, P. G. Hoel, D. H. Hyers, O. E. Lancaster, A. D. Michal, Knox Millsaps, M. K. Peabody, W. T. Puckett, W. C. Randels, G. E. F. Sherwood, R. H. Sorgenfrey, D. V. Steed, J. W. Swank, Gabor Szegö, A. H. Tappan, S. E. Urner, F. A. Valentine, František Wolf, E. R. Worthington.

The morning session was opened by the reading of paper 1 by Dr. C. H. Dix. By invitation of the Program Committee, Professor Gabor Szegö of Stanford University delivered an hour address entitled *On the capacity of a condenser*, with Professor Harry Bateman presiding.

By invitation of the Program Committee, Professor František Wolf of the University of California delivered an hour address in the afternoon on *Summability and uniqueness of trigonometric integrals*, with Professor D. H. Hyers presiding.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title.

1. C. H. Dix, C. Y. Fu, Mrs. E. W. McLemore: *The cubic Rayleigh wave equation.* (Abstract 51-1-24.)
2. R. A. Beaumont: *Groups with isomorphic proper subgroups.* (Abstract 50-11-260-*t*.)
3. E. F. Beckenbach: *A Looman-Menchoff theorem for Newtonian vectors.* (Abstract 51-1-13-*t*.)
4. H. W. Becker: *The composite umbra theorem.* (Abstract 51-1-1-*t*.)
5. H. W. Becker: *The hyper-umbra theorem.* (Abstract 51-1-2-*t*.)
6. P. J. Kelly: *Some properties of a certain interchange type of self-isometry.* (Abstract 50-11-285-*t*.)
7. E. J. Purcell: *Some Cremona involutions in n-dimensional space.* (Abstract 51-1-42-*t*.)
8. Gordon Overholtzer: *A new application of the Schur derivate.* (Abstract 50-9-204-*t*.)
9. Henry Scheffé: *A note on the Behrens-Fisher problem.* (Abstract 51-1-47-*t*.)

A. D. MICHAL, *Associate Secretary*

DAVID EUGENE SMITH—IN MEMORIAM

With the death of David Eugene Smith on July 29, 1944, at age eighty-four, the world lost one of the most colorful and influential figures in the fields of the history and teaching of mathematics. It is not possible to choose the one in which Professor Smith accomplished the most or reached the greatest number of individuals. He lives on in the lives of innumerable persons whom he inspired as teachers and whom he inspired as writers, as well as in his own writings. He will live on long after all these persons have gone, in the great library that he presented to Columbia University in 1931. In the limited space at our disposal, we shall try chiefly to give some idea of the immense value of this collection for the history of mathematics.

Let us first note briefly some facts concerning Doctor Smith's career. He was born at Cortland, N. Y., in January 1860, and here he received his early education to the extent that he spoke both Latin and Greek as a boy. When the Cortland Normal School opened, he was the first student to enroll, and at seventeen years of age he entered Syracuse University. During the years 1881–87, Syracuse conferred on him the degrees of Ph.B., Ph.M. and Ph.D. From 1881–84, he practiced law at Cortland, N. Y., before taking up the teaching of mathematics at the State Normal School in Cortland until 1891. For the next seven years, he was professor of mathematics at the Michigan State Normal College, Ypsilanti, and then for three years principal of the New York State Normal School at Brockport. In 1901, he accepted an appointment as professor of mathematics at Teachers College, Columbia University. Here he continued until his retirement as professor emeritus in 1926.

Already in 1900 he had commenced to attract attention by his writings and had started the publication of the series of elementary mathematical texts which were later enormously popular, some of them being translated into Spanish, Arabic, and Chinese. For several years about 1930, more than a million copies of his books were sold annually. It was not long after coming to Columbia University that Doctor Smith started the series of annual trips to different parts of the world, and the collection of items for the Library, now at the service of scholars in beautifully appointed quarters.

Professor Smith was a collector from his youth up. He once related that as a boy he had read a Victorian novel, *St. Elmo*, by Augusta J. Evans, published first in 1866. In this book he found a description of the furnishings of some "quaint and elegant rooms." A few phrases from this description will account for his saying to himself: "Sometime I am going to have a room like that." "On a *verd antique* table lay a satin cushion holding a vellum MS., bound in blue velvet, whose uncial letters were written in purple ink, powdered with gold-dust, while the margins were stiff with gilded illuminations; . . . A small Byzantine picture . . . hung over an etagère, . . . where lay a leaf from Nebuchadnezzar's diary, one of those Babylonian bricks . . . . Several handsome rosewood cases were filled with rare books—two in Pali—centuries old; and moth-eaten volumes and valuable MSS.—some in parchment, some bound in boards—recalled the days of astrology and alchemy . . . " In a not too modest fashion, Professor Smith reached the goal which he had set before himself.

Whatever the beginnings of his passion for old books and related material on the history of mathematics, it possessed him throughout a long life. A census made on April 22, 1940, shows that the David Eugene Smith Library contained the following items: Volumes 10,965, Documents 1,500, Autograph letters (before 1900) 4,000, Instruments 280, Portraits 3,250, Medals, Counters, and so on, 300—a total of 20,295. While the Library adds occasional items, the figures quoted here as well as in Osiris vol. 1 do not change substantially.

It may be well to note at the outset that the Smith collection as enumerated above, the Plimpton collection of incunabula, early textbooks and manuscripts, the Dale collection of books on weights and measures, are all housed in the same rooms in the Seth Low Library, and together constitute The Plimpton-Smith-Dale Library. Many of the mathematical items in the Plimpton collection were purchased by or on the advice of Professor Smith. These two great collectors had agreed not to duplicate in their acquisitions, with the result that they gave to Columbia University the greatest collection in America of works on mathematics up to 1600, and one of the greatest in the world. The interest of Professor Smith was the interest of a scholar in his field. This is illustrated by his own notes on the margins of the books, and by his annotations on slips attached to a thousand documents of the fourteenth, fifteenth and sixteenth centuries. There was here the interest of a real lover, for he came in to the Library every day, rain or shine, asked to see a book or just to look around.

Only a few high lights can be turned on to illuminate the figures given above. It means little to say that there are nearly eleven thousand volumes without indicating their nature. The comprehensiveness of this collection in books and periodicals dealing with the history of mathematics cannot be too strongly emphasized. It is safe to say that there is hardly a mathematician of any importance before 1900 who is not represented by some edition of his works. Here are all the outstanding books in the history of mathematics such as the one in which is made "the first serious attempt in England to write on the history of mathematics." This is *De Algebra Tractatus*; *Historicus & Practicus* by John Wallis. The collection contains his *Opera Mathematica*, 1693–1695, in vol. 2 of which this work is found. There are ninety-four other works on general history of mathematics, in Latin, Italian, French, German, English. In addition, there are innumerable works on the history of mathematics in some twenty-four countries. To these must be added the great number of biographies of individual men.

The Smith and Plimpton libraries together contain probably the most important collection of editions of Euclids in the United States and the largest collection of sixteenth century editions of the *Elements* in this country. Besides the first printed edition of 1482 which is in Latin, it contains editions in other languages including original Greek (from Greek manuscript), Italian, German, French, English. There are manuscripts, in addition. Among the "more than fifty Euclids printed before 1600 are the first printed edition of 1482; the Pacioli edition of 1509; the Peletarius translation of 1557; the first English edition, translated by Billingsley in 1570; four editions of the Clavius translation and the same number of Vivianis. Barrow, Oughtred, Deschalles and many others are represented by one or more editions."

The Galileo collection of books is a rich one in content if not in volume. The most outstanding works are the *Dialogo*, Florence, 1632, and *Discorsi e dimostrazioni matematiche, intorno à due nuoue scienze*, Leida, 1638. Other important works are *Sidereus nuncius*, Venice, 1610, London, 1682 [that is, 1683], *Della scienza mecanica*, Bologna, 1655, and *Trattato della sfera di Galileo Galilei*, Roma, 1656.

In the joint libraries, there is a fine collection of Newton material. Here are found: the first edition of his masterpiece, *Philosophiae naturalis principia mathematica*, 1687 (both first and second issues), 1713, 1714, 1739, 1822, 1846; *Arithmetica Universalis*, 1707, 1732, 1752; *Universal Arithmetick*, 1720, 1728; *A catalogue of the Portsmouth collection of books and papers written by or belonging to Sir Isaac Newton* . . . 1888 ("Many of the mathematical papers contain Newton's preparations for the *Principia*, and notes which sprang out of questions that were started by his own correspondents."); *Correspondence of Sir Isaac Newton and Professor Cotes, including letters of other eminent men*, . . . 1850; *Enumeratio linearum tertii ordinis* (in his *Optice*, 1706); *La methode des fluxions, et des suites infinies*, par M. Chevalier Newton, 1740; *Optice*, 1706, 1719, 1740; *The mathematical principles of natural philosophy* . . . Tr. into English by Andrew Motte, 1729; a

French translation of the *Principia*, 1756; an Italian translation of same, 1925; *Tractatus de quadratura curvarum*, 1706. Added to these works by Sir Isaac Newton himself is a long array of works about him with titles suggesting fascinating reading and study. The greater number of this list is in the Smith library and works of equal value with those in the Plimpton section.

Professor Smith was particularly interested in collecting Italian works. In the purchase of the library of Professor Jacoli of Venice in 1904 he obtained a library which was very valuable in the history of Italian mathematics, and there were very few important works on the subject which he did not have.

Among many treasured association copies of books are: G. Libri, *Histoire des Science Mathêmatiques en Italie* (this is one of eight or ten copies saved from the first edition. Libri had gone to the publisher to get copies for a few friends the day before its expected publication. A fire destroyed the whole edition. This copy contains Libri's corrections for the ultimate first edition); Joseph Raphson, *Analysis Æquatiorium Universalis*, 1690; (a presentation copy from Raphson "To M^{r} Isaack Newton [*sic*] Wth my most humble service. J. R."); Bombelli, *L'Algebra parte maggiore dell' aritmetica*, 1572 (first edition of his algebra. Autograph presentation copy by his grandson); *The Analyst*, or, *A Discourse to an Infidel Mathematician*, 1734 (this is Bishop Berkley's attack on Newton's fluxions. It is the presentation copy from Berkley to Sir Thomas Hammer, Speaker of the House of Commons). Then, too, there are two bound volumes of the proof sheets of the translation into English of the *Ganita Sāra Sangraha* of Mahāvīra, a gift from the translator, the late Professor Rángāchārya. Doctor Smith characterized this work as the most important contribution to our knowledge of Hindu mathematics made in recent times, and regarded it as one of the rarest books he had.

Among the printed material are almost two thousand pamphlets not yet evaluated. These cover reprints, memorials, dissertations, school programs with always an article of less magnitude than a dissertation but still a scholarly one. Professor Smith had a magnetic eye for the little known, and so here is assembled material on many lesser lights, as well as some well known. The collection is especially well equipped in the field of mathematical recreations. All the worthy contributors are represented as well as the most popular but not so worthy.

The books on mathematical and astronomical instruments of historical interest constitute probably the best collection in America. It contains some 265 volumes, all published previous to 1790. Of these, 90 appeared before 1600. The books are concretely illustrated by 280 instruments of historical interest. "Outstanding items include several celestial spheres of the seventeenth century. One of these is from Japan, made of *papier-mâché*; another is of Hindu origin, of bronze with realistic silver stars; while a third is from Persia, made by the grandson of Haddad, the emperor Humayun's chief astronomer. A telescope made by the famous instrument maker, Jesse Ramsden, about 1775, is still in working order. A rare group of old English tally sticks dating from about 1296 has an interesting history. They were found in the Chapel of the Pyx, Westminster, in 1906, where they had lain undisturbed for six hundred years . . . The ancient English tallies were ordered burned in 1834, and it is said that owing to the extra fires made up for this purpose the Houses of Parliament were destroyed . . . Astrolabes of intricate and delicate workmanship and from many lands show the development of this ancient scientific instrument . . . Compasses dating from the beginning of the Christian era, quadrants, protractors and sextants further enlarge the scope of the collection." (*Bibliotheca Columbiana*, June 1936.)

But the rich primary source-material, consisting of thousands of manuscripts and tablets, is undoubtedly the most valuable part of the Smith Library. In his *History of Mathematics*, a considerable number of such items was made available in facsimile to historians and students. The De Ricci *Census of Medieval and Renaissance Manuscripts in the United States and Canada*, 1937–40, gives descriptions of thirty-five of the manuscripts dated 1118–1650. It then continues: "The same collection contains several fragments of Greek papyri, one describing the area of a piece of land; also an extensive series of about one thousand early deeds of the XIVth, XVth, and XVIth centuries, mainly relating to France, and some fifty single leaves of manuscripts, mainly theological. These would be worth listing in detail."

By far the most valuable part of the manuscript material, however, is that from the East. The Indic manuscripts constitute the largest collection of any institution in the United States in the outstanding languages of India—(according to Horace I. Poleman of the Library of Congress who catalogued it). Besides the ones from India, there are manuscripts in Japanese, Chinese, Arabic, Persian.

The Indic manuscripts include the famous work on astronomy by Varāhamihira, of which "only six complete copies . . . in Sanskrit [are] known to Oriental scholars." There are sixteen manuscripts of the *Līlāvati* of Bhāskara, and one complete manuscript of the *Bija Gaṇita* by the same author. Of the two rare specimens of the best-known algebras of the Moslem civilization, one is the *'ilm al-jabr w'al* *muqabalah'* of al-Khowârizmî, a manuscript of the eighteenth century, and the other of the fourteenth century is one in Arabic of the treatise on algebra of Omar Khayyâm. There are two Oriental manuscripts of special interest on geometry. One is an Arabic translation of Books I–VI of Euclid's *Elements*, made originally *c.* 890 but this copy is dated 751 A.H. (1350/51 A.D.); "the second of the interesting manuscripts of Euclid is the Chinese version in several volumes of Matteo Ricci, . . . translated in 1603–1607 with the help of two learned mandarins." This copy was probably made *c.* 1800. "This was the first translation of Euclid attempted in the Far East." There is a Persian manuscript of the tables of Ulugh Beg, the prince astronomer of Samarkand, which bears the date 1214 A.H. (1836 A.D.). Of this, Doctor Smith says: "I consider this the finest mathematical manuscript that I bought in the East."

There are more than a hundred specimens of Japanese manuscripts as well as many books. In one of his dictated notes, Doctor Smith says: "In Japan [first visit in 1907], I simply laid down the rule that I wanted to buy every mathematical manuscript or printed book that could be found . . . I had all the important works on mathematics in manuscript or in first editions . . . In my library there is now a very complete collection of the important works on mathematics written or printed in Japanese." Would that it were possible to quote the accounts of the experiences in obtaining some of this material! It is almost fantastic at times. One wonders how Professor Smith survived the emotion of having long-sought or unexpected treasures fall into his hands like manna from Heaven.

It is in order to mention here the exquisitely written manuscript of Omar Khayyâm's *Rubâiyât* from the seventeenth century. And Professor Smith's own edition of the *Rubâiyât* must be noted. In this is, as the title page states, "*The Rubâiyât of Omar Khayyâm* set forth in meter . . . based upon a verbatim translation by Hashim Hussein." It was most beautifully illustrated by a Persian artist.

While making no outstanding original discoveries in the history of mathematics, Professor Smith made available many works by facsimiles, translations, annotations either alone or in collaboration with some able student. Among these are: *The Sumario Compendioso of Brother Juan Diez, the earliest mathematical work of the New World*; *The Geometry of René Descartes*; *Le Comput Manuel de Magister Anianus*; *A Source Book in Mathematics* [ed.]; and articles on the first printed and other arithmetics. *Rara Arithmetica,* A Catalogue of the Arithmetics written before the year MDC1 with a description of those in the library of George Arthur Plimpton of New York, is the only thing in its field and is the most important of his works from the librarian's point of view.

The Letters and Portraits are highly important but must be covered in a few sentences. The latest count on the number of autograph letters shows that there are 4056 from some 560 individuals (mainly those written before 1900—later correspondence not included). Among these individuals are found Arago (75), the Bernouillis (23), Carnot (16), Delambre (75), Descartes (1), Euler (16), Galileo (3), Newton (6), Quetelet (19), Voltaire (7). The choice of names is purely arbitrary. These letters throw light on the writers as human beings and as mathematicians. A series of articles *Among my autographs* by Professor Smith in the American Mathematical Monthly (between February 1921 and November to December 1922) presents not only letters, sometimes in facsimile, but with meticulous care an explanation of every item in the letters as well as some account of the writer. *De Moivre expresses himself* and *Legendre and Cauchy sponsor Abel* are two of the accounts. *Descartes's appreciation of Huygens the Elder* is the one precious specimen from the hand of Descartes.

The over three thousand portraits of mathematicians include representations of about a thousand individuals. There are several portraits of many of the men represented; for example, there are thirty of Arago, twenty of Johann Bernouilli, fifty of D'Alembert, eight of Descartes, and some hundred and fifty of Newton. Calls come from all over the world for representations of mathematicians and these have been supplied to many writers.

Professor Smith made other collections, the outstanding one of which is that of some eighty Korans. This also was presented to Columbia University. He himself writes: "My experience in buying mathematical books led me at one time to take an interest in the Korans. I have always felt that the most beautiful manuscripts in the world are the Korans, not that they are illustrated (which would be against the Mohammedan law) but because of the writing itself. I therefore purchased not only complete Korans, but some valuable pages from the older manuscripts . . . There are decorations around the edge of the pages but not of humans, only floral designs, etc. . . . Pages from the Koran of every century since the birth of Mohammed will be seen in the collection." Most of these Korans are on paper since the art of paper making was known very early in the Mohammedan countries (Samarkand, 751). There are two interesting exceptions, one on linen cloth and one on vellum. The most unique volume in the collection is the one on linen. It is a Persian manuscript of the eighteenth century and measures eight by thirteen inches. The opening pages are exquisitely decorated with the predominating blue that is as thrilling as a glorious sunset or a strain of great music. The most valuable volume is one of the nineteenth century. Even a slight examination by an Arabian scholar has shown this collection to be of textual value. A number of its treasures were loaned for an exhibit in the Metropolitan Museum of Art.

What has been written here gives only a slight hint of the riches in the Smith Library at Columbia University. That scholars shall make use of all this material gathered by Professor Smith will make it live as he made it live, an end to be devoutly hoped for.

In addition to the collections already presented, some others will be touched on. Among these is one of Early Diplomas of the sixteenth, seventeenth and eighteenth centuries, the earliest a diploma of the University of Bologna, dated 1564. Still another is "believed to be one of the finest Ikon collections in the United States." In it are rare old pieces, some of which were made as early as the ninth century. "Almost all . . . were gathered from ancient religious meeting places in Russia, Old Macedonia, Juga-Slavia [*sic*], and Bulgaria." A further list covers: illuminated manuscripts and calligraphic specimens; medals, coins, and counters, illustrating the history of mathematics; Sumerian and Babylonian tablets illustrating the use of the early numerals of Iraq; and, manuscripts and printed books on education which include manuscripts relating to the history of nursing.

Many readers have in their possession evidences of Doctor Smith's wonderful gifts as a letter-writer. These would make a story in themselves, and it is hoped that that story will some day be written. Then, too, among memories of readers are remarkable dinners and luncheons given by this most gracious host at his home in New York City and abroad at Congresses. Professor C. J. Keyser has recalled these so well that his statement is quoted: "What I remember most vividly was his extraordinary hospitality, his fine and many-sided culture and his gentle representation of the gracious manner of the long since vanished and almost forgotten ways of the mid-Victorian social life." There stand out from such occasions ones on his return from some corner of the globe, always with treasures and always with interesting accounts of their acquisition. Details of these talks have been brought back and strengthened through the privilege accorded the writer of reading notes dictated by Professor Smith, many of them as late as 1936. Let us recount one of these.

"Finally I came back to Colombo and called upon the high priest of the Buddhists in the outskirts of the city. I went to the monastery and was cordially received by two or three priests, but they told me it would be impossible for me to see the high priest because he was very sick and was not expected to live. I gave them my card telling my affiliation with the University and the American Mathematical Society, and asked that it be presented to him after I left. I noticed, however, that they had called in two or three other priests, and in the corner of the room they looked over my card and then one of them left the room and I was asked to wait a few moments. Soon he returned, saying that His Reverence, the High Priest, wished to see me. I passed through a long corridor, and was admitted to his bedroom, where I saw him lying on his bed in the darkness, the curtains having been lowered. It was rather an embarrassing situation to be going to a man's death bed and talking mathematics to him. I found him, however, one of the finest gentlemen I ever met. He was interested in my inquiry, and he finally said there was nothing in the library of Ceylon that related to mathematics or astronomy. There was, however, he said, a well-known book on astrology which had some mathematics in it, and he sent the servant out and he brought back a palm leaf manuscript which the high priest told me was well-known among all the scholars of Ceylon. He said that it would be impossible to buy one, but that it would give him great pleasure to have a copy made and sent back to me. This was done, and I received it somewhere along the route back to America . . . I had been seated in a small child's chair, the priest having told me that in the presence of His Reverence no one was allowed to sit, but that in this case, if I would sit in the chair of a child, I would be welcome to that rest. When the time arrived for leaving, I arose, and to my astonishment, the dying priest threw off the bed clothes, arose, and conducted me with great courtesy to the door. I treasure that copy that was made, not only for its own value, but also for the great kindness that this man, an invalid, showed me, a stranger. I was glad to learn afterwards that he recovered, and that he lived some years thereafter."

The most outstanding memories of the writer are those of visits with a group of twenty or thirty students which took place for many years at the close of each semester through June 1939. These were made to Professor Smith's home and later to the Smith Library by students from courses in the history of mathematics who had become somewhat familiar with the names and writings of the great workers in the field. When these gatherings were held in his beautiful, treasure-filled home, tea was served and was followed by a talk that was pure magic on the books and other objects assembled on tables at the front. (Professor Smith's talks to general audiences, such as the whole student body at a college chapel hour, and even on a topic like "Leonardo of Pisa," always held them spell-bound.) In the Smith Library, a display had been especially prepared and Professor Smith acted as guide par excellence in presenting these treasures. Notice of his death in the daily papers must have carried the thoughts of scores of women in New York City back to these rare experiences, to some of them a glimpse into an unknown world.

The direct association of Professor Smith with the American Mathematical Society was as vice president (1922), associate editor of the Bulletin (1902–20), Librarian (1902–20). In *A Semi-centennial History of the American Mathematical Society*, 1888–1938, p. 91, Dr. R. C. Archibald states: "With the election of Professor Smith as Librarian for 1902 enormous strides were almost immediately made in the Library's development . . . The Library's present excellence is almost wholly due to Professor Smith . . . " The extraordinary growth of the Library in this period is shown by increase in the number of volumes from 121 to 5,862 and in the number of periodicals on the exchange list from 64 to 123 (in 1912, 180).

Professor Smith's activities were legion and cannot all be named in this brief space, but a few others demand a place. He was the Founder of the History of Science Society and the Founder of the journal Scripta Mathematica and for one year (1920–1921) was president of the Mathematical Association of America. He was also a vice president of the International Commission on the Teaching of Mathematics (1908–1920), president (1928–1932), and honorary president (1932+). He was departmental editor in mathematics of the Fourteenth Edition of the Encyclopaedia Britannica.

Apart from being an international educational leader, Professor Smith was a great teacher. On the occasion of his seventy-fifth birthday, he was presented with a large collection of letters from former students. Among these was one from Miss Muriel Bowden which expressed fully their attitude toward him, and we close on this note.

Dedication

TO DAVID EUGENE SMITH

Who by his great scholarship has given to all his
students a love of sound learning;
Who by his glowing wisdom ever transformed the
dead letter into the living spirit;
Who by his sincerity and fortitude has inspired
and encouraged all those who have known
him in the quest for truth:

MASTER AND FRIEND.

NOTES

1. The first volume of Osiris, 1936, was dedicated to Doctor Smith and presented to him on his seventy-sixth birthday. It contains his portrait, a dedication to him by George Sarton, editor of Isis (official journal of the History of Science Society), a bibliography of his critical, historical and pedagogical writings, as well as thirty-six contributions by different authors and friends.

2. In *Bibliotheca Columbiana*, April 1933; December 1934; June 1936; August 1937, there will be found brief accounts of Doctor Smith's gifts to Columbia.

3. The Mathematics Teacher, May 1926, *In honor of Professor David Eugene Smith* contains addresses (by colleagues, former students and friends, given at a dinner at the time of his retirement), and also a reproduction of an oil painting of Doctor Smith by Leo Mielziner.

4. In *Columbia University Honorary Degrees. Appreciations by Nicholas Murray Butler*, 1933, p. 137, is found a tribute given at the time of the conferring on Doctor Smith of the honorary degree of Sc.D.

5. *Dr. David Eugen Smith, ein hervorrangender amerikanischer Reckenmethodiker*, vom Rudolf Knilling, Reportorium der Pädagogik vol. 52 (1898) pp. 561–569, is another tribute.

6. Brief but comprehensive biographies are found in (a) *Biography*, R. C. Archibald, Scripta Mathematica, April 1936, pp. 182–184; (b) *Seventy years of textbook publishing*, 1867–1937, Thomas B. Lawler, pp. 155–159.

The writer of this article wishes to express deep appreciation of the unfailing courtesy and scholarly assistance of the curator and assistants of The Plimpton-Smith-Dale Library. Acknowledgment is also made of invaluable help from other sources.

LAO GENEVRA SIMONS

BOOK REVIEW

Integration. By E. J. McShane. Princeton University Press, 1944. 8+392 pp. $6.00.

Modern mathematical analysis deals with many different methods of assigning values to integrals. If A and B are two such methods, perhaps those of Riemann and Lebesgue, we may use the symbols

$$(1)\qquad (A)\int_E f(x)dx, \qquad (B)\int_E f(x)dx$$

to denote the corresponding values of the integrals when E and $f(x)$ are such that the integrals exist, that is, when E and $f(x)$ are such that the methods assign values to the integrals. The following terminology, while not so firmly established in the theory of integration as in the theory of summability of infinite series, serves as a basis for precise statements in the theory of integration. Two methods A and B for evaluation of integrals are *consistent* if the two integrals in (1) are equal whenever both exist. The two methods A and B are *equivalent* if the two integrals in (1) both exist and are equal whenever E and $f(x)$ are such that at least one of the two exists. A method A *includes* a method B if existence of the second integral in (1) implies existence of the first and equality of the two. Thus A and B are equivalent if and only if each includes the other. The esteem attached to the notion of regularity in the theory of summability suggests that a method for evaluation of integrals should be called *regular* if it includes the Riemann method.

It could be argued that, in a textbook in applied mathematics in which there is a tacit understanding that only integrals consistent with each other are employed, there is no point in bothering with names of integrals. Whether the integrals be Riemann, Darboux, Lebesgue, Cauchy-Riemann, Cauchy-Lebesgue, Riemann-Stieltjes, and so on, does not matter. In such a book, the statement that a function is integrable means simply that it is integrable by one of the methods in the (undefined) family used. When an integrable function is given, one evaluates or estimates the integral, and proceeds to use the result. In such a book, the statement that a given function is not integrable could not be proved; there is always the possibility that there is some method in the undefined family by means of which the integral can be evaluated. There is no reason why one must stop when he has reached a general Banach integral by means of which every bounded function is integrable over every bounded interval over which it is defined.

A textbook on the theory of integration should, as the one under review does, make careful presentations of its definitions of integral. It should, as the one under review does, give proper attention to questions of consistency. It should, as the one under review does to some extent, make distinctions between methods which are not equivalent. It should, as the one under review fails to do in important cases pointed out below, make distinctions between equivalent methods which are conceptually different. That the author recognizes the desirability of making these distinctions is clearly indicated on p. 54 where he wrote "The reason for marking the integral sign with a prime in the definition is to avoid its confusion with other integrals later to be defined; even when applied to step-functions these later (and more useful) definitions will be conceptually different . . . and may even be numerically different."

This book deals mainly with Lebesgue and Lebesgue-Stieltjes integrals and with integrals equivalent to them. The author says in his preface that it is designed for students of little maturity. While less detailed, its style is similar to that of Hobson's *Theory of functions of a real variable.*

Chap. 1, pp. 1–51, is an introductory chapter giving a precise development of the fundamentals of the theory of sets in R_q (Euclidean space of q dimensions) and of real functions defined over such sets.

In chap. 2, pp. 52–100, Lebesgue integrals are defined, in terms of bounds of other integrals, without mention of measurable sets and measurable functions. The volume ΔI of an interval (or cell) I in R_q is defined to be the product of its dimensions. It is in accordance with the tenor of the book that detailed proofs of such theorems as the following are given. If $I_1, I_2, \cdots, I_n$ are closed non-overlapping intervals whose sum (union) is an interval I, then $\sum \Delta I_k = \Delta I$. Let I be an interval which is the union of disjoint subintervals $I_1, I_2, \cdots, I_n$, let $c_1, \cdots, c_n$ be constants, and let $s(x)$ be the step function for which $s(x) = c_k$ when x is in I_k. The *prime* integral of $s(x)$ over I is defined by

$$\int_I' s(x)dx = \sum_{k=1}^{n} c_k \,|\, I_k \,|.$$

A few properties of the prime integral are developed. The upper and lower Darboux (the author says Riemann) integrals over I of a function $f(x)$ bounded over I are defined to be, respectively, the greatest lower bound of $\int_I' s(x)dx$ for all step functions $s(x) \geqq f(x)$ over I, and the least upper bound of the same integral for all step functions $s(x) \leqq f(x)$ over I. This is a neat way of phrasing the classic definitions of Darboux. When these upper and lower integrals are equal, their common value (the Darboux integral) is called the Riemann integral. There is no mention of the Riemann sums used by Riemann to define his integral. Some properties of the Riemann integral are developed. Special integrals of semicontinuous functions are defined in terms of bounds of integrals of continuous functions in specified classes. Special upper and lower integrals of functions $f(x)$ defined over a finite closed interval I are defined in terms of bounds of integrals of semicontinuous functions in specified classes. If the latter upper and lower integrals are equal, the author defines their common value to be the Lebesgue integral of $f(x)$ over I. Just as the Darboux concept of integral is different from that of Riemann, so also this concept of integral is different from that of Lebesgue. It is not proved that this method of assigning values to integrals is equivalent to that of Lebesgue; unfortunately, the book does not mention Lebesgue's fundamental concept of the value of an integral. The nearest approach to this matter comes on p. 125 where it is shown that if $f(x)$ is bounded and measurable over a set E of finite measure, then $f(x)$ is Lebesgue integrable over E. At this point, it would be easy to give Lebesgue's own definition of integral and use it to prove the required equivalence. On the basis of the author's definition, the fundamental properties of the Lebesgue integral are derived.

In chap. 3, pp. 101–135, the theory of measurable sets is obtained by applying the theory of integration to characteristic functions. The theory of measurable functions is then developed. The classes L_p are defined, and the Hölder and Minkowski inequalities are proved. Chap. 4, pp. 136–187, gives the Fubini and other theorems on iterated integrals. Functions of sets are defined and discussed. Fatou's lemma is proved. Chap. 5, pp. 188–217, deals with differentiation of integrals and change of variable of integration. Chap. 6, pp. 218–241, deals with approximations to functions. In Chap. 7, pp. 242–311, Lebesgue-Stieltjes integrals are developed for functions defined over sets in R_q. Chap. 8, pp. 312–335, develops the Perron integral. Chap. 9, pp. 336–365, gives existence theorems for systems of differential equations. The final chapter, Chap. 10, pp. 366–382, deals with differentiation of multiple integrals. There is a good index.

R. P. AGNEW

NOTES

The editors of the Bulletin wish to make grateful public acknowledgment of the services rendered by the following persons who have referreed papers: C. R. Adams, R. P. Agnew, A. A. Albert, Reinhold Baer, M. A. Basoco, Harry Bateman, E. G. Begle, E. T. Bell, Stefan Bergman, P. G. Bergmann, L. M. Blumenthal, R. P. Boas, D. G. Bourgin, Richard Brauer, A. B. Brown, R. H. Bruck, R. C. Buck, S. S. Chern, Claude Chevalley, E. W. Chittenden, Alonzo Church, L. W. Cohen, M. M. Day, John DeCicco, J. L. Doob, Jesse Douglas, F. G. Dressel, Nelson Dunford, Samuel Eilenberg, Paul Erdös, R. H. Fox, K. O. Friedrichs, H. L. Garabedian, Kurt Gödel, W. H. Gottschalk, V. G. Grove, M. H. Heins, J. D. Hill, Einar Hille, Witold Hurewicz, S. B. Jackson, Nathan Jacobson, R. L. Jeffery, F. B. Jones, Mark Kac, Irving Kaplansky, E. P. Lane, C. G. Latimer, Walter Leighton, Norman Levinson, D. C. Lewis, L. H. Loomis, E. R. Lorch, N. H. McCoy, C. C. MacDuffee, W. T. Martin, W. E. Milne, C. N. Moore, F. D. Murnaghan, C. V. Newsom, Oystein Ore, J. C. Oxtoby, F. W. Perkins, Harry Pollard, Hans Rademacher, G. Y. Rainich, W. T. Reid, R. M. Robinson, Raphael Salem, G. E. Schweigert, Wladimir Seidel, I. S. Sokolnikoff, N. E. Steenrod, Otto Szász, Gabor Szegö, J. D. Tamarkin, Alfred Tarski, W. J. Thron, J. W. Tukey, H. L. Turrittin, S. M. Ulam, J. V. Uspensky, R. J. Walker, H. S. Wall, A. D. Wallace, Morgan Ward, Hassler Whitney, G. T. Whyburn, D. V. Widder, Jacob Wolfowitz, J. W. T. Youngs, Oscar Zariski, Antoni Zygmund.

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Board, who have been consulted regarding papers offered for publication in volumes 55 and 56: Reinhold Baer, P. O. Bell, Garrett Birkhoff, L. M. Blumenthal, R. P. Boas, Salomon Bochner, A. T. Brauer, R. H. Bruck, S. S. Chern, Claude Chevalley, W. Z. Chien, H. S. M. Coxeter, H. H. Goldstine, J. W. Green, Einar Hille, B. W. Jones, J. L. Kelley, C. B. Morrey, P. V. Reichelderfer, C. E. Rickart, Raphael Salem, O. F. G. Schilling, T. E. Sewell, C. L. Siegel, M. H. Stone, Otto Szász, Gabor Szegö, Alfred Tarski, André Weil, Hermann Weyl, L. R. Wilcox.

The Adolph Lomb Medal for 1944 has been awarded to Mr. R. C. Jones of the Bell Telephone Laboratories.

The Department of Mathematics of Brooklyn College, Professor

Jesse Douglas of Brooklyn College, and Professor John DeCicco of the Illinois Institute of Technology received awards at the First Honors Day Exercises at Brooklyn College, October 5, 1944.

Professor A. R. Crathorne of the University of Illinois has retired with the title professor emeritus.

Professor N. J. Lennes of Montana State University has been given the title professor emeritus.

Professor W. H. McCrea of Queen's University, Belfast, Northern Ireland, has been appointed to a professorship at the University of London.

Professor Harold Simpson of the University of London has retired and Dr. W. N. Bailey of the University of Manchester has been appointed to the chair of mathematics at the University of London.

Dr. A. J. Allen has been appointed professor of engineering at the University of Pittsburgh.

Dr. E. G. Begle of Yale University has been promoted to an assistant professorship.

Assistant Professor H. R. Branson of Howard University has been promoted to an associate professorship.

Associate Professor W. H. Brothers of Talladega College, Talladega, Alabama, has been promoted to a professorship.

Mr. J. O. Brown of Hampton Institute, Hampton, Virginia, has been promoted to an assistant professorship of physics.

Dr. E. W. Cannon of the University of Delaware has been promoted to an assistant professorship.

Dr. M. A. Coler has been appointed adjunct professor of chemical engineering at New York University.

Sister M. Loyola Conlan of the College of Mt. St. Vincent, New York City, has been promoted to a professorship.

Mr. R. E. Fullerton of Yale University has been appointed to an assistant professorship at the University of Oklahoma.

Dr. B. E. Gillam of the University of Missouri has been appointed to an assistant professorship at Drake University, Des Moines, Iowa.

Mr. F. F. Helton of the University of Illinois has been appointed to an associate professorship at Central College, Fayette, Missouri.

Dr. A. W. Jones of Yale University has been appointed to an assistant professorship at Michigan State College.

Assistant Professor J. F. Kubis of Fordham University has been promoted to an associate professorship.

Mr. Simon Lopata has been appointed lecturer in the evening session of Hunter College.

Associate Professor Florence M. Mears of George Washington University has been promoted to a professorship.

Assistant Professor R. H. Moorman of Tennessee Polytechnic Institute has been granted leave of absence to serve as acting associate professor at the College of Charleston.

Associate Professor Cronan Mullen of Siena College, Loudonville, New York, has been promoted to a professorship.

Assistant Professor R. J. Pitts of Fort Valley State College, Fort Valley, Georgia, has been promoted to an associate professorship.

Dr. G. M. Robison of Susquehanna University has been promoted to an assistant professorship.

Associate Professor J. S. Rosen of Eastern New Mexico College has been appointed to an assistant professorship at the University of Kansas City.

Assistant Professor A. E. Ross of St. Louis University has been promoted to an associate professorship.

Dr. T. H. Southard of Wayne University has been promoted to an assistant professorship.

Dr. C. W. Vickery, formerly with the University of Texas and with the War Department, is now a consulting engineer, specializing in industrial, actuarial, and quality control engineering in Washington, D. C.

Assistant Professor W. G. Warnock of the University of Alabama has been promoted to an associate professorship.

Associate Professor S. S. Wilks of Princeton University has been promoted to a professorship.

Mr. J. F. Wyckoff of Trinity College has been promoted to an assistant professorship.

The following persons are in residence at the Institute for Advanced Study during the current academic year: Professor S. S. Chern, Dr. Kurt Gödel, Dr. A. A. Grau, Dr. Olaf Helmer, Dr. Ning Hu, Professor Wolfgang Pauli, Professor Tibor Radó, Professor D. E. Richmond, Professor Robert Schatten, and Professor C. L. Siegel. Mr. E. G. Straus and Dr. Valentine Bargmann have appointments as assistants.

The following appointments to instructorships are announced: University of Buffalo: Mr. R. C. Luippold; University of Chicago: Mr. S. P. Hughart; Georgia School of Technology: Mr. S. B. Sommerville; Hillyer Junior College, Hartford, Connecticut: Mr. P. T. Bateman; Michigan State College: Dr. Margaret Mauch; University of Michigan: Dr. O. G. Owens; New Mexico College of Agriculture and Mechanic Arts: Mr. N. W. Wells; University of New Mexico: Miss Ethel M. Hove; Oregon State College: Dr. S. P. Avann; University of Pennsylvania: Mr. E. A. Knobelauch; Queens College: Dr. Ruth O. Goodman, Dr. J. C. R. Li; Rutgers University: Mr. Max LeLeiko; University of Saskatchewan: Dr. W. J. R. Crosby.

Sir Arthur Eddington, professor of astronomy at Cambridge University, died November 21, 1944.

Professor G. D. Birkhoff of Harvard University died November 12, 1944. He had been a member of the Society since 1906.

Professor Paul Capron, formerly of the United States Naval Academy, died September 29, 1944, at the age of sixty-nine years. He had been a member of the Society for forty years.

Professor J. A. Shohat of the University of Pennsylvania died on October 8, 1944. He had been a member of the Society for twenty years.

Professor Emeritus C. L. Thornburg of Lehigh University died October 14, 1944, at the age of eighty-five years. He had been a member of the Society for twenty-one years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial num- of the abstract.

ALGEBRA AND THEORY OF NUMBERS

1. H. W. Becker: *The composite umbra theorem.*

Let $U, V, W, \cdots$ be different umbrae, and let [ ] confine a polynomial umbra. It is well known that $[U+V]_n=(U+V)^n$, based on the generator equation, $e^{[U+V]}=e^U\cdot e^V\cdot[UV]_n$, needs analogous definition. $e^{[UV]}=e^{UV}=(e^{e^U-1})^{(V)}=e^{U\epsilon^*}(V)$, where $(V)_n$ is a Jordan factorial, and $[U\epsilon]_n=\phi_n(U)$ is the exponential polynomial of E. T. Bell (Ann. of Math. vol. 35 (1934) p. 263). Then $[UV]_n=[U\epsilon^*(V)]_n=\phi_n(U^*V)$, where * means that every term in U of weight m is multiplied by $(V)_m$. This is the composite umbra theorem. Such asymmetric composition is in general commutative, associative and distributive only for scalars, or for umbra iterates and inverses (calculated from $UU=U^{(2)}$, $UU^{(-1)}=1$, and so on). These decompositions greatly simplify, if $U_0=U_1=1$. Or they may be generalized, to an umbra form $[fU]_n$, where f is any function of any number of umbrae. Where the U are scalars, $[fU]_n$ reduces to $(fU)^n$, conveniently verifying the theorem and its consequences. The extension to any number of factors, $[UVW\cdots]$, is in close parallelism with the iterated exponential integers of E. T. Bell (Ann. of Math. vol. 39 (1938) p. 539), the classic instance. (Received October 28, 1944.)

2. H. W. Becker: *The hyper-umbra theorem.*

An umbra U is the representative of a series $U_0, \cdots, U_n, \cdots$. The umbra of an umbra, and so on, to m dimensions, or blanks, is called a hyper-umbra, and written ${}^mU={}^mU\{,\cdots,\}$. The fundamental umbra is ϵ, of generator $e^{e^t-1}=e^{t\epsilon}$. Its property $(\epsilon)=1$, where ( ) is a Jordan factorial, underlies the new operational transformation $e^E=eE^\epsilon$ in the finite difference calculus. Application to an umbra yields $e^{E_0}U\{0\}=eU\{\epsilon\}$. The classic instance is Dombinski's theorem, in the form $e^E0^r=e\epsilon_r$. The operation may be iterated, along each dimension of a hyper-umbra. Denote by ${}^me$ a continued exponential of the mth order. Then ${}^me^{E_0m}U\{0,\cdots,0\}={}^me^{mU}=e^{m\cdot m}U\{\epsilon,\cdots,\epsilon\}$. This is the hyper-umbra theorem. Where ${}^mU=Y$ is the cubic array whose typical cell is $(\overline{UZ}+X)^n$, this gives ${}^3e^Y=\exp\exp\exp Y=e^e\cdot e^{eU}$. Where ${}^mU=W$ is the square array of cells $\overline{UZ}_n=(U+\cdots+U)^n$ to ZU's, $e^{eW}=e^{eU}$. This is remarkable, in that the part is equipotent to the whole. If $U=1=$ the identity umbra, then $W=M=$ the table of all integer powers. Thus the power matrix is equipotent to unity. The theorem generalizes to ${}^me^{TE_0m}U\{0,\cdots,0\}={}^me^{mU\cdot T}=e^{mT*}\{\epsilon T,\cdots,\epsilon T\}$, where * denotes scalar or subscript multiplication according as T is ordinary or umbral. (Received October 28, 1944.)

3. R. H. Bruck: *Quasigroup theory. IV. Associatral series.* Preliminary report.

For a loop G, finite or with suitable chain conditions for normal subloops, define the *associator* $A=A(G)$ to be the maximal normal subloop of G contained in the three associators of G (see abstract 50-5-107). If H is any normal subloop of G define (i) $U=U(H)$ to be the unique normal subloop of G such that $U/H=A(G/H)$, and (ii) $L=L(H)$ to be the minimal normal subloop of G such that $U(L)\supset H$. Associatral series, and in particular the upper and lower associatral series, may be defined analogously to central series. If any associatral series extends from 1 to G so do the upper and lower series, and the latter have equal length a, the associatral class of G. The *associatizer* $L(G)$ may be used for a theory of associatral solubility analogous to ordinary (central) solubility. (Received November 6, 1944.)

4. C. J. Everett: *The basis theorem for vector spaces over rings.*

A vector space of n basis elements over a ring with unit has the property that every proper subspace has a basis of at most n elements if and only if the ring has no zero divisors and is a right-principal-ideal ring. This perfects Theorem F of the author's *Vector spaces over rings*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 312–316. (Received October 26, 1944.)

5. C. J. Everett and S. M. Ulam: *Projective algebra.* I.

For a subset A of a direct product (X, Y) of two sets, define $x(A)$ as the set of all (x, y_0) for which there exists a y such that $(x, y)\in A$, and $y(A)$ similarly, where (x_0, y_0) is a fixed point of (X, Y). Define the direct product $A\times B$ for $A\subset(X, y_0)$, $B\subset(x_0, Y)$ in the usual way. If a boolean algebra $\mathfrak{B}$ of subsets of $(X, Y)=I$ is closed under projection and product, it is called a *projective algebra of* (X, Y) and has properties 1. $x(A\cup B)=x(A)\cup x(B)$; 2. $xy(I)=(x_0, y_0)=yx(I)$ is an atom in $\mathfrak{B}$; 3. $x(A)=0$ if and only if $A=0$; 4. $xx(A)=x(A)$; 5. $x(A\times B)=A$, $y(A\times B)=B$, and $A\times B$ contains all sets with these projections; 6. $(X, y_0)\times(x_0, y_0)=(X, y_0)$; 7. direct product is distributive with respect to union; with similar properties for y in 1, 3, 4, 6. An abstract boolean algebra with mappings $x(A)$, $y(A)$, and product satisfying these properties as postulates is called a *projective algebra*. It is proved that every such algebra is embeddable in a complete ordered projective algebra, and that every projective algebra of *all* subsets of a set is representable as a projective algebra of some (X, Y). (Received October 26, 1944.)

6. N. J. Fine: *Congruence properties of the elementary symmetric functions.*

Define $u_k(n)$ as the elementary symmetric function of order k in n independent variables. Let $P_n(p, k, a)$ be the probability that $u_k(n)$ be congruent to a (mod p), where p is any prime. It is proved that $\lim P_n(p, k, a)=P(p, k, a)$ exists. If s is the number of ones in the dyadic expansion of k, then $P(2, k, 1)=1/2^s$. Simple recursion formulas are given for $P(3, k, a)$. The following general theorems are proved: (i) $P(p, k, a)=1/p$ for $k=rp^t$, with $0<r<p$, all p and t. (ii) $P(p, k, 0)>1/p$ for $k=(p-1)p^t+R$, with $0<R<p^t$. (iii) $P(p, k, 0)>C(p)>0$ for all positive k. It is conjectured that equidistribution holds only if $k=rp^t$; that $P(p, k, 0)$ is not less than $1/p$ for all p and positive k; that $P(p, kp, a)=P(p, k, a)$; finally, that $\lim\sup P(p, k, 0)=1$. (Received October 26, 1944.)

7. A. P. Hillman: *On derivatives of differential polynomials.*

Proofs of the following results are given. Let F be a differential polynomial effectively involving the unknowns $y_1, \cdots, y_n$. Then F holds no (essential) component of the manifold of its derivative F_1. Every component of the rth derivative F_r of F is the general solution of a differential polynomial of order at least r in at least one of the unknowns. For r sufficiently large, F_r is algebraically irreducible, its manifold is irreducible, and the perfect ideal it generates is prime. (Received November 30, 1944.)

8. Nathan Jacobson: *A problem on algebraic algebras.*

The question considered in this paper is the following one: Let $\mathfrak{A}$ be an algebra over a field Φ such that (1) every element of $\mathfrak{A}$ satisfies an equation of degree not greater than N with coefficients in Φ and (2) $\mathfrak{A}$ has a finite number of generators. Then is $\mathfrak{A}$ necessarily an algebra with a finite basis? This is a special case of a problem recently proposed by Kurosh and it is analogous to the well known Burnside problem in the theory of groups. A partial solution of the problem is obtained in this paper. It is shown that the question can be answered in the affirmative for algebras that are semi-simple in the sense that they contain no nil ideals not equal to 0. This makes it possible to reduce the problem to the more special one in which (1) is replaced by (1′) every element of $\mathfrak{A}$ satisfies the equation $a^N=0$. It is shown also that an affirmative answer to a special case of Burnside's problem on groups would imply an affirmative answer to our problem for algebras over a field of prime characteristic. (Received October 13, 1944.)

9. Irving Kaplansky: *The commutativity of generalized Boolean rings.*

Stone's observation that a ring in which $x^2=x$ is necessarily commutative is extended to more general rings, and in particular to the p-rings of McCoy and Montgomery (Duke Math. J. vol. 3 (1937) pp. 455–459). It is shown that a ring of characteristic p in which $x^n=x$ is commutative if $n=p^r$ with r a power of 2; and regardless of the characteristic, it is commutative if the regular polygon of $n-1$ sides is constructible by ruler and compass. The lowest case where commutativity remains in doubt is $n=8$. (Received November 30, 1944.)

10. Fred Kiokemeister: *The Asano postulates for orders in a linear algebra.*

Let g be a domain of integrity with unique factorization of ideals into products of prime ideals, and let P be the quotient field of g. An investigation of the g-submodules of the linear algebra A over P leads to the theorem that if R is an order in A, the maximal chain condition and the modified minimal chain condition hold for regular ideals in R. The Asano postulates for an arithmetic are shown to hold in A. (Received October 25, 1944.)

11. Fred Kiokemeister and G. W. Whitehead: *A coset theory for left loops.*

A left loop is a multiplicative system Q with two-sided identity such that the equation $ax=b$ has a unique solution for every $a, b \in Q$. A subset H of Q is an admissible left subloop if it is closed under multiplication, the solution of the equation $ax=b$

is in H whenever $a, b \in H$, and $x(yH) = (xy)H$ for every $x, y \in Q$. Under these circumstances Q has an expansion in left cosets of H, and the system Q/H of left cosets is made into a left loop under a suitable definition of multiplication. The group $\mathfrak{L}$ spanned by the left multiplications (that is, the permutations $L_x(a) = ax$) of Q is introduced, and an isomorphism of Q with $\mathfrak{L}/\mathfrak{L}_0$ (where $\mathfrak{L}_0$ is the subgroup of $\mathfrak{L}$ consisting of all permutations keeping the identity E fixed) is established. It is shown that the admissible left subloops of Q are in 1:1 correspondence with the subgroups $\mathfrak{M} \supset \mathfrak{L}_0$ of $\mathfrak{L}$, and isomorphisms $H \cong \mathfrak{M}/\mathfrak{L}_0$, $Q/H \cong \mathfrak{L}/\mathfrak{M}$ are established. An extension theory is developed: given left loops H and K, a construction is given for all left loops Q such that $Q/H = K$. Necessary and sufficient conditions are given (when H is a group) that Q shall be a group, and specialization of H to be normal yields the Schreier extension theory. (Received October 20, 1944.)

12. Seymour Sherman: *Complex polynomials and polygonal domains.*

Theorems of Sturm, Routh, and Hurwitz have been generalized so as to provide a finite numerical algorithm for finding the number of such roots of a polynomial with complex coefficients as lie on a generalized polygon or linear transformation thereof. By this means a finite procedure is given for determining the number of roots of a polynomial lying in a quadrant, half-plane, circle, or circular sector. Such problems have proved of interest recently in connection with airplane flutter (S. Sherman, Jane DiPaola, and H. Frissel, *Routh's discriminant, flutter, and ground resonance,* abstract 50-7-190) and econometric business cycle analysis (P. A. Samuelson, *Conditions that the roots of a polynomial be less than unity in absolute value,* Annals of Mathematical Statistics vol. 12 (1941)). (Received October 17, 1944.)

ANALYSIS

13. E. F. Beckenbach: *A Looman-Menchoff theorem for Newtonian vectors.*

It is shown that if the vector function $X(x, y, z)$ is continuous in the finite domain D, if except at most at the points of a denumerable set of points in D, $X(x, y, z)$ is totally differentiable in the planes parallel to the coordinate planes, and if the curl and divergence of $X(x, y, z)$ vanish almost everywhere in D, the $X(x, y, z)$ has continuous partial derivatives of all orders. (Received October 28, 1944.)

14. R. E. Fullerton: *Linear operators with range in a space of differentiable functions.*

The Banach space $C^n(0, 1)$ is defined to be the space of functions possessing n continuous derivations over the interval (0, 1) with norm $\|f\| = \text{l.u.b.}_{0 \leq t \leq 1}\text{l.u.b.}_{k \leq n} |f^{(k)}(t)|$. If $Tx = f$ is a bounded linear operator from a Banach space X to $C^n(0, 1)$, Tx is representable in the form $\bar{x}_t x$ where $\bar{x}_t$ is a function defined from (0, 1) to the space $\bar{x}$ conjugate to X. In this paper, necessary and sufficient conditions that $\bar{x}_t$ represent such an operator are found. Both bounded and completely continuous operators are investigated. Particular attention is devoted to representations of operators from sequence spaces and Lebesgue spaces to the space $C^n(0, 1)$. In all cases the expression for the norm of the operator is obtained in terms of the function $\bar{x}_t$. (Received October 20, 1944.)

15. Einar Hille: *The commutative n-parameter case of semi-groups of linear transformations.*

Let $T(s)$ be a family of linear bounded transformations on a B-space $\mathfrak{X}$ to itself such that $T(s)T(t)=T(s+t)$. Here $s=\{s_1, s_2, \cdots, s_n\}$ and $T(s)$ is defined in S: $0 \leqq s_k < \infty$, $\sum s_k > 0$; parameter addition being vector addition. If $T(s)$ is strongly measurable in S then it is strongly continuous. If $\|T(s)\| \leqq M$, $\sum s_k < 1$, and if $U_s T(s)[\mathfrak{X}]$ is dense in $\mathfrak{X}$, then $T(s) \to I$ when $s \to 0$. From this it follows that $T(s)$ is the direct product of not more than n one-parameter commuting semi-groups. Conversely the direct product of such semi-groups is a semi-group of the type considered here. If $A_1, A_2, \cdots, A_n$ are the infinitesimal generators of the factors, then the Lie ring of $T(s)$ is the set $\sum \alpha_k A_k$ with $\alpha_k \geqq 0$. (Received October 28, 1944.)

16. H. K. Hughes: *The asymptotic developments of a class of entire functions.*

In an earlier paper (Bull. Amer. Math. Soc. vol. 50 (1944) pp. 425–430) the author established a general theorem which furnishes the asymptotic expansions, valid for large $|z|$, of an entire function defined by a Maclaurin series of the form $\sum_{n=0}^{\infty} g(n)z^n$, where the coefficient $g(n)$ of z^n satisfies certain conditions. In the present paper, the theorem is applied to find the asymptotic expansions of the function $f(z)$ for which $g(n)$ is the reciprocal of the product of m gamma functions of the form $\Gamma(\alpha_j n+a_j)$; $1 \leqq j \leqq m$; $m \geqq 2$; $\alpha_j > 0$; a_j any constants, real or complex. The main part of the paper consists of showing that the present $g(n)$ satisfies the condition stated in the hypotheses of the theorem. This requires the introduction of a special lemma concerning the asymptotic representation of a fraction whose numerator and denominator are each the product of m gamma functions of the type described above, with certain relations existing among the constants. (Received October 13, 1944.)

17. C. T. Loo: *Note on the strong summability of Fourier series.*

Let $f(x)$ be of period 2π and of the class L^p, $p>1$. Let $k=2, 3, \cdots$ be an integer less than p and suppose that for some x_0 and for some S, $\int_0^t |f(x_0+u)+f(x_0-u)-2S|^p du = o(t^p)$ as $t \to 0$. Then the partial sums $S_n(x)$ of the Fourier series of f satisfy the relation $\sum_{m=0}^{n} |S_{m^k}(x_0)-S|^2 = o(n)$. (Received October 16, 1944.)

18. C. T. Loo: *Two Tauberian theorems in the theory of Fourier series.*

Let $f(x)$ be integrable L and of period 2π. Let $a_0/2+\sum_1^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_0^{\infty} A_n(x)$ be the Fourier series of f, and let $\sigma_n^{\alpha}(x)$ denote the αth Cesàro means of this series. Let, for some fixed x_0 and S, $\phi(t)=\{f(x_0+t)+f(x_0-t)-2S\}/2$, $\phi_\alpha(t) = \alpha t^{-\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)du$. Then (i) if $\alpha>0$ and if $\sigma_n^{\alpha}(x_0)-S=o(1/\log n)$, we have $\phi_{1+\alpha}(t) \to 0$; (ii) if $\sigma_n^{\alpha}(x_0)-S=O(n^{-\epsilon})$ and if $A_n(x_0)=O(n^{-\delta})$ with $\alpha>0$, $\epsilon>0$, $\delta>0$, $\epsilon+\delta>1$, we have $\phi_\alpha(t) \to 0$. (Received October 16, 1944.)

19. W. T. Reid: *A matrix differential equation of Riccati type.*

This paper is concerned with the matrix differential equation (I): $W'+WA(x)+D(x)W+WB(x)W=C(x)$, where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are given $n \times n$ square matrices whose elements are continuous functions on the interval $a \leqq x \leqq b$. The first part of the paper contains generalizations of well known theorems on solutions of a single ordinary differential equation of Riccati type; in particular, the theorems established provide decided extensions of results proved by W. M. Whyburn (Amer. J. Math. vol. 56 (1934) pp. 587–592) for the equation $W'+WW=C(x)$. In the second part of the paper the matrix equation (I) is used to present a compact discussion of the Legendre transformation of the second variation for simple integral problems of the calculus of variations. (Received October 19, 1944.)

20. A. R. Schweitzer: *Functional relations valid in the domains of abstract groups and Grassmann's space analysis.* II.

The relations stated in abstract 50-7-185 are generalized as follows: 1. Autodistributivity: $f(u_1, u_2, \cdots, u_{n+1})=f(u, t_1, t_2, \cdots, t_n)$ where $u_i=f(x_i, t_1, t_2, \cdots, t_n)$ and $u=f(x_1, x_2, \cdots, x_{n+1})$. 2. Restricted distributivity: $f(v_0, v_1, v_2, \cdots, v_n)=\phi(v, t_1, t_2, \cdots, t_n)$ where $v_0=\phi(x, t_1, t_2, \cdots, t_n)$, $v_i=\phi(t_i, t_1, t_2, \cdots, t_n)$ and $v=f(x, t_1, t_2, \cdots, t_n)$. 3. Reflexiveness: $f(x, x, \cdots, x)=x$. 4. Duals, in f and ϕ, of the preceding postulates. 5. Closure of the set S of elements x_i under the compositions f and ϕ. Various systems associated with the above postulates are discussed. For example: 6. The postulate $f\{\phi(x, t_1, t_2, \cdots, t_n), t_1, t_2, \cdots, t_n\}=f(t_1, t_2, \cdots, t_n)$ and its dual are adjoined, with definition: $f(x_1)=x_1$, $f(x_1, x_2)=f\{x_1, f(x_2)\}, \cdots$, $f(x_1, x_2, \cdots, x_{n+1})=f\{x_1, f(x_2, \cdots, x_{n+1})\}$, and dually for ϕ. 7. Postulate 2 is replaced by full distributivity. Postulates 1–6 are valid in the domain of the algebra of logic if $f(x_1, x_2)=x_1\ x_2$ and $\phi(x_1, x_2)=x_1+x_2$. In particular, any associative, commutative and reflexive function $f(x_1, x_2)$ is autodistributive. (Received October 20, 1944.)

21. W. S. Snyder: *Derivatives of set functions.*

Let Γ be a class of subsets T of the xy-plane, subject to the following conditions: (A) The sets T lie in a bounded portion of the xy-plane. (B) $|T|=|\overline{T}|>0$ for every set T. (C) The parameter of regularity of each T exceeds a fixed positive number ρ. Let f be a real-valued set function defined on Γ. The upper (lower)-derivatives of f at a point p are defined by $u(f, p)=\lim\sup_{|T|\to 0}(f(T)/|T|)$, $p\in T$ $(l(f, p)=-u(-f, p))$. The derivatives are shown to be measurable point functions. Necessary and sufficient conditions of an $\epsilon-\delta_\epsilon$ variety are obtained (a) for the derivatives to be summable, (b) for a unique, finite derivative to exist almost-everywhere, (c) for f to be an indefinite Lebesgue integral. Similar theorems are proved for functions defined on non-regular classes. The results are then extended to several other varieties of derivatives, in particular to the derivative considered by R. C. Young (*Functions of* $\Sigma, \cdots$, Math. Ann vol. 29 (1928) pp. 171–216). (Received October 20, 1944)

22. W. J. Trjitzinsky: *Singular integral equations with Cauchy kernels.*

The theory of equations $\alpha(t)\phi(t)+\int\kappa(t, y)\phi(y)dy/(y-t)=f(t)$ (α, f, κ of Holder class along L, integrations in sense of Cauchy principal values over L) in recent times has been brought to a high degree of perfection, notably by M. I. Mushelishvili and I. Vecoua, in the case when L consists of a finite number of suitably regular open or closed arcs (in the complex plane of y) without common points. In the present work the theory is developed in the more general situation, presenting substantial new difficulties, when the arcs may have common points. (Received October 20, 1944)

23. Antoni Zygmund: *On smooth functions.*

Let $f(x)$ be continuous, of period 2π. It is *uniformly smooth* (symbolically: $f\in\lambda^*$), if

$\Delta_h^2 f(x) = f(x+h) + f(x-h) - 2f(x) = o(h)$ uniformly in x. If $\Delta_h^2 f = O(h)$, $f \in \Lambda^*$. If $f \in \text{Lip } 1$, then $f \in \Lambda^*$. The converse is false, since there are $f \in \Lambda^*$ nowhere differentiable. The modulus of continuity of an $f \in \Lambda^*$ is $O(\delta \log \delta)$. The class Λ^* is sometimes more natural than Lip 1. (i) A necessary and sufficient condition that the best approximation $E_n[f] = O(n^{-k})$ $(k = 1, 2, \cdots)$ is $f^{(k-1)} \in \Lambda^*$. (ii) If $f \in \Lambda^*$, so does the conjugate function $\bar{f}$. (iii) Let f^α, f_α denote the αth derivative and integral of f $(0 < \alpha < 1)$. If $f \in \Lambda^*$, then $f^\alpha \in \text{Lip } (1-\alpha)$. If $f \in \text{Lip } \alpha$ then $f_{1-\alpha} \in \Lambda^*$. (iv) A necessary and sufficient condition that a harmonic function $f(r, x)$, $0 \leqq r < 1$, be the Poisson integral of an $f \in \Lambda^*$ is $\partial^2 f(r, x)/\partial x^2 = O\{1/(1-r)\}$. If $\Delta_h^2 f(x) = o(h)$ for each x, f is *smooth* $(f \in \lambda)$. A $g(x)$ defined over a set E is said to satisfy condition D, if $g(x)$ takes all intermediate values. (v) If $f \in \lambda$, $f'(x)$ exists in a dense set (Rajchman) and satisfies condition D. (vi) The sum of a trigonometric series with coefficients $o(1/n)$ satisfies condition D. (vii) If $g(x)$ is continuous, $\tilde{g}(x)$ satisfies condition D. (Received October 28, 1944.)

APPLIED MATHEMATICS

24. C. H. Dix, C. Y. Fu, Mrs. E. W. McLemore: *The cubic Rayleigh wave equation.*

Consider a plane compressional wave incident on the free plane surface of a semi-infinite elastic medium. The incident and reflected compressional amplitudes are respectively A and B. Then $B/A = N(i, s)/D(i, s)$ where $s = \lambda/\mu$, $ND = 16(s+1)w^3 - 8(3s+4)w^2 + 8(s+2)w - (s+2)$ and $w = \sin^2 r_1$, $r_1 =$ reflection angle of shear wave. Zeros of ND show two or no i's (also zeros of N) corresponding to no reflected compressional wave. The third zero is a zero of D and gives the reciprocal of the important solution of the Rayleigh wave cubic. The cubic curves all pass through one fixed point whose coordinates are $w = 1.0957$, $ND = -1.83927$. For a fixed s, ND has the same value when $w = 0$ that it has when $w = 1$. If s corresponds to a small Poisson ratio then i for the larger zero of N will be very close to 90°, giving no reflection of compressional type whereas for $i = 90°$ all reflected energy is compressional. There is a discontinuity when $s = 0$ where $B/A = +1$ while $B/A = -1$ for s greater than 0. (Received October 27, 1944.)

25. H. W. Eves: *A geometrical note on the isocenter.*

Consider a central projection of plane p on plane p', L being the center of projection, and adopt the convention that angular directions on p (or p') are positive if they are counterclockwise when p (or p') is viewed from L. A point on p is called a positive isocenter on p if all angles on p having the point for vertex are invariant under the projection, and a point on p is called a negative isocenter if all angles on p having the point for vertex project into equal but oppositely directed angles on p'. It is shown that a tilted photograph possesses one and only one positive isocenter and one and only one negative isocenter. These points are geometrically located on the picture. Now the positive isocenter has long been known in photogrammetry, but the existence of the negative isocenter does not seem to have been noticed before. With the combined aid of these two isocenters a simple graphical procedure is developed for rectifying a tilted photograph. The mapping process can then be continued by the method of radial plotting. (Received October 19, 1944.)

26. H. W. Eves: *Analytical and graphical rectification of a tilted photograph.*

An aerial photograph fails to be a perfect map of the ground photographed because of unavoidable tilt in the photograph and because of the presence of relief on the ground. To convert the photograph into a true map of the ground, then, it must be corrected for both tilt and relief. This correction program usually follows the order of first correcting for tilt, and then correcting that result for relief. The present paper is believed to be a new analytical and graphical treatment of the first, and more difficult, part of the program, that of correcting a photograph for map displacements induced by tilt. The graphical method devised is particularly applicable to high oblique photographs, and seems superior to the perspective grid method of the Canadian Government in that it is faster and does not require that the terrain be flat. Some linkage motions are indicated for carrying out the graphical solution on high obliques. The analytical results obtained are employed in discussing the curves which appear on parallax correction graphs. (Received November 25, 1944.)

27. Walter Kohn: *Contour integration in the theory of the spherical pendulum and the heavy symmetrical top.*

This paper is concerned with new developments of the method of complex integration as applied to the motions of the spherical pendulum and the heavy symmetrical top. Let $\phi(t)$ be the angle of azimuth at time t and Φ the advance in azimuth corresponding to the passage from the lowest of the highest level of motion. By decomposing the expression for $\phi'(t)$, the Puiseux inequality for the spherical pendulum, namely $\phi > \pi/2$, is derived in a new and very simple manner. Next, an adaption of Weinstein's vertical line of integration (Amer. Math. Monthly vol. 49 (1942) p. 521) yields a new result which includes the Puiseux inequality. Similar methods are applied to the different types of motion of a heavy symmetrical top, and bounds for Φ are determined in each case. These bounds are proved to be actually the greatest lower and least upper bounds respectively. It is found that all important characteristics of the motion of the spherical pendulum, such as those of Halphen, Puiseux and the level-inequality characteristics, are shared by the heavy top whose precession is retrograde. The spherical pendulum is a limiting case of such a top. (Received October 11, 1944.)

28. A. N. Lowan and H. E. Salzer: *Formulas for inverse interpolation within a square grid in the complex plane.*

The results obtained in the article by H. E. Salzer, *A new formula for inverse interpolation*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 513–516, are applicable to the problem of inverse interpolation of complex functions which are tabulated at equidistant intervals along any straight line in the complex plane, provided that h stands for the complex interval and p, given by (5), is a complex number from which is obtained the unknown argument as a_0+ph. But when the function is tabulated over a square grid, just as in the case of direct interpolation (see A. N. Lowan and H. E. Salzer, *Coefficients for interpolation within a square grid in the complex plane*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 23 (1944) pp. 156–166), a closer approximation is obtained by choosing the points along the corners of a square. For the cases where it suffices to employ quadratic and cubic direct interpolation with the points chosen as in the latter article, formula (5) in the former article is still applicable, where for the quadratic case $r=2(f_p-f_0)/\Delta$, $s=[(-1+i)f_{-1}-2if_0+(1+i)f_1]/\Delta$, where $\Delta=(1-i)f_{-1}+2(i-1)f_0+(1-i)f_1$, $t=u=v=w=0$, and for the cubic case $r=2(f_p-f_0)/\Delta$, $s=[(-3+i)f_{-1}-4if_0+(3+i)f_1+2if_2]/\Delta$, $t=[(1-i)f_{-1}+(1+i)f_0+(-1+i)f_1+(-1-i)f_2]/\Delta$, where $\Delta=2f_{-1}+(-3+3i)f_0-2if_1+(1-i)f_2$, $u=v=w=0$. (Received October 20, 1944.)

29. F. J. Murray: *Gears, cams and differentials.*

The problem of obtaining, mechanically, the value of a function, $F(x_1, \cdots, x_m)$, without integrating or differentiating, is considered. In the machine the $x_1, \cdots, x_m$ enter as rotations of corresponding shafts and the value of F appears as a rotation. A set of gears yields an output which is a fixed multiple of its imput. The output of a differential is the average of its two imputs. A device which yields a prescribed function of one variable will be called a cam. F can be mechanized by n cams and various gears and differentials, if there exist constants, $\alpha_{i,k}, \beta_{i,j}, y_k, \delta_i, i, j=1, \cdots, n$; $k=1, \cdots, m$, and n functions $J_i(t)$, such that $F=\sum_{k=1}^{m} y_k x_k + \sum_{j=1}^{n} \delta_j J_j(I_j)$ for the set of I_i's defined by the system of equations $I_i=\sum_{k=1}^{m} \alpha_{i,k} x_k + \sum_{j=1}^{n} \beta_{i,j} J_j(I_j)$, $i=1, \cdots, n$. The I_i's are the imputs of the various cams. A system of partial differential equations on F which constitute necessary conditions for this are derived and sufficient conditions are obtained from a part of these when certain determinants are not zero. The difficulties of this theory arise in showing that these determinants are not identically zero. This, however, is done. It is also shown that there are analytic functions of two or more variables which cannot be mechanized by gears, cams, and differentials. (Received November 13, 1944.)

30. F. J. Murray: *On solving a set of n linear equations.*

Consider a system of n linear equations in n unknowns, $\sum_k a_{ik} x_k = b_i$. If we form the function $F_i(x_1, \cdots, x_n) = \sum_i (b_i - \sum_k a_{i,k} x_k)^2$ then the gradient of F at $(0, \cdots, 0)$ lies along the direction in which x_k is negatively proportional to $\sum_i b_i a_{i,k}$. Consequently the vectors $(y_1, \cdots, y_n)$ with $y_k = \lambda \sum_i b_i a_{i,k}$ lie along a path which is initially one of quickest descent for F. If λ is chosen so as to make F a minimum along this path, an approximation for the solutions of the given system of equations is obtained. This can be used as a step of an iterative process for solving the equations. The error of the nth iterate depends on the dispersion of the reciprocals of the characteristic roots of the matrix, $(\sum_j a_{i,j} a_{j,k})_{i,k=1,\cdots,n}$. The gradient notion can also be used to design machines with very desirable characteristics for solving such systems. (Received October 23, 1944.)

31. Isaac Opatowski: *A botanical application of Jensen's inequality.*

The paper is a mathematical elaboration of some ideas connected with the theory of form of plants of N. Rashevsky (Bulletin of Mathematical Biophysics vol. 5 (1943) pp. 33–47, 69–73). Its purpose is to show that the function of branches of a tree, which is usually explained on exclusively biological bases, may be also interpreted from a mechanical viewpoint, as some kind of tendency of the tree to shape itself in such a manner as to increase its own strength. The theory is based on some experimental relations established by the U. S. Forest Products Laboratory, and consists of a comparison of a tree (T) of n branches of length L_i $(i=1, \cdots, n)$ with a hypothetical tree (H) of one branch only which from a purely biological viewpoint is identical with (T), except for a different shape. The ratio of the maximum bending stress in the branches of (T) to a similar stress in (H) is $(\sum_i L_i^3)^5/(\sum_i L_i^5)^3$, which is greater than 1, by Jensen's inequality. The paper is a part of an article to be published in the December 1944 issue of the Bulletin of Mathematical Biophysics. (Received October 18, 1944.)

32. Ida Roettinger: *An operational approach to the solution of boundary value problems by generalized Fourier series.*

The idea of G. Doetsch (Math. Ann. vol. 112 (1935) pp. 52–68) to consider the set of Fourier coefficients of a function as the finite Fourier transformation of this function is applied to the set of coefficients in the expansion of a function according to characteristic functions of certain Sturm-Liouville problems. Results known for the sin nx- and cos nx-transformations (see H. Kniess, Math. Zeit. vol. 44 (1938) pp. 266–291; see also the author's abstract 50-5-155) are extended to these more general transformations. The main idea in this generalization is the introduction of almost periodic functions in the form of "almost periodic extensions" of a function defined on a finite interval. Some applications of these transformations to boundary value problems are given. In particular the solution of the problem of an elastically supported string is expressed in closed form by means of an almost periodic extension; this form of solution gives an intrinsic characterization of the motion of the string. (Received October 17, 1944.)

33. W. H. Roever: *Second derivatives of the potential function of the earth's weight field of force.*

The author shows that, in considering the statical and dynamical phenomena which take place in the earth's weight field of force, both the first and the second partial derivatives of the potential function of this field of force come into play. The moment actuating the Eötvös torsion balance involves some of these derivatives, and these are thus experimentally determined and found to considerably exceed in value those corresponding to the generally accepted potential function used in geodesy. Those second derivatives which occur in the curvature of the lines of force are of the order of magnitude of ω^2, where ω is the angular velocity of the earth's rotation. However, those which occur in the curvatures of the level surfaces are considerably larger and come into play in exterior ballistics if the convergence of the verticals is considered. The differential equations of motion of a trajectory are also derived using for the potential functions of the weight field an analytic approximation involving both the first and second derivatives of this function. In setting up the frame of reference in exterior ballistics it is shown that ω^2 *cannot* be neglected. (Received November 24, 1944.)

34. H. E. Salzer: *Inverse interpolation for eight-, nine-, ten-, and eleven-point direct interpolation.*

In the paper *A new formula for inverse interpolation*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 513–516, the author obtained all the terms involving the first six powers of (f_p-f_0) in the expansion for p in terms of f_p and the tabular values corresponding to the cases where three- to seven-point direct interpolation was required. In the present paper that same formula for p is extended to include all the terms involving the first ten powers of (f_p-f_0) and, as before, quantities are defined in terms of f_p and the tabular entries to provide for inverse interpolation when eight-, nine-, ten-, or eleven-point formulas are required for direct interpolation. Although the greater part of the expression for p will hardly ever be needed in most practical problems, due to the rapidity of convergence and smallness of many of its terms, its full use can provide unusual accuracy in solving equations (both real and complex) up to the tenth degree when the values of the polynomials are tabulated near the root at equal intervals. (Received November 14, 1944.)

35. J. J. Stoker: *Nonlinear theory of curved elastic sheets.*

The membrane theory of thin elastic shells, which is based on the assumption that bending stresses can be neglected, is relatively simple from the mathematical point of view since the stresses can be determined independently of the strains. However, it is not possible then, in general, to satisfy boundary conditions which refer to displacements—such as, for example, the condition of a fixed edge. The present paper presents a theory of thin shells which neglects bending stresses but which, nevertheless, makes it possible to satisfy various types of boundary conditions which are reasonable from a physical point of view, such as that of a fixed edge. This is accomplished by taking into account certain of the quadratic terms in the expressions for the strains as functions of the displacements, in a manner analogous to that employed in deriving the von Kármán equations for bending of thin plates. (Received October 4, 1944.)

ERGODIC THEORY

36. P. R. Halmos: *On an incompressible transformation.*

E. Hopf has introduced a very strong notion of incompressibility for one-to-one measurable transformations on a measure space and showed that a transformation is incompressible in that sense if and only if it possesses a positive finite invariant integral. Recently Hurewicz has shown that under the assumption of a much weaker notion of incompressibility a very elegant generalization of Birkhoff's ergodic theorem is valid. The purpose of this note is to point out the following two facts: (1) If a transformation does possess an invariant integral, the Hurewicz theorem can be made to follow from known results for measure preserving transformations. (This fact is not immediately obvious only because of the rather peculiar formation of the Hurewicz means.) (2) There exists a one-to-one measurable transformation on a measure space which has the weak but not the strong property of incompressibility. (This fact in addition to answering a question explicitly raised by Hopf also serves to show that Hurewicz's theorem is indeed an extension of Birkhoff's.) (Received October 16, 1944.)

GEOMETRY

37. Felix Bernstein: *The swastika and the Sicilian triskelon from the standpoint of "higher geometry."*

From the standpoint of higher geometry as defined by Felix Klein, the swastika and the triskelon are interpreted by the crystallographic groups of the plane, which are known groups composed of translations and rotations, with an angle of rotation of $360/n$ degrees, where n is restricted to the values 2, 3, 4, 6. The known fundamental domains (F.D.) of these groups with one center of symmetry are altered here into F.D. with two centers of symmetry. The broken line of greatest length connecting the two centers is called an arm and the smallest region whose boundary consists of arms only is called a blitz according to its shape. A blitz and its images produced by the operations of the group in the case $n=4$ fill the whole plane with swastikas, in the case $n=6$ with a like set of triskelons. With the aid of a properly generalized swastika it is possible, in an analogous manner, to fill euclidean space. By certain alterations of given F.D., a proof of the Pythagorean Theorem is obtained. (Received December 1, 1944.)

38. John DeCicco: *Survey of polygenic functions.*

The author presents a general outline of the theory of the first and second derivatives of a general polygenic function. This leads to the geometry of the related circles, limaçons, and cardioids. (When the function is monogenic, these related curves degenerate into points.) In addition to summarizing the already published material, many new theorems are included. (Received October 26, 1944.)

39. V. G. Grove: *Quadrics associated with a curve on a surface.*

The quadrics of Darboux, Moutard and Davis, the conjugal quadrics, the asymptotic osculating quadrics and many other quadrics belong to a certain family of quadrics. This paper seeks to characterize all of the members of this family in terms of cross-ratios. In so doing a generalization is obtained for Bell's R-associate of a line in the tangent plane. Some special quadrics of the pencil are characterized and new characterizations of the pan-geodesics are obtained. (Received October 7, 1944.)

40. C. C. Hsiung: *A ternary of plane curvilinear elements with a common singular point.*

This paper studies three curves having a common singular point of different kinds and a common tangent at the point. A projective invariant is found and characterizations are found for the invariant for various kinds of singularities. (Received October 7, 1944.)

41. Edward Kasner: *Multi-valued symmetries.*

The author studies conformal symmetry in a general algebraic curve. This is equivalent to Schwarzian reflection for an analytic curve. For an algebraic curve of degree n, the operation T is in general of degree n^2. The degrees of the powers of T are studied in detail. In the special case of a conic, the results are noteworthy. If the base curve is a potential curve (obeys the Laplace equation), symmetry is easily constructible. Satellite curves discussed in a previous paper are related to the present theory. (Received October 26, 1944.)

42. E. J. Purcell: *Some Cremona involutions in n-dimensional space.*

A previous paper (E. J. Purcell, *Variety congruences of order one in n-dimensional space*, Amer. J. Math. vol. 66 (1944) pp. 621–635) discusses linear k-parameter systems of varieties in n-dimensional projective space (k any positive integer not greater than n). Each variety of such a system is of dimension $n-k$ and order h (h any positive integer). Through a generic point of $[n]$ one and only one variety of the system passes. When $n=k$ and $h=2$, a generic variety of the system is a pair of points. Each point determines the pair to which it belongs and the system consists of the pairs of a rational Cremona involution in $[n]$. This paper treats a type $(n)_n$ Cremona involution in $[n]$. When $n=2$, the involution is Geiser's. When $n=3$, the involution is due to Sharpe and Snyder. (Received October 25, 1944.)

STATISTICS AND PROBABILITY

43. T. R. Hollcroft: *The probability of repetitions.*

The probability of repetitions is concerned with repetitions only and not with the particular numbers that are repeated. For example, let one number be drawn at a time from ten and replaced after each draw. Eight may be drawn as follows: 4 3 7 6 4 7 6 7. This set contains one triple and two double repetitions. The double repetition 4 4 counts merely as a pair as does 6 6 or any other two of a kind drawn. This set is represented by the symbol (3221). The repetitions and number of trials in the set characterize the set. Let n be the number of trials and p be the probability of drawing a given number in one trial. The total number of sets of repetitions for any n are given in a table. The total number of ways in which each set can be drawn is then found and finally general formulas for the probability that a given set will occur in n trials. Relations among probabilities are obtained. A special case of one of these states that two pairs are $3(n-3)/4$ times as probable as one triple. Similar probability formulas are derived in case the draws are made without replacement. (Received October 10, 1944.)

44. Mark Kac: *Random walk in the presence of absorbing barriers.*

The solution of the random walk problem in the presence of absorbing barriers depends on calculating probabilities of the form $P(n;\ p,\ q)=\text{Prob.}\{-q\leqq X_1\leqq p,\ -q\leqq X_1+X_2\leqq p,\ \cdots,\ -q\leqq X_1\cdots X_n\leqq p\}$, where $X_1, X_2, \cdots$ are identically distributed random variables and $p\geqq 0$, $q\geqq 0$. The present paper illustrates a method by means of which explicit formulas can be obtained. If each X assumes values 1 and -1 with probability 1/2 and if p and q are integers one obtains $P(n;\ p,\ q)=2(p+q+2)^{-1}\sum \cos^n \pi j/p+q+2 \sin \pi(q+1)j/p+q+2 \cot \pi j/2(p+q+2)$, where j runs from 1 to $p+q+1$ through odd integers only. If the density of the probability distribution of each x is $\exp(-|x|)/2$, $P(n;\ p,\ q)$ can be expressed in terms of the roots of the transcendental equation $\tan(p+q)y=-2y/1-y^2$. The method of solution depends on the use of the eigenvalue theory of matrices (discrete case) and integral equations (continuous case). (Received November 21, 1944.)

45. A. T. Lonseth: *A note on relative errors in systems of linear equations.*

A nonsingular system of linear equations is considered, the coefficients being subject to error. Under the assumption of uniformly bounded errors in the coefficients, a bound is found for the relative error in each solution-component. (This note is to appear soon in the Annals of Mathematical Statistics.) (Received October 19, 1944.)

46. A. T. Lonseth: *Error-limitation for the method of least squares.*

Let $y(t)$ be such that $\int_0^1 y^2dt=\|y\|^2$ exists. Let $x(t)$ satisfy $Tx=y$, where T is an additive, homogeneous bi-unique transformation possessing the bounded inverse T^{-1} with bound $M(T^{-1})$: that is, $\|T^{-1}y\|\leqq M(T^{-1})\|y\|$ for every y with finite $\|y\|$. Omitting boundary conditions, which enter if T is a differential operator, an estimate $x_n^0(t)$ is obtained as follows (method of least squares): given a set of n functions $\phi_k(t)$ such that $\|T\phi_k\|$ exists for $k=1, 2, \cdots, n$, values $a_1^0, a_2^0, \cdots, a_n^0$ of parameters $a_1, a_2, \cdots, a_n$ are determined so as to minimize $F(a_1,\ a_2, \cdots,\ a_n)=\int_0^1(Tx_n-y)^2dt$, where $x_n=\sum_{i=1}^{i=n}a_i\phi_i$. Then $x_n^0=\sum_{i=1}^{i=n}a_i^0\phi_i$. It is shown in this paper that the *global error* $\|x_n^0-x\|\leqq M(T^{-1})F(a_1^0, a_2^0, \cdots, a_n^0)$, where $F^2(a_1^0, a_2^0, \cdots, a_n^0)=\|y\|^2-\sum_{i=1}^{i=n}a_i^0\int_0^1 yT\phi_i dt$. (The integrals under summation occur in the linear equations for the minimizing parameters.) For linear integral equations of Fredholm type and second kind the *local error* $|x_n^0(t)-x(t)|$ is similarly bounded. If T is continuous, hence itself bounded, the *accuracy* of the approximation can be limited: $\|x_n^0-x\|\geqq M(T)^{-1}F(a_1^0, a_2^0, \cdots, a_n^0)$. (Received October 19, 1944.)

47. Henry Scheffé: *A note on the Behrens-Fisher problem.*

Let $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_n)$ be samples from two normal universes. The problem is the comparison of the means of the universes when the ratio of their variances is unknown. Solutions based on the t-distribution were studied in a previous paper (Annals of Mathematical Statistics vol. 14 (1943) pp. 35–44), and a very convenient one was singled out. This, however, did not have the desirable "symmetry" property, that is, invariance under permutations of the x's among themselves and of the y's among themselves. This note outlines a proof that there exists no "symmetric" solution based on the t-distribution. (Received October 26, 1944.)

TOPOLOGY

48. L. M. Blumenthal: *Generalized elliptic spaces and quadratic forms.*

Continuing his study of generalized elliptic spaces (abstract 49-11-304) the writer obtains quadratic form theorems concerning invariance of rank under sign changes of coefficients. Thus, for example, if a quadratic form in more than three variables, with the coefficients of the squared terms 1 and those of the products x_ix_j $(i<j)$ between -1 and 1, is positive definite of rank 2, then each positive definite form obtained by a change of sign of coefficients of terms x_ix_j is also of rank 2. The "natural" extension of this theorem to forms in more than four variables with rank 3 is not valid. By investigating the different kinds of equilateral sets contained in n-dimensional elliptic space, the writer shows that these spaces have neither congruence order $n+3$ nor $n+4$ (except for $n=1$). Since, for example, the ordinary elliptic plane contains an equilateral 6-point it failed to have congruence order 6. (Received October 23, 1944.)

49. R. H. Fox: *An application of the complete homotopy group.*

The fundamental group $\tau_n(Y)=\pi_1(Y^T)$ of the space of continuous mappings of the $(n-1)$-dimensional torus $T=T_{n-1}$ into a topological space Y was introduced in a previous communication (abstract 49-11-306). J. H. C. Whitehead has proved two theorems (Proc. London Math. Soc. (2) vol. 45 (1939) p. 281 and Ann. of Math. vol. 42 (1941) p. 418) about homotopy groups. When these are combined and restated in terms of the groups τ_n the result is as follows: Let K^* be a complex and let K be a complex obtained from K^* by removing the interior $\sigma-\dot{\sigma}$ of a principal n-dimensional simplex σ, where $n>2$. The nucleus of the injection homomorphism $\tau_n(K)\to\tau_n(K^*)$ is precisely the invariant subgroup of $\tau_n(K)$ which is generated by the image of the injection homomorphism $\tau_n(\dot{\sigma})\to\tau_n(K)$. This reformulation seems to have more intuitive content and suggests an attack on the harder problem where dim $\sigma>n$. (Received October 28, 1944.)

50. R. H. Fox: *On topologies for function spaces.*

To be useful a topology for the set F of continuous functions from X to Y should have one or more of the following properties: (1) A function from $X\times T$ to Y is continuous if and only if the corresponding function from T to F is continuous. (2) The sectioning operation to or from F is continuous. (3) The function $\phi(x, f)=f(x)$ from $X\times F$ to Y is continuous. For any compact set $A\subset X$ and open set $W\subset Y$ let $M(A, W)$ denote the set of continuous functions $f\in F$ such that $f(A)\subset W$. The topology determined by the subbasis $\{M(A, W)\}$ satisfies (1) and (2) if $X\times T$ is separable Hausdorff. If X is normal but not locally compact and Y is an arc, no topology satisfies both (1) and (3). (Received October 21, 1944.)

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EVALUATION OF VARIOUS WIENER INTEGRALS BY USE OF CERTAIN STURM-LIOUVILLE DIFFERENTIAL EQUATIONS

R. H. CAMERON AND W. T. MARTIN

Introduction. Wiener [1][1] has introduced a measure in the space C consisting of all real-valued functions $x(t)$ continuous on $0 \leqq t \leqq 1$ and vanishing at $t=0$. He first defines the measure of a quasi-interval—a quasi-interval is the set of all functions of C which satisfy relations of the form

$$a_j < x(t_j) < b_j, \qquad j = 1, \cdots, n,$$

where n is a positive integer, a_j and b_j are real numbers, $-\infty \leqq a_j < b_j \leqq \infty$, and $t_1, \cdots, t_n$ are n ordered points on $0 < t \leqq 1$, $0 < t_1 < \cdots < t_n \leqq 1$. The Wiener measure of this quasi-interval is defined to be

$$\frac{1}{(\pi^n t_1(t_2 - t_1) \cdots (t_n - t_{n-1}))^{1/2}} \int_{a_1}^{b_1} \cdots$$

$$\cdot \int_{a_n}^{b_n} \exp\left(-\frac{s_1^2}{t_1} - \frac{(s_2 - s_1)^2}{t_2 - t_1} - \cdots - \frac{(s_n - s_{n-1})^2}{t_n - t_{n-1}}\right) ds_1 \cdots ds_n,$$

the multiplier before the integrals having been so chosen as to make the measure of the entire space C equal to unity. With this definition of measure for quasi-intervals, the Wiener measure on C can be defined in a manner entirely similar to that used for ordinary Lebesgue measure, and a theory of (Wiener) integration can be defined over C, the integral of a functional $F[x]$ being a number. We write

$$\int_S^W F[x] d_w x \tag{0.1}$$

to denote the Wiener integral of a functional $F[x]$ over a subset S of the space C.

In several papers the authors have recently considered various aspects of the theory; in the present paper we shall evaluate several Wiener integrals. In particular, we shall evaluate integrals of the form

$$\int_C^W \exp\left(\lambda \int_0^1 p(t)[x(t) + g(t)]^2 dt\right) d_w x \tag{0.2}$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

or, more generally,

$$(0.3) \qquad \int_C^W \exp\left(\lambda \int_0^1 p(t)x^2(t)dt\right) F\left[\int_0^1 g(t)x(t)dt\right] d_w x,$$

where $p(t)$ is a positive-valued continuous function on $0 \leqq t \leqq 1$, $g(t)$ is real and of class L_2 on $0 \leqq t \leqq 1$, and λ is real and less than a suitable positive constant λ_0 which depends only upon the function $p(t)$. In (0.3) the function $F(u)$ will be a fairly general Lebesgue measurable function on $-\infty < u < \infty$.

In a recent paper [2] we have shown that

$$(0.4) \qquad \int_C^W \exp\left(\lambda \int_0^1 x^2(t)dt\right) d_w x = (\sec \lambda^{1/2})^{1/2}, \qquad 0 \leqq \lambda < \pi^2/4.$$

This integral was evaluated by making use of a suitable linear transformation of C into itself; namely, the transformation

$$(0.5) \qquad y(t) = x(t) + \lambda^{1/2} \int_0^t \tan \lambda^{1/2}(s-1) \cdot x(s) ds.$$

Under this transformation we found that

$$(0.6) \quad 1 = \int_C^W 1 \cdot d_w y = (\cos \lambda^{1/2})^{1/2} \int_C^W \exp\left(\lambda \int_0^1 x^2(t)dt\right) d_w x, \qquad 0 \leqq \lambda < \pi^2/4,$$

and from this (0.4) was obtained. The evaluation of integrals of the form

$$(0.7) \qquad \int_C^W \exp\left(\lambda \int_0^1 p(t)x^2(t)dt\right) d_w x, \qquad -\infty < \lambda < \lambda_0,$$

can be carried out in a similar fashion using suitable linear transformations; the particular linear transformation to be used in each case will be determined by a consideration of the second order differential equation

$$(0.8) \qquad f''(t) + \lambda p(t) f(t) = 0,$$

and the value λ_0 will be the least characteristic value of the equation (0.8) corresponding to the boundary values $f(0) = f'(1) = 0$.

The evaluation of integrals of the form (0.3) will use the same range of ideas, together with certain results recently obtained. Integrals of the form (0.2) are special cases of (0.3).

The evaluation of (0.7) is given by the following theorem.

THEOREM 1. *Let $p(t)$ be continuous and positive* (>0) *on* $0\leqq t\leqq 1$ *and let* λ_0 *be the least characteristic value of* (0.8) *subject to the boundary conditions*

$$f(0) = f'(1) = 0. \tag{0.9}$$

Then if $-\infty<\lambda<\lambda_0$ *and* $f_\lambda(t)$ *is any non-trivial solution of* (0.8) *satisfying* $f_\lambda'(1)=0$, *we have*

$$\int_C^W \exp\left(\lambda\int_0^1 p(t)x^2(t)dt\right)d_wx = \left(\frac{f_\lambda(1)}{f_\lambda(0)}\right)^{1/2}. \tag{0.10}$$

Consequently, if $\lambda<\lambda_0$ *and* $f^{(1)}(t)$ *and* $f^{(2)}(t)$ *are any two linearly independent solutions of* (0.8), *we have*

$$\begin{aligned}\int_C^W &\exp\left(\lambda\int_0^1 p(t)x^2(t)dt\right)d_wx\\ &= \left(\frac{f^{(2)\prime}(a)f^{(1)}(a)-f^{(1)\prime}(a)f^{(2)}(a)}{f^{(2)\prime}(1)f^{(1)}(0)-f^{(1)\prime}(1)f^{(2)}(0)}\right)^{1/2},\end{aligned} \tag{0.11}$$

where a is any convenient point in $0\leqq a\leqq 1$.

Remark 1. By the hypotheses of Theorem 1, λ_0 must exist as a positive number.

Remark 2. For every value of λ there always exist (non-trivial) solutions of (0.8) satisfying $f_\lambda'(1)=0$.

Remark 3. The numerator inside the square root sign in (0.11) is a Wronskian whose value is actually independent of a, $0\leqq a\leqq 1$.

Theorem 1 is a special case of Theorem 1a, which we shall prove in §1.

THEOREM 1a. *Let $p(t)$ be continuous and positive on* $0\leqq t\leqq 1$, *and let* λ_0 *be the least characteristic value of the differential equation* (0.8) *subject to the boundary conditions* (0.9). *Then if $F(x)$ is any Wiener measurable functional, if* $\lambda<\lambda_0$, *and if* $f_\lambda(t)$ *is any non-trivial solution of* (0.8) *satisfying* $f_\lambda'(1)=0$, *we have*

$$\begin{aligned}\int_C^W &F(x)\exp\left(\lambda\int_0^1 p(t)x^2(t)dt\right)d_wx\\ &= \left(\frac{f_\lambda(1)}{f_\lambda(0)}\right)^{1/2}\int_C^W F\left[y(\cdot)+f_\lambda(\cdot)\int_0^{(\cdot)}\frac{f_\lambda'(s)}{[f_\lambda(s)]^2}y(s)ds\right]d_wy,\end{aligned} \tag{0.12}$$

where the existence of either Wiener integral implies the existence of the other.

Remark 4. The radical on the right-hand side of (0.12) can be replaced by the radical on the right-hand side of (0.11).

By using Theorem 1a and certain results proved recently we shall obtained in §2 the following result.

THEOREM 2. *Let* $g(t)$ *be real and of class* L_2 *on* $0 \leqq t \leqq 1$, *let* $p(t)$ *be a positive-valued continuous function on* $0 \leqq t \leqq 1$, *let* λ_0 *be the least characteristic value of the system* (0.8), (0.9) *and let* $f_\lambda(t)$ *be any non-trivial solution of* (0.8) *satisfying* $f_\lambda'(1) = 0$. *Then*

$$\begin{aligned}(0.13)\qquad &\int_C^W \exp\left(\lambda \int_0^1 p(t)x^2(t)dt\right) F\left[\int_0^1 g(t)x(t)dt\right] d_w x \\ &\qquad = \left(\frac{f_\lambda(1)}{\pi \cdot f_\lambda(0)}\right)^{1/2} \int_{-\infty}^{\infty} F(bu)e^{-u^2}du, \qquad \lambda < \lambda_0,\end{aligned}$$

where

$$(0.14)\qquad b^2 = \int_0^1 \left[\frac{1}{f_\lambda(t)} \int_1^t g(s)f_\lambda(s)ds\right]^2 dt,$$

the result holding and the Wiener integral being convergent whenever

$$(0.15)\qquad F(bu)e^{-u^2} \in L_1(-\infty, \infty).$$

In the final sections we consider a few choices of $p(t)$ for which the differential equation (0.8) has solutions in terms of well known functions. Various other choices may occur to the reader.

1. Proofs of Theorems 1a and 1. We shall need certain well known properties of the differential equation (0.8). First, by classical Sturm-Liouville theory it is known that there is a least characteristic value λ_0 of the system [(0.8), (0.9)], it is known that λ_0 is positive, and it is known that the solution $f(t)$ of the system [(0.8), (0.9)] with $\lambda = \lambda_0$ is nonvanishing in $0 < t \leqq 1$. These facts are all given for example in [3]. We need one other (known) property; namely, the following:

RESULT 1. *Every non-trivial solution* $f_\lambda(t)$ *of* (0.8) *corresponding to any real value of* $\lambda < \lambda_0$ *and satisfying the single boundary condition*

$$(1.1)\qquad f_\lambda'(1) = 0$$

is nonvanishing in $0 \leqq t \leqq 1$.

For the sake of completeness we shall give a classical proof of this result. For the proof we use the identity

$$(1.2)\qquad [f'f_\lambda - ff_\lambda']_t^1 = (\lambda - \lambda_0)\int_t^1 p(s)f(s)f_\lambda(s)ds,$$

where $0 \leqq t \leqq 1$, $\lambda < \lambda_0$, $f_\lambda(t)$ is a non-trivial solution of (0.8) satisfying (1.1), and $f(t)$ is a solution of the system [(0.8), (0.9)] with $\lambda = \lambda_0$. Now $f_\lambda(t)$ cannot vanish at $t=0$ since λ_0 is the least characteristic value of the system [(0.8), (0.9)]. Also it cannot vanish at $t=1$ for if it did it would be identically zero. If it vanishes at all in $0 \leqq t \leqq 1$ its zeros are isolated and we shall denote by t_0 its greatest zero in (0, 1). With this value of $t_0 (<1)$ relation (1.2) becomes

$$(1.3) \qquad f(t_0) f_\lambda'(t_0) = (\lambda - \lambda_0) \int_{t_0}^{1} p(s) f(s) f_\lambda(s) ds.$$

Without losing generality we assume

$$(1.4) \qquad f_\lambda(t) > 0, \qquad t_0 < t \leqq 1,$$

$$(1.5) \qquad f(t) > 0, \qquad 0 < t \leqq 1.$$

Then (1.3) yields the result that $f_\lambda'(t_0)$ is negative. Since $f_\lambda(t_0)$ is assumed zero this contradicts (1.4). Hence there is no greatest zero of $f_\lambda(t)$ in $0 < t \leqq 1$. This yields Result 1.

We now proceed to the proof of Theorem 1a. For this purpose we consider the linear transformation

$$(1.6) \qquad y(t) = x(t) - \int_0^t \frac{f_\lambda'(s)}{f_\lambda(s)} x(s) ds,$$

where λ is fixed, $-\infty < \lambda < \lambda_0$, and $f_\lambda(t)$ is any non-trivial solution of (0.8) satisfying (1.1). (By Result 1, the denominator of the integral in (1.6) does not vanish.) Obviously the transformation (1.6) takes the space C into a part of C. We shall show that it actually takes C into the whole of C in a 1-1 manner. First

$$(1.7) \qquad \begin{aligned} \frac{f_\lambda'(t)}{[f_\lambda(t)]^2} y(t) &= \frac{f_\lambda'(t)}{[f_\lambda(t)]^2} x(t) - \frac{f_\lambda'(t)}{[f_\lambda(t)]^2} \int_0^t \frac{f_\lambda'(s)}{f_\lambda(s)} x(s) ds \\ &= -\frac{d}{dt}\left[\frac{1}{f_\lambda(t)} \int_0^t \frac{f_\lambda'(s)}{f_\lambda(s)} x(s) ds\right] \end{aligned}$$

so that

$$(1.8) \qquad \int_0^t \frac{f_\lambda'(s)}{[f_\lambda(s)]^2} y(s) ds = -\frac{1}{f_\lambda(t)} \int_0^t \frac{f_\lambda'(s)}{f_\lambda(s)} x(s) ds.$$

Multiplying (1.8) by $f_\lambda(t)$ and adding to (1.6), we obtain

$$(1.9) \qquad y(t) + f_\lambda(t) \int_0^t \frac{f_\lambda'(s)}{[f_\lambda(s)]^2} y(s) ds = x(t).$$

Thus to every function $x(\cdot)$ of C there corresponds a function $y(\cdot)$ of C by (1.6), and this $y(\cdot)$ satisfies the relation (1.9). Conversely, it is easily seen that if $y(\cdot)$ is any function of C and if $x(\cdot)$ is defined by (1.9), then $x(\cdot)$ and $y(\cdot)$ again satisfy (1.6). Hence (1.6) is a 1-1 transformation of C onto itself. This could also be seen from the fact that (1.6) is a Volterra transformation. We shall consider it, however, as a Fredholm transformation whose kernel vanishes when $t<s$, and calculate the Fredholm determinant using the half (or arithmetic mean) value along the diagonal:

$$\begin{aligned} D &= \exp\left(\frac{1}{2}\int_0^1 \left\{-\frac{f_\lambda'(s)}{f_\lambda(s)}\right\} ds\right) \\ &= \exp\left(-\frac{1}{2}\left[\log f_\lambda(1) - \log f_\lambda(0)\right]\right) = \left(\frac{f_\lambda(0)}{f_\lambda(1)}\right)^{1/2}. \end{aligned} \tag{1.10}$$

In Theorem I of a recent paper [2] we have considered the behavior of Wiener integrals under Fredholm and Volterra transformations,

$$y(t) = x(t) + \int_0^1 K(t, s)x(s)ds,$$

where

$$K(t, s) = \begin{cases} K^{(1)}(t, s) & \text{when} \quad 0 \leqq t < s,\ 0 < s \leqq 1, \\ K^{(2)}(t, s) & \text{when} \quad s < t \leqq 1,\ 0 \leqq s < 1, \\ K^{(1)}(s, s)/2 + K^{(2)}(s, s)/2 & \text{when} \quad t = s, \qquad 0 \leqq s \leqq 1. \end{cases}$$

We quote for reference the special case of this theorem in which $K^{(1)}(t, s) \equiv 0$, so that the transformation is a pure Volterra transformation:

THEOREM A. *Let $K(t, s)$ be continuous on the closed triangle $[0 \leqq s \leqq t, 0 \leqq t \leqq 1]$, and let it satisfy the following five conditions*:

A. *For almost all s, $K(t, s)$ is absolutely continuous in t, $s \leqq t \leqq 1$.*

B. *For almost all s, $\partial K(t, s)/\partial t$ is essentially of bounded variation, $s \leqq t \leqq 1$. More precisely, there exists a measurable function $\tilde{K}(t, s)$ which is of bounded variation in t for each s and which for almost all t, s in the triangle $[s \leqq t \leqq 1, 0 \leqq s \leqq 1]$ is equal to $\partial K(t, s)/\partial t$. (In the remaining statements, the hypotheses will be considered satisfied when $\partial K/\partial t$ is replaced by such a $\tilde{K}$.)*

C. $\int_0^1 \sup_{s \leqq t \leqq 1} |\partial K(t, s)/\partial t| ds < \infty$.

D. $\int_0^1 \operatorname{var}_{s \leqq t \leqq 1} [\partial K(t, s)/\partial t] ds < \infty$.

E. *$K(s, s)$ is of bounded variation $0 \leqq s \leqq 1$.*

Let S be a Wiener measurable subset of C, and let TS be the image of S under the transformation

$$\text{T:} \qquad y(t) = x(t) + \int_0^t K(t, s)x(s)ds.$$

Then if $\mathcal{J}(y)$ *is any Wiener measurable functional for which either member of the following equation exists, the other member exists and the equality holds:*

$$\int_{TS}^{W} \mathcal{J}[y]d_w y = \mathcal{D}\cdot \int_{S}^{W} \mathcal{J}\left[x(\cdot) + \int_0^{(\cdot)} K(\cdot, s)x(s)ds\right]\exp(-\Phi[x])d_w x;$$

where

$$\begin{aligned}\Phi[x] = &\int_0^1 \left[\frac{d}{dt}\int_0^t K(t, s)x(s)ds\right]^2 dt \\ &+ 2\int_0^1 \left[\int_0^t \frac{\partial}{\partial t} K(t, s)x(s)ds\right] dx(t) \\ &+ \int_0^1 K(t, t)d\{[x(t)]^2\}\end{aligned}$$

and $\mathcal{D}$ *is the Fredholm determinant of the transformation, using the half (or arithmetic mean) value of the kernel on the diagonal:*

$$\begin{aligned}\mathcal{D} &= \sum_{n=0}^{\infty} \frac{1}{n!}\int_0^1 \cdots \int_0^1 \frac{K(s_1, s_1)}{2}\,\frac{K(s_2, s_2)}{2} \cdots \frac{K(s_n, s_n)}{2} ds_1 ds_2 \cdots ds_n \\ &= \exp\left(\frac{1}{2}\int_0^1 K(s, s)ds\right).\end{aligned}$$

It is clear that the linear transformation (1.6) for $\lambda < \lambda_0$ satisfies the hypotheses of Theorem A, and hence by that theorem with $\mathcal{J}[y] = F[x] = F[T^{-1}(y)]$, we have

$$\begin{aligned}&\int_C^W F\left[y(\cdot) + f_\lambda(\cdot)\int_0^{(\cdot)} \frac{f_\lambda'(s)}{\{f_\lambda(s)\}^2} y(s)ds\right] d_w y \\ (1.11) \qquad &= D\int_C^W F[x] \exp\left\{-\int_0^1 \left[\frac{f_\lambda'(s)}{f_\lambda(s)} x(s)\right]^2 ds \right. \\ &\qquad \left. + \int_0^1 \frac{f_\lambda'(s)}{f_\lambda(s)} d[x(s)]^2\right\} d_w x,\end{aligned}$$

where the existence of either side implies the existence of the other.

On integrating by parts, simplifying, using the boundary condition (1.1) and the differential equation (0.8), we obtain that (1.11) is equal to

$$
\begin{aligned}
&D\int_C^W F[x]\exp\left\{-\int_0^1\left[\frac{f_\lambda'(s)}{f_\lambda(s)}x(s)\right]^2 ds-\int_0^1 x^2(s)d\left[\frac{f_\lambda'(s)}{f_\lambda(s)}\right]\right\}d_wx\\
&\quad=D\int_C^W F[x]\exp\left\{-\int_0^1\frac{[f_\lambda'(s)]^2+f_\lambda(s)f_\lambda''(s)-[f_\lambda'(s)]^2}{[f_\lambda(s)]^2}x^2(s)ds\right\}d_wx\\
(1.12)\qquad&\quad=D\int_C^W F[x]\exp\left\{-\int_0^1\frac{f_\lambda''(s)}{f_\lambda(s)}x^2(s)ds\right\}d_wx\\
&\quad=D\int_C^W F[x]\exp\left(\lambda\int_0^1 p(s)x^2(s)ds\right)d_wx.
\end{aligned}
$$

This, together with (1.10), yields Theorem 1a, including the existence of the integrals. Relation (0.10) 1 is obtained by specializing the function F in Theorem 1 to be identically unity, $F(u)\equiv 1$.

Now take $f_\lambda(t)=f^{(2)\prime}(1)f^{(1)}(t)-f^{(1)\prime}(1)f^{(2)}(t)$ where $f^{(1)}(t)$ and $f^{(2)}(t)$ are any two linearly independent solutions of (0.8). This function obviously satisfies the condition $f_\lambda'(1)=0$ and yields (0.11) with $a=1$. But since the differential equation (0.8) has no term in $f'(t)$, its Wronskian is constant, and hence we may use any convenient point a instead of $a=1$. This yields Theorem 1.

2. **Proof of Theorem 2.** In a recent paper [4, Lemma 3] we gave a new proof of a special case of a theorem of Paley and Wiener [5]. We proved:[2]

Let $m(t)$ be real and of bounded variation on $0\leqq t\leqq 1$, and let $m(1)=0$. Let

$$(2.1)\qquad b^2=\int_0^1 m^2(t)dt$$

and let $F(u)$ be a (real or complex-valued) measurable functional defined on $-\infty<u<\infty$. Then a necessary and sufficient condition that

$$(2.2)\qquad F\left[\int_0^1 x(t)dm(t)\right]$$

be a Wiener summable functional of $x(\cdot)$ over C is that

$$(2.3)\qquad e^{-u^2}F(bu)$$

be of class L_1 on $-\infty<u<\infty$. Moreover, if this condition is satisfied,

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[2] *Added in proof.* In a letter to one of the authors, Professor J. L. Doob recently pointed out that this result, for an arbitrary function $m(t)$ of L_2, follows at once from probability considerations. The probability argument uses the known fact that $\int_0^1 m(t)dx(t)$ is a Gaussian variable with mean zero and dispersion $b^2/2$.

$$(2.4)\qquad \int_C^W F\left[\int_0^1 x(t)dm(t)\right]d_wx = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} e^{-u^2}F(bu)du.$$

On integrating by parts and using the fact that $m(1)=0$ we find from (2.4) that

$$(2.5)\qquad \int_C^W F\left[\int_0^1 m(t)dx(t)\right]d_wx = \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} e^{-u^2}F(bu)du$$

holds under the same conditions.

We pass now to the proof of Theorem 2 itself. We first consider the integral which occurs in the left-hand member of (0.13); namely,

$$(2.6)\qquad \int_C^W \exp\left(\lambda\int_0^1 p(t)x^2(t)dt\right)F\left[\int_0^1 g(t)x(t)dt\right]d_wx,$$

where $g(t)$ and $p(t)$ are real, $g(t)$ of class $L_2(0, 1)$ and $p(t)$ positive and continuous, $0\leqq t\leqq 1$. On applying relation (0.12) of Theorem 1a for $\lambda<\lambda_0$ we find that (2.6) is equal to

$$(2.7)\qquad \left(\frac{f_\lambda(1)}{f_\lambda(0)}\right)^{1/2}\int_C^W F\left[\int_0^1 g(t)y(t)dt + \int_0^1 g(t)f_\lambda(t)\left\{\int_0^t \frac{f_\lambda'(s)}{\{f_\lambda(s)\}^2}y(s)ds\right\}dt\right]d_wy,$$

the equality holding for every measurable function F for which either (2.6) or (2.7) exists. We now consider the argument of F in (2.7) and integrate twice by parts; we obtain

$$(2.8)\qquad \begin{aligned} &\int_0^1 g(t)y(t)dt + \int_0^1 g(t)f_\lambda(t)\left\{\int_0^t \frac{f_\lambda'(s)}{\{f_\lambda(s)\}^2}y(s)ds\right\}dt \\ &= \int_0^1 g(t)y(t)dt \\ &\qquad + \int_0^1 g(t)f_\lambda(t)\left\{-\left[\frac{y(s)}{f_\lambda(s)}\right]_0^t + \int_0^t \frac{dy(s)}{f_\lambda(s)}\right\}dt \\ &= \int_0^1 g(t)y(t)dt - \int_0^1 g(t)f_\lambda(t)\cdot\frac{y(t)}{f_\lambda(t)}dt \\ &\qquad + \int_0^1 g(t)f_\lambda(t)\left\{\int_0^t \frac{dy(s)}{f_\lambda(s)}\right\}dt \\ &= \int_0^1 g(t)f_\lambda(t)\left\{\int_0^t \frac{dy(s)}{f_\lambda(s)}\right\}dt \\ &= \left[G(t, \lambda)\int_0^t \frac{dy(s)}{f_\lambda(s)}\right]_0^1 - \int_0^1 \frac{G(t, \lambda)}{f_\lambda(t)}dy(t), \end{aligned}$$

where

$$G(t,\lambda)=\int_1^t g(s)f_\lambda(s)ds. \tag{2.9}$$

Now $G(t,\lambda)$ vanishes at $t=1$, and $\int_0^t dy(s)/f_\lambda(s)$ vanishes at $t=0$. Hence (2.8) becomes

$$-\int_0^1 \frac{G(t,\lambda)}{f_\lambda(t)}dy(t) \tag{2.10}$$

and hence the quantities (2.6) and (2.7) are equal to

$$\left(\frac{f_\lambda(1)}{f_\lambda(0)}\right)^{1/2}\int_C^W F\left[-\int_0^1 \frac{G(t,\lambda)}{f_\lambda(t)}dy(t)\right]d_w y \tag{2.11}$$

provided any one of the (Wiener) integrals in (2.6), (2.7), or (2.11) exists. Now the function $G(t,\lambda)/f_\lambda(t)$ is of bounded variation on $0\leqq t\leqq 1$, and it vanishes at $t=1$. Hence the result (2.5) applies to (2.11), yielding the result that the integral in (2.11) is equal to

$$\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} F(bu)\cdot e^{-u^2}du, \tag{2.12}$$

where

$$b^2=\int_0^1\left[\frac{G(t,\lambda)}{f_\lambda(t)}\right]^2 dt=\int_0^1\left[\frac{1}{f_\lambda(t)}\int_1^t g(s)f_\lambda(s)ds\right]^2 dt. \tag{2.13}$$

This yields Theorem 2, including the summability of the integral.

3. A corollary to Theorem 2. On specializing the function $F(u)$ of Theorem 2 to be

$$F(u)=e^{2\lambda u} \tag{3.1}$$

and replacing $g(t)$ by $p(t)g(t)$, we obtain from Theorem 2 the relation

$$\begin{aligned}&\int_C^W \exp\left(\lambda\int_0^1 [p(t)x^2(t)+2p(t)g(t)x(t)]dt\right)d_w x\\ &\quad=\left(\frac{f_\lambda(1)}{f_\lambda(0)}\right)^{1/2}\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} e^{2\lambda\beta u}e^{-u^2}du=\left(\frac{f_\lambda(1)}{f_\lambda(0)}\right)^{1/2}e^{\lambda^2\beta^2},\end{aligned} \tag{3.2}$$

where

$$\beta^2=\int_0^1\left[\frac{1}{f_\lambda(t)}\int_1^t p(s)g(s)f_\lambda(s)ds\right]^2 dt. \tag{3.3}$$

On multiplying the first and third members of (3.2) by $\exp(\lambda\int_0^1 p(t)g^2(t)dt)$ we obtain the following corollary.

Corollary 1.

$$\int_C^W \exp\left(\lambda \int_0^1 p(t)\{x(t)+g(t)\}^2 dt\right) d_w x = \left(\frac{f_\lambda(1)}{f_\lambda(0)}\right)^{1/2} \exp\left(\lambda^2\beta^2 + \lambda\int_0^1 p(t)g^2(t)dt\right), \tag{3.4}$$

the equality holding for $\lambda<\lambda_0$ *and for every real-valued function* $g(t)$ *belonging to* L_2 *on* $0<t<1$. *The constant* β *in* (3.4) *is defined in* (3.3).

This evaluates the expression (0.2) mentioned in the introduction. In the remaining sections we consider briefly a few examples.

4. **Example 1,** $p(t)\equiv 1$. For $p(t)\equiv 1$, the differential equation (0.8) becomes

$$f''(t) + \lambda f(t) = 0 \tag{4.1}$$

and the least characteristic value is

$$\lambda_0 = \pi^2/4. \tag{4.2}$$

A non-trivial solution of (4.1) satisfying $f_\lambda'(1)=0$ is

$$f_\lambda(t) = \cos\lambda^{1/2}(t-1). \tag{4.3}$$

Relation (0.10) of Theorem 1 yields

$$\int_C^W \exp\left(\lambda\int_0^1 x^2(t)dt\right) = \left(\frac{1}{\cos\lambda^{1/2}}\right)^{1/2}, \quad -\infty<\lambda<\pi^2/4; \tag{4.4}$$

which agrees with result (0.4) stated in the introduction.

For use elsewhere we state explicitly Corollary 1 of §4 for $p(t)\equiv 1$.

Corollary 2. *Let* $g(t)$ *be a real-valued function belonging to* L_2 *on* $0<t<1$, *and let* $-\infty<\lambda<\pi^2/4$. *Then*

$$\begin{aligned}\int_C^W \exp\left(\lambda\int_0^1 [x(t)+g(t)]^2 dt\right) d_w x &= (\sec\lambda^{1/2})^{1/2}\exp\left(\lambda^2\int_0^1 [G_0(t,\lambda)\sec\lambda^{1/2}(t-1)]^2 dt\right)\\ &\quad\cdot\exp\left(\lambda\int_0^1 g^2(t)dt\right),\end{aligned} \tag{4.5}$$

where

$$G_0(t, \lambda) = \int_1^t g(s) \cos \lambda^{1/2}(s-1) ds.$$

5. **Example 2,** $p(t)=(t+\alpha)^{-2}$, $0<\alpha<\infty$. With this value of $p(t)$ the differential equation (0.8) becomes

$$f''(t) + (\lambda/(t+\alpha)^2)f(t) = 0 \tag{5.1}$$

and if we write

$$\lambda = 1/4 - \mu^2, \qquad 0 < \mu < \infty, \tag{5.2}$$

then two linearly independent solutions are

$$(t+\alpha)^{\mu+1/2} \tag{5.3}$$

and

$$(t+\alpha)^{-\mu+1/2}. \tag{5.4}$$

Also it is easily seen that $\lambda_0 \geqq 1/4$ since no linear combination of (5.3) and (5.4) satisfies the boundary condition

$$f(0) = f'(1) = 0. \tag{5.5}$$

Hence relation (0.11) of Theorem 1 yields

$$\begin{aligned} &\int_C^W \exp\left((1/4-\mu^2)\int_0^1 \frac{x^2(t)}{(t+\alpha)^2}dt\right)d_wx \\ &\qquad = \left(\frac{2\mu\alpha^{\mu-1/2}(1+\alpha)^{\mu+1/2}}{(\mu-1/2)\alpha^{2\mu}+(\mu+1/2)(1+\alpha)^{2\mu}}\right)^{1/2}, \\ &\qquad\qquad 0<\mu<\infty,\ 0<\alpha<\infty. \end{aligned} \tag{5.6}$$

6. **Example 3,** $p(t)=e^{\alpha t}$, α real, $\alpha\neq 0$. With this choice of $p(t)$ the differential equation (0.8) becomes

$$f''(t) + \lambda e^{\alpha t} f(t) = 0. \tag{6.1}$$

We solve (6.1) by use of a series with undetermined coefficients—let

$$f(t) = \sum_{n=0}^{\infty} a_n e^{n\alpha t}. \tag{6.2}$$

Then (formally)

$$f''(t) = \sum_{n=0}^{\infty} a_n \cdot n^2\alpha^2 e^{n\alpha t}$$

and (6.1) leads to

$$\sum_{n=0}^{\infty} a_n n^2 \alpha^2 e^{n\alpha t} = -\lambda \sum_{n=0}^{\infty} a_n e^{(n+1)\alpha t} = -\lambda \sum_{n=1}^{\infty} a_{n-1} e^{n\alpha t}.$$

On equating coefficients we find

$$a_n n^2 \alpha^2 = -\lambda a_{n-1}, \qquad n = 1, 2, 3, \cdots,$$

or

$$a_n = -\lambda \frac{a_{n-1}}{n^2\alpha^2} = \frac{(-\lambda)^n}{(n!)^2 \alpha^{2n}} a_0. \tag{6.3}$$

Choosing $a_0 = 1$ we find

$$f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{\alpha^{2n}} \frac{e^{n\alpha t}}{(n!)^2} \tag{6.4}$$

and it is easily seen that this is an actual solution, not just a formal one. Also the series (6.4) is the classical Bessel function of order zero,

$$J_0(2\lambda^{1/2} e^{\alpha t/2}/\alpha). \tag{6.5}$$

The function (6.5) is a solution of (6.1) for every value of λ, $-\infty < \lambda < \infty$. We note that $J_0(z)$ is an even (entire) function and hence there is no difficulty in (6.5) when λ is negative.

A second linearly independent solution of (6.1) for λ real and not equal to 0 is given by

$$f^{(2)}(t) = Y_0(2\lambda^{1/2} e^{\alpha t/2}/\alpha), \tag{6.6}$$

where $Y_0(z)$ is the Bessel function of the second kind of order zero, as introduced by Weber and Schläfli. The general Bessel function of the second kind is defined by the equation

$$Y_n(z) = \frac{J_n(z) \cos n\pi - J_{-n}(z)}{\sin n\pi} \tag{6.7}$$

or by the limit of this expression when n is an integer. See [6, pp. 370, 371.] Denoting (6.5) by $f^{(1)}(t)$ we next calculate the expression

$$f^{(2)\prime}(a) f^{(1)}(a) - f^{(1)\prime}(a) f^{(2)}(a), \tag{6.8}$$

which occurs in relation (0.11) of Theorem 1. It is easily seen that (6.8) is equal to

$$\begin{aligned} &\lambda^{1/2} e^{\alpha a/2} \cdot \text{Wronskian } [J_0(z), Y_0(z)]_{z=2\lambda^{1/2} e^{\alpha a/2}/\alpha} \\ &\qquad = \lambda^{1/2} e^{\alpha a/2} [J_0(z) Y_0'(z) - J_0'(z) Y_0(z)]_{z=2\lambda^{1/2} e^{\alpha a/2}/\alpha}. \end{aligned} \tag{6.9}$$

By (6.7), for n not an integer, we have

$$\begin{aligned}\text{Wronskian }(J_n(z), Y_n(z)) &= J_n(z)Y_n'(z) - J_n'(z)Y_n(z)\\ &= \frac{1}{\sin n\pi}\left[J_n(z)J_n'(z)\cos n\pi - J_n(z)J_{-n}'(z)\right.\\ &\qquad\left.- J_n'(z)J_n(z)\cos n\pi + J_n(z)J_{-n}'(z)\right]\\ &= \frac{1}{\sin n\pi}\left[J_n'(z)J_{-n}(z) - J_n(z)J_{-n}'(z)\right].\end{aligned}\tag{6.10}$$

Now it is known that

$$J_n(z)J_{-n}'(z) - J_{-n}(z)J_n'(z) = -2\sin n\pi/\pi z,\tag{6.11}$$

see [6, pp. 360, 380]. Hence (6.10) becomes

$$\text{Wronskian }(J_n(z), Y_n(z)) = \frac{1}{\sin n\pi}\,\frac{2\sin n\pi}{\pi z} = \frac{2}{\pi z},\tag{6.12}$$

this holding whenever n is not an integer. On taking limits one can easily show that

$$\text{Wronskian }(J_n(z), Y_n(z)) = 2/\pi z\tag{6.13}$$

still prevails when n is an integer. Hence (6.8) and (6.9) are equal to

$$\lambda^{1/2}e^{\alpha a/2}\frac{2}{\pi}\,\frac{\alpha}{2\lambda^{1/2}e^{\alpha a/2}} = \frac{\alpha}{\pi}.\tag{6.14}$$

On using this in relation (0.11) of Theorem 1 we find

$$\begin{aligned}&\int_C^W \exp\left(\lambda\int_0^1 x^2(t)e^{\alpha t}dt\right)d_wx\\ &\quad= \frac{\alpha^{1/2}}{\pi^{1/2}\lambda^{1/4}e^{\alpha/4}}\cdot\left[Y_0'\left(\frac{2\lambda^{1/2}e^{\alpha/2}}{\alpha}\right)J_0\left(\frac{2\lambda^{1/2}}{\alpha}\right)\right.\\ &\qquad\left.- J_0'\left(\frac{2\lambda^{1/2}e^{\alpha/2}}{\alpha}\right)Y_0\left(\frac{2\lambda^{1/2}}{\alpha}\right)\right]^{-1/2},\end{aligned}\tag{6.15}$$

this holding for α real and different from zero, and for $-\infty<\lambda<0$ or $0<\lambda<\lambda_0$, where λ_0 is the least characteristic value of the differential equation (6.1) with the boundary conditions $f(0)=f'(1)=0$.

7. **Example 4,** $p(t)=(t+\alpha)^\beta$, $0<\alpha<\infty$, $\beta\neq-2$. In this example we let β range over all real values other than the value -2. This value has already been handled in Example 2 given in §5. With this choice of $p(t)$, the differential equation (0.8) becomes

$$f''(t) + \lambda(t+\alpha)^\beta f(t) = 0.\tag{7.1}$$

We take as a trial solution of (7.1) the series

$$f(t) = \sum_{n=0}^{\infty} a_n (t+\alpha)^{(\beta+2)n+1}. \tag{7.2}$$

Then formally

$$\begin{aligned} f''(t) &= \sum_{n=1}^{\infty} a_n n(\beta+2)[n(\beta+2)+1](t+\alpha)^{(\beta+2)n-1} \\ &= -\lambda \sum_{n=0}^{\infty} a_n (t+\alpha)^{(\beta+2)n+\beta+1} \\ &= -\lambda \sum_{n=1}^{\infty} a_{n-1} (t+\alpha)^{(\beta+2)n-1}. \end{aligned} \tag{7.3}$$

Thus we find the recurrence relations

$$n(\beta+2)[n(\beta+2)+1]a_n = -\lambda a_{n-1} \tag{7.4}$$

or

$$\begin{aligned} a_n &= -\frac{\lambda a_{n-1}}{(\beta+2)^2(n+1/(\beta+2))n} \\ &= \frac{(-1)^n a_0 \lambda^n}{(\beta+2)^{2n}(n+1/(\beta+2))(n-1+1/(\beta+2)) \cdots (1+1/(\beta+2))n!}. \end{aligned} \tag{7.5}$$

This suggests that we take

$$f^{(1)}(t) = (t+\alpha)^{1/2} J_{1/(\beta+2)}\left[\frac{2(t+\alpha)^{\beta/2+1}}{\beta+2}\lambda^{1/2}\right], \tag{7.6}$$

$$f^{(2)}(t) = (t+\alpha)^{1/2} Y_{1/(\beta+2)}\left[\frac{2(t+\alpha)^{\beta/2+1}\lambda^{1/2}}{\beta+2}\right], \tag{7.7}$$

and it can be shown easily that these represent two linearly independent solutions of (7.1), for λ real and different from zero. As an example we shall show that $f^{(1)}(t)$ satisfies the differential equation (7.1).

For convenience in writing we set

$$z = \frac{2(t+\alpha)^{\beta/2+1}}{\beta+2}\lambda^{1/2}. \tag{7.8}$$

With this notation we find from (7.6)

$$f^{(1)\prime}(t) = (1/2)(t+\alpha)^{-1/2} J_{1/(\beta+2)}(z) + \lambda^{1/2}(t+\alpha)^{\beta/2+1/2} J'_{1/(\beta+2)}(z) \tag{7.9}$$

and

$$
\begin{aligned}
f^{(1)''}(t) = & - (1/4)(t+\alpha)^{-3/2}J_{1/(\beta+2)}(z) \\
& + \lambda^{1/2}(t+\alpha)^{\beta/2-1/2}(\beta/2+1)J'_{1/(\beta+2)}(z) \\
& + \lambda(t+\alpha)^{\beta+1/2}J''_{1/(\beta+2)}(z).
\end{aligned}
\tag{7.10}
$$

Hence

$$
\begin{aligned}
& f^{(1)''}(t) + \lambda(t+\alpha)^{\beta}f^{(1)}(t) \\
& = \lambda(t+\alpha)^{\beta+1/2}\left[J''_{1/(\beta+2)}(z) + \frac{1}{z}J'_{1/(\beta+2)}(z)\right. \\
& \qquad \left. - \frac{1}{z^2(\beta+2)^2}J_{1/(\beta+2)}(z) + J_{1/(\beta+2)}(z)\right]
\end{aligned}
\tag{7.11}
$$

and this is zero by virtue of Bessel's differential equation. Hence $f^{(1)}(t)$ defined as in (7.6) is a solution of (7.1). Similarly (7.7) is also a solution of (7.1), and it is linearly independent.

Next we calculate

$$
\begin{aligned}
& f^{(2)'}(a)f^{(1)}(a) - f^{(1)'}(a)f^{(2)}(a) \\
& = \lambda^{1/2}(a+\alpha)^{\beta/2+1}\left[Y'_{1/(\beta+2)}(z)J_{1/(\beta+2)}(z) - J'_{1/(\beta+2)}(z)Y_{1/(\beta+2)}(z)\right].
\end{aligned}
\tag{7.12}
$$

By (6.13) and (7.8) we find that this becomes

$$
\begin{aligned}
& f^{(2)'}(a)f^{(1)}(a) - f^{(1)'}(a)f^{(2)}(a) \\
& = \lambda^{1/2}(a+\alpha)^{\beta/2+1}\frac{2}{\pi}\,\frac{\beta+2}{2(t+\alpha)^{\beta/2+1}\lambda^{1/2}} = \frac{\beta+2}{\pi}.
\end{aligned}
\tag{7.13}
$$

Also by (7.9) and the recurrence formula for Bessel functions we find

$$
\begin{aligned}
f^{(1)'}(t) & = \lambda^{1/2}(t+\alpha)^{\beta/2+1/2}\left[J'_{1/(\beta+2)}(z) + \frac{1}{z(\beta+2)}J_{1/(\beta+2)}(z)\right] \\
& = \lambda^{1/2}(t+\alpha)^{\beta/2+1/2}J_{1/(\beta+2)-1}(z).
\end{aligned}
\tag{7.14}
$$

Similarly

$$
f^{(2)'}(t) = \lambda^{1/2}(t+\alpha)^{\beta/2+1/2}Y_{1/(\beta+2)-1}(z).
\tag{7.15}
$$

Hence relation (0.11) of Theorem 1 yields

$$
\begin{aligned}
& \int_C^W \exp\left(\lambda\int_0^1 (t+\alpha)^{\beta}x^2(t)dt\right)d_wx \\
& = \frac{(\beta+2)^{1/2}}{\pi^{1/2}\lambda^{1/4}(1+\alpha)^{(\beta+1)/4}\alpha^{1/4}} \\
& \quad \cdot\left[Y_{1/(\beta+2)-1}(z_1)J_{1/(\beta+2)}(z_0) - J_{1/(\beta+2)-1}(z_1)Y_{1/(\beta+2)}(z_0)\right]^{-1/2},
\end{aligned}
\tag{7.16}
$$

where

$$z_0 = 2\,\frac{\alpha^{\beta/2+1}\lambda^{1/2}}{\beta+2}, \qquad z_1 = 2\,\frac{(1+\alpha)^{\beta/2+1}\lambda^{1/2}}{\beta+2}.$$

The relation (7.16) holds for $\beta \neq -2$, $0<\alpha<\infty$, and $-\infty<\lambda<\lambda_0$, $\lambda\neq 0$.

It is perhaps interesting to put $\beta=0$ in (7.16). We obtain

$$\begin{aligned}(7.17)\quad \int_C^W \exp\left(\lambda\int_0^1 x^2(t)dt\right)d_wx = {} & \frac{2^{1/2}}{\pi^{1/2}\lambda^{1/4}(1+\alpha)^{1/4}\alpha^{1/4}} \\ & \cdot\big[Y_{-1/2}((1+\alpha)\lambda^{1/2})J_{1/2}(\alpha\lambda^{1/2}) \\ & \qquad - J_{-1/2}((1+\alpha)\lambda^{1/2})Y_{1/2}(\alpha\lambda^{1/2})\big]^{-1/2}.\end{aligned}$$

Now [6, pp. 364, 370]

$$J_{1/2}(u) = (2/\pi u)^{1/2}\sin u, \qquad J_{-1/2}(u) = (2/\pi u)^{1/2}\cos u,$$

$$Y_{1/2}(u) = \frac{J_{1/2}(u)\cos(\pi/2) - J_{-1/2}(u)}{\sin(\pi/2)} = -\left(\frac{2}{\pi u}\right)^{1/2}\cos u,$$

$$Y_{-1/2}(u) = \frac{J_{-1/2}(u)\cos(-\pi/2) - J_{1/2}(u)}{\sin(-\pi/2)} = +\left(\frac{2}{\pi u}\right)^{1/2}\sin u.$$

Hence (7.17) becomes

$$\begin{aligned}(7.18)\quad \int_C^W \exp\left(\lambda\int_0^1 x^2(t)dt\right)d_wx = {} & \frac{2^{1/2}}{\pi^{1/2}\lambda^{1/4}(1+\alpha)^{1/4}\alpha^{1/4}} \\ & \cdot\left[\frac{2}{\pi(1+\alpha)^{1/2}\lambda^{1/2}\alpha^{1/2}}\left\{\sin((1+\alpha)\lambda^{1/2})\sin(\alpha\lambda^{1/2})\right.\right. \\ & \left.\left. + \cos((1+\alpha)\lambda^{1/2})\cos(\alpha\lambda^{1/2})\right\}\right]^{-1/2} = \frac{1}{(\cos\lambda^{1/2})^{1/2}},\end{aligned}$$

which yields our earlier result.

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A GENESIS FOR CESÀRO METHODS

RALPH PALMER AGNEW

1. Introduction. The Cesàro methods C_r, introduced by Cesàro[1] because of their applicability to Cauchy products of series, constitute the most publicized class of methods of summability.

The regular Nörlund methods[2] of summability constitute one of the two most publicized general classes of consistent methods of summability. The regular Hurwitz-Silverman-Hausdorff methods constitute the other.

This note proves the following theorem.

THEOREM. *The Cesàro methods are the only methods of summability, regular or not, which are both Nörlund methods and Hurwitz-Silverman-Hausdorff methods.*

Thus if the Cesàro methods had not been previously introduced into mathematical literature, they could be defined and exploited as the unique class of methods of summability enjoying all of the properties of Nörlund methods and all of the properties of Hurwitz-Silverman-Hausdorff methods.

In §4, it is shown that the only methods which are both Riesz methods and Hurwitz-Silverman-Hausdorff methods are methods Γ_r closely related to the methods C_r.

2. Nörlund methods. Each sequence $p_0, p_1, \cdots$ of real or complex constants for which $P_n \equiv p_0 + p_1 + \cdots + p_n \neq 0$ for each n defines a Nörlund method of summability by means of which a sequence $s_0, s_1, \cdots$ is summable to σ if $\sigma_n \rightarrow \sigma$ where

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[1] E. Cesàro, *Sur la multiplication des series*, Bull. Sci. Math. (2) vol. 14 (1890) pp. 114–120.

[2] N. E. Nörlund, *Sur une application des fonctions permutables*, Lunds Universitets Arsskrift (2) vol. 16 (1919).

$$(1)\qquad \sigma_n = (p_n s_0 + p_{n-1} s_1 + \cdots + p_1 s_{n-1} + p_0 s_n)/P_n.$$

The class of Nörlund transformations (1) is identical with the class of triangular matrix transformations

$$(2)\qquad \sigma_n = \sum_{k=0}^{n} a_{nk} s_k$$

for which

$$(3)\qquad a_{nn} \neq 0, \qquad n = 0, 1, 2, \cdots,$$

$$(4)\qquad \sum_{k=0}^{n} a_{nk} = 1, \qquad n = 0, 1, 2, \cdots,$$

and, for each $q=0, 1, 2, \cdots$, there is a constant b_q such that

$$(5)\qquad a_{n,n-q} = b_q a_{nn}, \qquad n \geqq q.$$

3. **Hurwitz-Silverman-Hausdorff methods.** These methods (hereafter the HSH methods) constitute the class of triangular matrix transformations which commute with the arithmetic mean transformation

$$(6)\qquad \sigma_n = (s_0 + s_1 + \cdots + s_n)/(n + 1)$$

and hence also with each other.

As Hurwitz and Silverman[3] and Hausdorff[4] have shown, with each method HSH there is associated a generating sequence $\lambda_0, \lambda_1, \cdots$ such that the transformation takes the form

$$(7)\qquad \sigma_n = \sum_{j=0}^{n} (-1)^j C_{n,j} \lambda_j \sum_{k=0}^{j} (-1)^k C_{j,k} s_k$$

or

$$(8)\qquad \sigma_n = \sum_{k=0}^{n} \left[C_{n,k} \sum_{j=0}^{n-k} (-1)^j C_{n-k,j} \lambda_{j+k} \right] s_k.$$

Assume that (8) is a Nörlund transformation, and let the quantity in brackets in (8) be denoted by a_{nk}. Then, $a_{nn}=\lambda_n$ so, by (3) and (4), $\lambda_n \neq 0$ for each n and $\lambda_0=1$. Moreover

$$(9)\qquad a_{n,n-1} = n(\lambda_{n-1} - \lambda_n)$$

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[3] W. A. Hurwitz and L. L. Silverman, *On the consistency and equivalence of certain definitions of summability*, Trans. Amer. Math. Soc. vol. 18 (1917) pp. 1–20.

[4] F. Hausdorff, *Summationsmethoden und Momentfolgen*. I and II, Math. Zeit. vol. 9 (1921) pp. 74–109 and 280–299.

and hence (5) with $q=1$ guarantees existence of a constant (which we call r instead of q_1) such that

$$n(\lambda_{n-1}-\lambda_n)=r\lambda_n, \qquad n\geqq 1. \tag{10}$$

Here r cannot be a negative integer; otherwise one could set $n=-r$ in (10) and contradict $\lambda_{-r-1}\neq 0$. Therefore

$$\lambda_n=(n/(n+r))\lambda_{n-1}, \qquad n=1,2,\cdots. \tag{11}$$

Since $\lambda_0=1$, (11) implies that

$$\lambda_1=1/(1+r), \quad \lambda_2=(2/(2+r))\lambda_1=1\cdot 2/(1+r)(2+r) \tag{12}$$

and in general

$$\lambda_n=1\cdot 2\cdot 3\cdots n/(1+r)(2+r)\cdots(n+r)=n!r!/(n+r)!, \qquad n=0,1,\cdots, \tag{13}$$

where $r!$ is the factorial function, $\Gamma(r+1)$, defined for all complex r except $-1, -2, -3, \cdots$.

By use of the familiar identity

$$\sum_{k=0}^{n} C_{n-k+r-1,r-1}=C_{n+r,r}, \qquad r\neq -1,-2,\cdots, \tag{14}$$

we obtain the familiar fact that for each $r\neq -1, -2, \cdots$ the Cesàro transformation C_r,

$$\sigma_n=\sum_{k=0}^{n}[C_{n-k+r-1,r-1}/C_{n+r,r}]s_k, \tag{15}$$

is the one and only Nörlund transformation for which $a_{nn}=\lambda_n=n!r!/(n+r)!$. This completes the proof of the fact that the only HSH transformations which are Nörlund transformations are the Cesàro transformations.

It is only when $r=0$ or r is a complex number with a positive real part that C_r is regular and can be written in the Hausdorff form

$$\sigma_n=\int_0^1\sum_{k=0}^{n}C_{n,k}t^k(1-t)^{n-k}s_k d\chi(t) \tag{16}$$

where

$$\chi(t)=1-(1-t)^r. \tag{17}$$

For each complex r not a negative integer, the more general Hurwitz-Silverman transformation (8) becomes C_r when $\lambda_n=n!r!/(n+r)!$.

4. **Riesz methods.** Each sequence $p_0, p_1, \cdots$ for which $P_n \equiv p_0 + p_1 + \cdots + p_n \neq 0$ for each n determines a Riesz transformation

$$\sigma_n = (p_0 s_0 + p_1 s_1 + \cdots + p_n s_n)/P_n, \qquad n = 0, 1, 2, \cdots . \tag{18}$$

If one of the two transformations

$$\sigma_n' = \sum_{k=0}^{n} a_{nk} s_k, \qquad \sigma_n'' = \sum_{k=0}^{n} a_{n,n-k} s_k \tag{19}$$

is a Nörlund transformation, the other is a Riesz transformation. It can be shown, by a suitable modification of our treatment of HSH and Nörlund methods, that if A is both an HSH method and a Riesz method, then there is a constant r not a negative integer such that A has the form

$$\sigma_n = \sum_{k=0}^{n} [C_{k+r-1,r-1}/C_{n+r,r}] s_k. \tag{20}$$

Thus *the only methods which are simultaneously HSH methods and Riesz methods are the methods* Γ_r *defined by* (20). For each $r \neq -1, -2, \cdots$, the transformation Γ_r is obtained from C_r by reversing the order in which the elements $a_{nk}^{(r)}$ are applied to $s_0, s_1, \cdots$. The HSH sequence λ_n generating Γ_r is $\lambda_0 = a_{00} = 1$ and

$$\lambda_n = a_{nn} = r/(n + r), \qquad n > 0. \tag{21}$$

The following discussion applies only to *regular* transformations. The elements a_{nk} of the matrix of the Hausdorff transformation generated by $\chi(t)$ are given by

$$a_{n,k} = \int_0^1 C_{n,k} t^k (1 - t)^{n-k} d\chi(t). \tag{22}$$

If (22) holds, then

$$\begin{aligned} a_{n,n-k} &= \int_0^1 C_{n,k} t^{n-k} (1 - t)^k d\chi(t) \\ &= -\int_0^1 C_{n,k} u^k (1 - u)^{n-k} d_u \chi(1 - u) \end{aligned}$$

and hence

$$a_{n,n-k} = \int_0^1 C_{n,k} t^k (1 - t)^{n-k} d_t [1 - \chi(1 - t)]. \tag{23}$$

It follows that if $\chi(t)$ generates one of the transformations in (19), then $\chi_1(t) \equiv 1 - \chi(1 - t)$ generates the other. Suppose now that $\chi(t)$

generates a regular Riesz method. Then $1-\chi(1-t)$ generates a Nörlund method, which must be a Cesàro method C_r. Hence $1-\chi(1-t)=1-(1-t)^r$ and $\chi(t)=t^r$. Since t^r generates a regular HSH method only when $\Re r>0$, and since t^r generates the regular Riesz method Γ_r when $\Re>0$, we have proved the following result. *The methods Γ_r for which $\Re r>0$ are the only regular methods of summability which are simultaneously Riesz methods and HSH methods.*

The identity

$$\frac{r!n!}{(n+r)!}=\frac{(r-1)!n!}{(n+r-1)!}\,\frac{r}{n+r}, \qquad r\neq 0,\,-1,\cdots, \tag{24}$$

implies the identities $C_r=C_{r-1}\Gamma_r=\Gamma_r C_{r-1}$ and $\Gamma_r=C_r C_{r-1}^{-1}=C_{r-1}^{-1}C_r$ involving the methods Γ_r and C_r; this is a consequence of the fact (Hurwitz-Silverman, loc. cit. p. 7) that if λ_n' generates A' and λ_n'' generates A'', then $\lambda_n'\lambda_n''$ generates $A'A''$. From the fact that C_r and the Hölder method H_r are equivalent $(C_r\sim H_r)$ when $\Re r>-1$ we obtain, when $\Re r>0$, the familiar formulas

$$C_r\sim H_r=H_{r-1}H_1\sim C_{r-1}H_1=C_1C_{r-1} \tag{25}$$

and $C_rC_{r-1}^{-1}\sim C_1$. This gives the fact, proved by Hausdorff, loc. cit., that Γ_r is equivalent to C_1 when $\Re r>0$.

CORNELL UNIVERSITY

A DIFFERENTIAL INEQUALITY

R. P. BOAS, JR.

The following theorem was discovered by S. B. Jackson in connection with a problem in differential geometry.

THEOREM 1. *If $f(x)$ is of class C^2 in $(0, a)$, $f(0)=f'(0)=0$, and $f''(x) \leqq K|f'(x)| + L|f(x)|$ in $(0, a)$, where K and L are constants, then $f(x) \leqq 0$ in some interval $(0, b)$.*

If $f(x)$ is in addition analytic at 0, Theorem 1 becomes trivial, since if a_n is the first nonvanishing coefficient of the power series of $f(x)$, $x^{2-n}f''(x)$ approaches a nonzero limit, while $x^{2-n}f'(x)$ and $x^{2-n}f(x)$ approach zero, as $x \to 0$, and consequently $a_n < 0$.

I shall prove the following more general theorem.

THEOREM 2. *If $f'(x)$ is absolutely continuous in $(0, a)$, $f(0)=f'(0)=0$, and*

$$(1) \qquad f''(x) \leqq K(x)\,|f'(x)| + x^{-1}L(x)\,|f(x)|$$

almost everywhere in $(0, a)$, where $K(x)$ and $L(x)$ are non-negative and integrable in $(0, a)$, then either $f(x) \equiv 0$ in some interval $(0, b)$, or $f'(x) < 0$ in $0 < x < \min(a, c)$, where c is such that

$$(2) \qquad \int_0^x \{K(t) + L(t)\}\,dt < 1, \qquad 0 < x < c.$$

Since $f(0)=0$, $f(x)$ is negative in $(0, c)$ when $f'(x) < 0$ in $(0, c)$. Theorem 1 is contained in the special case $K(t)=K$, $L(t)=Lt$.

Theorem 2 is the best possible result of its kind; for, if $\int_0^x K(t)dt$ diverges and $K(t)$ is positive and continuous in $t > 0$, the function $f(x)$ defined by $f'(x) = 1/\int_x^1 K(t)dt$, $f'(0)=f(0)=0$, is positive in $x > 0$ and satisfies (1) for all x such that $\int_x^1 K(t)dt > 1$.

Assume that $f(x)$ is not identically zero in any interval $(0, b)$, and write $M(x) = \max_{0 \leqq t \leqq x}|f'(t)|$, so that $M(x) > 0$ for $x > 0$ and $M(x)$ is nondecreasing.

We observe that for $0 \leqq x \leqq a$,

$$(3) \qquad |f(x)| = \left| \int_0^x f'(t)dt \right| \leqq xM(x).$$

There are points x_n such that $x_n \downarrow 0$ and $M(x_n) = |f'(x_n)|$. Suppose that $f'(x_n) > 0$ for some n. Let (a_n, b_n) be the largest interval, contain-

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ing x_n, in which $f'(x)>0$; since $f'(x)$ is continuous, there is such an interval, and $f'(a_n)=0$. Consequently we have, using (1) and (3),

$$\begin{aligned} 0 < M(x_n) = f'(x_n) &= \int_{a_n}^{x_n} f''(t)dt \\ &\leqq \int_{a_n}^{x_n} \{K(t)f'(t) + t^{-1}L(t)\,|\,f(t)\,|\,\}dt \\ &\leqq M(x_n)\int_0^{x_n} \{K(t) + L(t)\}dt. \end{aligned}$$

This leads to a contradiction if $x_n<c$, where c is defined by (2). Hence we must have $f'(x_n)<0$ for $x_n<c$.

There is a largest interval (a_n, b_n), containing x_n, in which $f'(x)<0$. Suppose first that $a_n>0$ for every n; then the intervals (a_n, b_n) are separated by other intervals in which $f'(x)\geqq 0$, and consequently $f'(b_n)=0$, $b_n\to 0$. Then we have for $a_n<x<b_n$

$$\begin{aligned} 0 < -f'(x) &= \int_x^{b_n} f''(t)dt \\ &\leqq \int_x^{b_n} \{-K(t)f'(t) + t^{-1}L(t)\,|\,f(t)\,|\,\}dt \qquad (4) \\ &\leqq M(b_n)\int_0^{b_n} \{K(t) + L(t)\}dt. \end{aligned}$$

Since $f'(b_n)=0$, there is a point x_n' in (x_n, b_n) such that $-f'(x_n')=M(b_n)$. Taking $x=x_n'$ in (4), we have

$$0 < M(b_n) \leqq M(b_n)\int_0^{b_n} \{K(t) + L(t)\}dt, \tag{5}$$

and again there is a contradiction when $b_n<c$.

Hence $a_n=0$ for some n. If $b_n\geqq a$, we have $f'(x)<0$ in $0<x<a$; if $b_n<a$, we have $f'(b_n)=0$, and then (4) and (5) hold. But (5) is contradictory unless $b_n\geqq c$. Hence we have $f'(x)<0$ in $0<x<\min(a, c)$.

We have incidentally established the following result about a function and its first derivative.

THEOREM 3. *If* $f(x)$ *is absolutely continuous in* $(0, a)$, $f(0)=0$, *and* $f'(x)\leqq K(x)|f(x)|$, *where* $K(x)$ *is non-negative and integrable in* $(0, a)$, *then either* $f(x)\equiv 0$ *in some interval* $(0, b)$, *or* $f(x)<0$ *in* $0<x<\min(a, c)$, *where* $\int_0^x K(t)dt<1$ *if* $x<c$.

HARVARD UNIVERSITY

NOTE ON THE EXPANSION OF A POWER SERIES INTO A CONTINUED FRACTION

H. S. WALL

1. **Introduction.** In view of the fact that the continued fraction frequently furnishes a method for summing a slowly convergent or even divergent power series, it is desirable to have a simple algorithem for obtaining the continued fraction. We present here such an algorithm based upon the fact that the process for constructing a sequence of orthogonal polynomials can be so arranged that it gives simultaneously a continued fraction expansion for a power series. It has been known at least since Tschebycheff that the problem of constructing a sequence of orthogonal polynomials is related to the problem of expanding a power series into a continued fraction. However, the fact that the two problems are actually identical does not seem to have been emphasized.

2. **The expansion of a power series into a *J*-fraction.** A continued fraction of the form

$$\frac{a_0}{b_1 + z -} \; \frac{a_1}{b_2 + z -} \; \frac{a_2}{b_3 + z -} \cdots \tag{2.1}$$

is called a *J-fraction*. The a_p and b_p are constants, and z is a complex variable. We shall suppose that the a_p are different from zero. We denote by $A_p(z)$ and $B_p(z)$ the pth numerator and denominator, respectively, of the J-fraction, so that $A_p(z)/B_p(z)$ is its pth approximant. The usual recurrence formulas

$$\begin{aligned} A_0 = 0,\ A_1 = a_0, \quad & A_p = (b_p + z)A_{p-1} - a_{p-1}A_{p-2}, \\ & \qquad\qquad\qquad p = 2, 3, 4, \cdots, \\ B_0 = 1,\ B_1 = b_1 + z,\ & B_p = (b_p + z)B_{p-1} - a_{p-1}B_{p-2}, \end{aligned} \tag{2.2}$$

show that $A_p(z)$ is a polynomial of degree $p-1$, and $B_p(z)$ is a polynomial of degree p:

$$\begin{aligned} A_p(z) &= \alpha_{p,0}z^{p-1} + \alpha_{p,1}z^{p-2} + \cdots + \alpha_{p,p-1}, \\ B_p(z) &= \beta_{p,0}z^{p} + \beta_{p,1}z^{p-1} + \cdots + \beta_{p,p}. \end{aligned} \tag{2.3}$$

We note that

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$$\beta_{p,0} = 1, \qquad \beta_{p,1} = b_1 + b_2 + \cdots + b_p. \tag{2.4}$$

By means of (2.2) we readily obtain the determinant formula

$$A_p(z)B_{p-1}(z) - A_{p-1}(z)B_p(z) = a_0a_1 \cdots a_{p-1}; \qquad p = 1, 2, 3, \cdots . \tag{2.5}$$

Consequently we find, with the aid of (2.4), that

$$\frac{A_{n+1}(z)}{B_{n+1}(z)} - \frac{A_n(z)}{B_n(z)} = \frac{a_0a_1 \cdots a_n}{z^{2n+1}} + \frac{h_n}{z^{2n+2}} + \cdots$$

where

$$h_n = -a_0a_1 \cdots a_n(b_1 + b_2 + \cdots + b_{n+1}). \tag{2.6}$$

It follows that there exists a power series

$$P(1/z) = c_0/z + c_1/z^2 + c_2/z^3 + \cdots \tag{2.7}$$

such that the expansion in descending powers of z of $A_n(z)/B_n(z)$ agrees term by term with $P(1/z)$ for the first $2n$ terms ($n=1, 2, 3, \cdots$). This uniquely determined power series is called the equivalent power series of the J-fraction.

We shall now write down formulas connecting the various constants, $\alpha_{p,q}$, $\beta_{p,q}$, c_p, a_p and b_p. These formulas serve as an algorithm for expanding a given power series $P(1/z)$ into a J-fraction, and, conversely, for obtaining the equivalent power series of a given J-fraction.

$$\begin{aligned}
&\beta_{00} = 1, \qquad c_0\beta_{00} = a_0, \qquad c_1\beta_{00} = h_0 = -a_0b_1;\\
&b_1 = -h_0/a_0, \qquad (\beta_{10}, \beta_{11}) = (1, b_1),\\
&\quad (c_2, c_1)\begin{pmatrix}\beta_{10}\\ \beta_{11}\end{pmatrix} = a_0a_1, \qquad (c_3, c_2)\begin{pmatrix}\beta_{10}\\ \beta_{11}\end{pmatrix} = h_1 = -a_0a_1(b_1 + b_2);\\
&b_2 = h_0/a_0 - h_1/a_0a_1,\\
&\quad (\beta_{20}, \beta_{21}, \beta_{22}) = (\beta_{10}, \beta_{11})\begin{pmatrix}1, & b_2, & 0\\ 0, & 1, & b_2\end{pmatrix} - a_1(0, 0, \beta_{00}),\\
&\quad (c_4, c_3, c_2)\begin{pmatrix}\beta_{20}\\ \beta_{21}\\ \beta_{22}\end{pmatrix} = a_0a_1a_2,\\
&\quad (c_5, c_4, c_3)\begin{pmatrix}\beta_{20}\\ \beta_{21}\\ \beta_{22}\end{pmatrix} = h_2 = -a_0a_1a_2(b_1 + b_2 + b_3);
\end{aligned} \tag{2.8}$$

$$b_3 = h_1/a_0a_1 - h_2/a_0a_1a_2,$$

$$(\beta_{30}, \beta_{31}, \beta_{32}, \beta_{33}) = (\beta_{20}, \beta_{21}, \beta_{22}) \begin{pmatrix} 1, & b_3, & 0, & 0 \\ 0, & 1, & b_3, & 0 \\ 0, & 0, & 1, & b_3 \end{pmatrix}$$
$$- a_2(0, 0, \beta_{10}, \beta_{11}),$$

. . . . ;

$$(2.9)\quad (\alpha_{n,0}, \alpha_{n,1}, \cdots, \alpha_{n,n-1}) = (\beta_{n,0}, \beta_{n,1}, \cdots, \beta_{n,n-1}) \cdot \begin{bmatrix} c_0, & c_1, & \cdots, & c_{n-1} \\ 0, & c_0, & \cdots, & c_{n-2} \\ \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \cdots, & c_0 \end{bmatrix}, \qquad n = 1, 2, 3, \cdots, (\beta_{n,0} = 1).$$

By way of illustration, we shall obtain the third approximant of the J-fraction for the function $P(1/z) = \log(1+1/z)$. Here $c_p = (-1)^p/(p+1)$, $p = 0, 1, 2, \cdots$. We then have:

$$\beta_{00} = 1, \qquad c_0 = a_0 = 1;$$

$$b_1 = 1/2, \quad (\beta_{10}, \beta_{11}) = (1, 1/2),$$

$$(1/3, -1/2)\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = 1/12 = a_1,$$

$$(-1/4, 1/3)\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} = -1/15 = h_1;$$

$$b_2 = 1/2, \quad (\beta_{20}, \beta_{21}, \beta_{22}) = (1, 1/2)\begin{pmatrix} 1, & 1/2, & 0 \\ 0, & 1, & 1/2 \end{pmatrix} - (1/12)(1, 0, 1)$$

$$= (1, 1, 1/6),$$

$$(1/5, -1/4, 1/3)\begin{pmatrix} 1 \\ 1 \\ 1/6 \end{pmatrix} = 1/180 = a_0a_1a_2, \quad a_2 = 1/15,$$

$$(-1/6, 1/5, 1/4)\begin{pmatrix} 1 \\ 1 \\ 1/6 \end{pmatrix} = -1/120 = h_2;$$

$$b_3 = 1/2, \qquad (\beta_{30}, \beta_{31}, \beta_{32}, \beta_{33}) = (1, 3/2, 3/5, 1/20);$$

$$(\alpha_{30}, \alpha_{31}, \alpha_{32}) = (1, 1, 11/60).$$

Consequently, the third approximant of the J-fraction is

$$\frac{A_3(z)}{B_3(z)}=\frac{1}{1/2+z}-\frac{1/12}{1/2+z}-\frac{1/15}{1/2+z}$$
$$=\frac{z^2+z+(11/60)}{z^3+(3/2)z^2+(3/5)z+(1/20)}.$$

We remark that for $z=1$ this gives $\log 2=.69312\cdots$, which is exact to *four* decimal places. Only six coefficients of the power series were used in the computation.

By the same method we find that the seventh approximant of the J-fraction expansion of the divergent power series

$$\frac{B_1}{1\cdot 2\cdot z}-\frac{B_3}{3\cdot 4\cdot z^3}+\frac{B_5}{5\cdot 6\cdot z^5}-\cdots,$$

where $B_1=1/6$, $B_3=1/30$, $B_5=1/42$, $\cdots$ are the Bernoulli numbers, is

$$\frac{1/12}{z}+\frac{1/30}{z}+\frac{53/210}{z}+\frac{195/371}{z}+\frac{22999/22737}{z}$$
$$+\frac{29944523/19733142}{z}+\frac{109535241009/48264275462}{z}.$$

Stieltjes [3, p. 521][1] proved that this J-fraction converges for $R(z)>0$ to the remainder $J(z)$ in Stirling's formula $\log\Gamma(z)=(z-1/2)\log z-z+(1/2)\log(2\pi)+J(z)$. He remarked that the law of formation of the coefficients in the J-fraction seems to be extremely complicated.

3. **Proof of the formulas (2.8) and (2.9).** We shall first prove that the formulas (2.8) constitute an arrangement of the algorithm for constructing a sequence of polynomials $B_n(z)=z^n+\beta_{n,1}z^{n-1}+\cdots+\beta_{n,n}$ which are orthogonal relative to a certain operator S. We define S to be the operator which replaces every z^p by c_p in any polynomial upon which it operates:

$$S(\beta_0z^n+\beta_1z^{n-1}+\cdots+\beta_n)=S(\beta_0z^n+\beta_1z^{n-1}+\cdots+\beta_nz^0)$$
$$=\beta_0c_n+\beta_1c_{n-1}+\cdots+\beta_nc_0,$$

where $c_0, c_1, \cdots, c_p, \cdots$ are given constants. Two polynomials B_p and B_q are said to be *orthogonal* if $S(B_pB_q)=0$ when the degrees p and q are unequal. We shall prove the following theorem:

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

THEOREM A. *Let m be a positive integer, and put*

$$\Delta_p = \begin{vmatrix} c_0, & c_1, & \cdots, & c_p \\ c_1, & c_2, & \cdots, & c_{p+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_p, & c_{p+1}, & \cdots, & c_{2p} \end{vmatrix}, \qquad p = 0, 1, 2, \cdots.$$

There exists a sequence of polynomials $B_n(z) = z^n + \beta_{n,1}z^{n-1} + \cdots + \beta_{n,n}$, $n = 0, 1, 2, \cdots, m$, such that

$$(3.1) \qquad S(B_pB_q)\begin{cases} = 0 & \text{if } p \neq q,\ p \leqq m,\ q \leqq m, \\ \neq 0 & \text{if } p = q < m, \end{cases}$$

if and only if $\Delta_p \neq 0$ for $p = 0, 1, 2, \cdots, m-1$. The polynomials are uniquely determined by the formulas

$$(3.2) \qquad B_{-1} = 0, \quad B_0 = 1, \quad B_p = (b_p + z)B_{p-1} - a_{p-1}B_{p-2}, \qquad p = 1, 2, 3, \cdots, m,$$

where

$$(3.3) \qquad \begin{aligned} S(z^pB_p) &= a_0a_1 \cdots a_p \neq 0, \\ S(z^{p+1}B_p) &= -a_0a_1 \cdots a_p(b_1 + b_2 + \cdots + b_{p+1}), \end{aligned} \qquad p = 0, 1, 2, \cdots, m-1.$$

PROOF. We suppose first that $\Delta_p \neq 0$ for $p = 0, 1, \cdots, m-1$, and shall prove that the required polynomials exist uniquely, and are given recurrently by (3.2) and (3.3). Since $B_0 = 1$, we have: $S(B_0^2) = S(1) = S(z^0) = c_0 = \Delta_0 \neq 0$. Let $B_1 = b_1 + z$. Then, $S(B_1) = b_1c_0 + c_1 = 0$ if and only if

$$S(B_0) = a_0, \qquad S(zB_0) = -a_0b_1.$$

Using induction, suppose that $B_0, B_1, \cdots, B_n$, $n < m$, have been uniquely determined such that (3.1) holds for $p \leqq n$, $q \leqq n$, (3.2) holds for $p \leqq n$, and (3.3) holds for $p \leqq n-1$. Now, an arbitrary polynomial of degree $n+1$ in which the coefficient of z^{n+1} is unity can be expressed uniquely in the form $B_{n+1} = (z + b_{n+1})B_n - a_nB_{n-1} + k_0B_0 + k_1B_1 + \cdots + k_{n-2}B_{n-2}$, where $b_{n+1}, a_n, k_0, k_1, \cdots, k_{n-2}$ are suitable constants. The conditions $S(z^pB_{n+1}) = 0$, $p = 0, 1, \cdots, n-2$, give in succession: $k_0a_0 = 0$, $k_1a_0a_1 = 0, \cdots, k_{n-2}a_0a_1 \cdots a_{n-2} = 0$, so that, since $a_p \neq 0$ for $p = 0, 1, \cdots, n-2$, we must have $k_0 = k_1 = \cdots = k_{n-2} = 0$. From the conditions $S(z^{n-1}B_{n+1}) = 0$ and $S(z^nB_{n+1}) = 0$, we then find that $S(z^nB_n) = a_0a_1 \cdots a_n$ and $S(z^{n+1}B_n) = -a_0a_1 \cdots a_n(b_1 + b_2 + \cdots + b_{n+1})$. Then, from the system of equations: $S(z^pB_n) = 0$, $p = 0, 1, \cdots, n-1$, $S(z^nB_n) = a_0a_1 \cdots a_n$, we find at once that

$$(3.4) \qquad \Delta_n = a_0 a_1 \cdots a_n \Delta_{n-1},$$

and, inasmuch as $n<m$, we see that $a_n \neq 0$. Consequently, B_{n+1} is uniquely determined, and (3.2), (3.3) hold for $p=n+1$ and $p=n$, respectively. Also, $S(B_pB_q)=0$ for $p\neq q$, $p\leqq n+1$, $q\leqq n+1$. Moreover, if $n+1<m$, then $S(B_{n+1}^2)=S(z^{n+1}B_{n+1})\neq 0$, for otherwise we would have $\Delta_{n+1}=0$. We have proved that the condition $\Delta_p\neq 0$, $p=0, 1, \cdots, m-1$, is *sufficient* for the polynomials to exist (uniquely) and satisfy the stated conditions.

Conversely, the condition is *necessary*. For, it is obviously necessary that $\Delta_0=c_0\neq 0$; and if $S(z^pB_n)=0$, for $p=0, 1, 2, \cdots, n-1$, $S(z^nB_n)=g_n\neq 0$, $n<m$, then the relation $\Delta_n=g_n\Delta_{n-1}$ must hold, and hence $\Delta_p\neq 0$, $p=0, 1, 2, \cdots, m-1$.

One will now readily see that the polynomials B_p given by (3.2) and (3.3) are the same as those given by (2.8).

THEOREM B. *Let* $\Delta_p\neq 0$, $p=0, 1, 2, \cdots$, *and define polynomials* $A_n(z)=\alpha_{n,0}z^{n-1}+\alpha_{n,1}z^{n-2}+\cdots+\alpha_{n,n-1}$ *by means of* (2.9). *Then,*

$$(3.5) \quad \frac{A_n(z)}{B_n(z)} = \frac{a_0}{b_1+z} - \frac{a_1}{b_2+z} - \cdots - \frac{a_{n-1}}{b_n+z}, \quad n = 1, 2, 3, \cdots,$$

and we have the formal power series identity

$$(3.6) \qquad P(1/z)B_n(z) - A_n(z) = \frac{a_0a_1\cdots a_n}{z^{n+1}} + \frac{h_n}{z^{n+2}} + \cdots,$$

where $h_n=-a_0a_1\cdots a_n(b_1+b_2+\cdots+b_{n+1})$ *and* $P(1/z)=\sum(c_p/z^{p+1})$.

PROOF. Let us *define* polynomials $A_n(z)$ by means of the formulas $A_{-1}=-1$, $A_0=0$, $A_p=(b_p+z)A_{p-1}-a_{p-1}A_{p-2}$, $p=1, 2, 3, \cdots$. From these recurrence formulas and (3.2) it follows that (3.5) holds. Furthermore, we may conclude from the determinant formula (2.5) that there exists a power series $P^*(1/z)=\sum(c_p^*/z^{p+1})$ such that

$$P^*(1/z)B_n(z) - A_n(z) = \frac{a_0a_1\cdots a_n}{z^{n+1}} + \frac{h_n}{z^{n+2}} + \cdots.$$

On equating coefficients of corresponding powers of z on either side of this identity we find that precisely the relations (3.3) hold but with c_p replaced by c_p^*. Inasmuch as those relations determine the c_p uniquely in terms of the a_p and b_p, we conclude that $c_p^*=c_p$, $p=0, 1, 2, \cdots$, or $P^*(1/z)=P(1/z)$, so that (3.6) holds. The relation (2.9) may now be obtained by equating the coefficients of $z^0, z^1, \cdots, z^{n-1}$ on either side of the identity (3.6).

This completes the proof of the formulas (2.8) and (2.9) connecting the constants $\alpha_{p,q}$, $\beta_{p,q}$, c_p, a_p, b_p of a J-fraction and its equivalent power series.

4. The expansion of a power series into an S-fraction. If we replace z by $1/z$ in the power series (2.7) and in its J-fraction expansion (2.1), the series becomes

$$P(z) = c_0z + c_1z^2 + c_2z^3 + \cdots, \tag{4.1}$$

and the J-fraction becomes

$$\frac{a_0z}{1 + b_1z} - \frac{a_1z^2}{1 + b_2z} - \frac{a_2z^2}{1 + b_3z} - \cdots. \tag{4.2}$$

An important special case arises when all the b_p are equal to zero. For, in this case it is evident that $P(z)/z$ contains only *even* powers of z. If we change the notation and replace c_{2n} by c_n, we see that the power series

$$c_0z + 0z^2 + c_1z^3 + 0z^4 + c_2z^5 + \cdots \tag{4.3}$$

has the expansion

$$\frac{a_0z}{1} - \frac{a_1z^2}{1} - \frac{a_2z^2}{1} - \cdots. \tag{4.4}$$

Let us now remove a factor z from both (4.3) and (4.4), and subsequently replace z^2 by z. Afterwards, we again multiply both the series and continued fraction by z and then replace z by $1/z$. The series then becomes

$$\frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots, \tag{4.5}$$

and the continued fraction becomes

$$\frac{a_0}{z} - \frac{a_1}{1} - \frac{a_2}{z} - \frac{a_3}{1} - \frac{a_4}{z} - \cdots. \tag{4.6}$$

Conversely, if the power series (4.5) has a continued fraction expansion of the form (4.6), then the power series (4.3) has a continued fraction expansion of the form (4.2) in which the b_p are all equal to zero. We shall call (4.6) an *S-fraction* since it is the form of continued fraction preferred by Stieltjes.

From the preceding it follows that the condition for (4.5) to have an S-fraction expansion (4.6) in which $a_p \neq 0$, $p = 0, 1, 2, \cdots$, is the same as the condition for (4.3) to have an expansion (4.2). This condition is that the determinants

$$c_0, \quad \begin{vmatrix} c_0, & 0 \\ 0, & c_1 \end{vmatrix}, \quad \begin{vmatrix} c_0, & 0, & c_1 \\ 0, & c_1, & 0 \\ c_1, & 0, & c_2 \end{vmatrix}, \quad \begin{vmatrix} c_0, & 0, & c_1, & 0 \\ 0, & c_1, & 0, & c_2 \\ c_1, & 0, & c_2, & 0 \\ 0, & c_1, & 0, & c_2 \end{vmatrix}, \cdots$$

be different from zero. From this we readily conclude the well known result that the power series (4.5) has an S-fraction expansion if and only if the determinants

$$\Delta_p = \begin{vmatrix} c_0, & c_1, & \cdots, & c_p \\ c_1, & c_2, & \cdots, & c_{p+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_p, & c_{p+1}, & \cdots, & c_{2p} \end{vmatrix}, \qquad \Omega_p = \begin{vmatrix} c_1, & c_2, & \cdots, & c_{p+1} \\ c_2, & c_3, & \cdots, & c_{p+2} \\ \cdot & \cdot & \cdot & \cdot \\ c_{p+1}, & c_{p+2}, & \cdots, & c_{2p+1} \end{vmatrix}$$

$$(p = 0, 1, 2, \cdots)$$

are all different from zero.

It is immediately evident that the algorithm of §2 can be used to compute the coefficients in (4.6) if we there replace c_{2n} by c_n and c_{2n+1} by 0.

5. A theorem of Stieltjes. A remarkable formulation of the problem of expanding a power series into a continued fraction was given by Stieltjes [3, p. 184]. Rogers [2] rediscovered part of the result of Stieltjes in a slightly different form. We offer the following formulation of the theorem.

The problem of expanding the power series

$$\frac{1}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \frac{c_3}{z^4} + \cdots$$

into a continued fraction

$$\frac{1}{b_1 + z} - \frac{a_1}{b_2 + z} - \frac{a_2}{b_3 + z} - \cdots$$

is equivalent to the problem of securing a power series identity of the form

$$Q(x + y) = Q(x)Q(y) + a_1Q_1(x)Q_1(y) + a_1a_2Q_2(x)Q_2(y) + \cdots,$$

where $a_p \neq 0$, $p = 1, 2, 3, \cdots$,

$$Q_n(z) = \frac{z^n}{n!} + \pi_{n,n+1}\frac{z^{n+1}}{(n+1)!} + \pi_{n,n+2}\frac{z^{n+2}}{(n+2)!} + \cdots,$$

and

$$Q_0(z) = Q(z) = 1 + c_1\frac{z}{1!} + c_2\frac{z^2}{2!} + c_3\frac{z^3}{3!} + \cdots.$$

The formulas connecting the various constants are:

$$\pi_{0,0} = 1, \qquad \pi_{p,q} = 0 \quad \text{for} \quad p > q;$$

$$(\pi_{0,q}, \pi_{1,q}, \pi_{2,q}, \cdots)$$

$$= (\pi_{0,q-1}, \pi_{1,q-1}, \pi_{2,q-1}, \cdots)\left\{\begin{matrix} b_1, & 1, & 0, & 0, \cdots \\ a_1, & b_2, & 1, & 0, \cdots \\ 0, & a_2, & b_3, & 1, \cdots \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot \end{matrix}\right\};$$

$$b_1 = \pi_{0,1}, \qquad b_p = \pi_{p-1,p} - \pi_{p-2,p-1}, \qquad p = 2, 3, 4, \cdots;$$

$$c_{p+q} = \pi_{0,p}\pi_{0,q} + a_1\pi_{1,p}\pi_{1,q} + a_1a_2\pi_{2,p}\pi_{2,q} + \cdots.$$

This combines the idea of Rogers with a formulation of Stieltjes' algorithm particularly adapted to the J-fraction. A part of this is given in [1, pp. 328–329]. We omit the proof.

Both Stieltjes and Rogers gave the example:

$$\int_0^\infty \operatorname{sech}^k u\, e^{-zu}du = \frac{1}{z+}\,\frac{1\cdot k}{z+}\,\frac{2(k+1)}{z\quad+}\,\frac{3(k+2)}{z\quad+\cdots}.$$

This can be obtained almost by inspection from the identity

$$\operatorname{sech}^k(x+y) = (\cosh x \cosh y + \sinh x \sinh y)^{-k}.$$

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A COMBINATORIAL FORMULA WITH SOME APPLICATIONS

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The aim of this note is to present a combinatorial formula and state its applications to partitions, number of solutions, and Dirichlet's integral.

Let $\theta_1(x), \cdots, \theta_n(x)$ be n arbitrary functions of x and let ${}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}([\theta_1]\cdots[\theta_n])$ be defined by

$$(1)\qquad {}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}([\theta_1]\cdots[\theta_n]) = \sum_{x_1+\cdots+x_n=m,\, a_1\leqq x_1\leqq b_1\cdots a_n\leqq b_n} \theta_1(x_1)\cdots\theta_n(x_n),$$

where $a_1, \cdots, a_n, b_1, \cdots, b_n, m$ are all integers and the right-hand side of (1) is summed over all different integral solutions of $x_1+\cdots+x_n=m$ with $a_1\leqq x_1\leqq b_1, \cdots, a_n\leqq x_n\leqq b_n$.

More generally we define

$$(2)\qquad \begin{aligned} &{}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}([\theta_1]^{n_1}\cdots[\theta_k]^{n_k}) \\ &= \sum_{x_{11}+\cdots+x_{1n_1}+\cdots+x_{k1}+\cdots+x_{kn_k}=m,\, a_1\leqq x_{1i}\leqq b_1, \cdots, a_k\leqq x_{ki}\leqq b_k} \theta_1(x_{11}) \\ &\qquad\cdots\theta_1(x_{1n_1})\cdots\theta_k(x_{k1})\cdots\theta_k(x_{kn_k}). \end{aligned}$$

We make the following conventions:[1]

(A) ${}_m\mathfrak{S}_a^b([\theta]^n)=0$ for $m<na$ or $m>nb$.

(B) ${}_m\mathfrak{S}([\theta]^0)=0$ if $m\neq 0$, ${}_m\mathfrak{S}([\theta]^0)=1$ if $m=0$.

(C) If $a_1=\cdots=a_n=a$, $b_1=\cdots=b_n=b$, we write

$${}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}([\theta_1]\cdots[\theta_n]) \quad\text{as}\quad {}_m\mathfrak{S}_a^b([\theta_1]\cdots[\theta_n]).$$

We now show that[2]

$$(3)\qquad \begin{aligned} &\sum_{\nu_1=0}^{n_1}\sum_{\nu_2=0}^{n_2}\cdots\sum_{\nu_k=0}^{n_k}(-1)^{\nu_1+\nu_2+\cdots+\nu_k}C_{n_1,\nu_1}C_{n_2,\nu_2}C_{n_k,\nu_k} \\ &\qquad\cdot {}_{m'}\mathfrak{S}_1^{\infty}([\phi_1]^{\nu_1}[\psi_1]^{n_1-\nu_1}\cdots[\phi_k]^{\nu_k}[\psi_k]^{n_k-\nu_k}) \\ &\qquad = {}_m\mathfrak{S}_{a_1\cdots a_k}^{b_1\cdots b_k}([\theta_1]^{n_1}\cdots[\theta_k]^{n_k}), \end{aligned}$$

where $\theta_1, \cdots, \theta_k$ are k arbitrary functions of x and $\phi_i(x)=\theta_i(x+b_i)$,

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[1] These conventions are used in proving (3) and other formulas.

[2] The formula (3) shows that all the strong restrictions for x can be removed.

$\psi_i(x)=\theta_i(x+a_i-1)$, $m'=m'(\nu_1\cdots\nu_k)=m-\sum b_i\nu_i-\sum(a_i-1)(n_i-\nu_i)$ $(i=1,\cdots,k)$.

PROOF. By definition we have

$$\begin{aligned}{}_{m'}\mathfrak{S}_1^\infty([\phi_1]^{\nu_1}[\psi_1]^{n_1-\nu_1}\cdots[\phi_k]^{\nu_k}[\psi_k]^{n_k-\nu_k})&\\ =\sum_{m_1+\cdots+m_{2k}=m}{}_{m_1}\mathfrak{S}_{b_1+1}^\infty([\theta_1]^{\nu_1})\,{}_{m_2}\mathfrak{S}_{a_1}^\infty([\theta_1]^{n_1-\nu_1})&\\ \cdots{}_{m_{2k-1}}\mathfrak{S}_{b_k+1}^\infty([\theta_k]^{\nu_k})\,{}_{m_{2k}}\mathfrak{S}_{a_k}^\infty([\theta_k]^{n_k-\nu_k}).&\end{aligned}$$

Let $T_i=\theta_i(x_{i1})\cdots\theta_i(x_{in_i})$ $(i=1,\cdots,k)$, and let $T=T_1\cdots T_k$ be a term contained in the right-hand side of (3), that is, $x_1\geqq a_1,\cdots,x_k\geqq a_k$. Without loss of generality we may assume $x_{11},\cdots,x_{1t_1}\geqq b_1+1;\cdots;x_{k1},\cdots,x_{kt_k}\geqq b_k+1$. Since the necessary and sufficient condition for

$$T_i\in{}_{m_{2i-1}}\mathfrak{S}_{b_i+1}^\infty([\theta_i]^{\nu_i})\,{}_{m_{2i}}\mathfrak{S}_{a_i}^\infty([\theta_i]^{n_i-\nu_i})\qquad(1\leqq i\leqq k)$$

is that there is a term $\theta_i(x_1)\cdots\theta_i(x_{\nu_i})$ of ${}_{m_{2i-1}}\mathfrak{S}_{b_i+1}^\infty$ contained in T_i as a part while the other part $T_i/\theta_i(x_1)\cdots\theta_i(x_{\nu_i})$ is contained in ${}_{m_{2i}}\mathfrak{S}_{a_i}^\infty$, the number of occurrences of T in the left-hand side of (3) is therefore given by

$$\left\{\sum_{\nu_1=0}^{t_1}(-1)^{\nu_1}C_{t_1,\nu_1}\right\}\left\{\sum_{\nu_2=0}^{t_2}(-1)^{\nu_2}C_{t_2,\nu_2}\right\}\cdots\left\{\sum_{\nu_k=0}^{t_k}(-1)^{\nu_k}C_{t_k,\nu_k}\right\}$$
$$=\begin{cases}0 & \text{if } t_1,\cdots,t_k \text{ are not all zero,}\\ 1 & \text{if } t_1=\cdots=t_k=0.\end{cases}$$

We see that the term T generally vanishes except when $a_j\leqq x_{ji}\leqq b_j$ $(j=1,\cdots,k)$. Hence (3) is proved.

It is directly deduced from (3) by putting $n_1=\cdots=n_k=1$, $n_1+\cdots+n_k=n$ that

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\sum_{(\alpha_1\cdots\alpha_k)\in(1\cdots n)}{}_{m'}\mathfrak{S}_1^\infty([\phi_{\alpha_1}]\cdots[\phi_{\alpha_k}][\psi_{\alpha_{k+1}}]\cdots[\psi_{\alpha_n}])&\\ ={}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}([\theta_1]\cdots[\theta_n]),&\end{aligned}\tag{4}$$

where $(\alpha_1\cdots\alpha_k\cdots\alpha_n)=(1\cdots n)$, $m'=m'(\alpha_1\cdots\alpha_k)=m+n-k-(b_{\alpha_1}+\cdots+b_{\alpha_k}+a_{\alpha_{k+1}}+\cdots+a_{\alpha_n})$.

Let F be an arbitrary function of $\theta_1,\cdots,\theta_n$ and let ${}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}\{F(\theta_1,\cdots,\theta_n)\}$ be defined by

$$\begin{aligned}{}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}\{F(\theta_1,\cdots,\theta_n)\}&\\ =\sum_{x_1+\cdots+x_n=m,\,a_1\leqq x_1\leqq b_1,\cdots,a_n\leqq x_n\leqq b_n}F(\theta_1(x_1),\cdots,\theta_n(x_n)).&\end{aligned}$$

Then, the formula (4) can be written more generally as

$$(5)\quad \sum_{k=0}^{n}(-1)^k \sum_{(\alpha_1\cdots\alpha_k)\in(1\cdots n)} {}_{m'}\mathfrak{S}_1^{\infty}\{F(\phi_{\alpha_1},\cdots,\phi_{\alpha_k},\psi_{\alpha_{k+1}},\cdots,\psi_{\alpha_n})\} = {}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}\{F(\theta_1,\cdots,\theta_n)\}.$$

If, for every $(\alpha_1\cdots\alpha_k)$, the limit

$$\lim_{m\to\infty} {}_{m'}\mathfrak{S}_1^{\infty}\{F(\phi_{\alpha_1},\cdots,\phi_{\alpha_k},\psi_{\alpha_{k+1}},\cdots,\psi_{\alpha_n})\} \qquad (m' = m'(\alpha_1\cdots\alpha_k))$$

exists, then we have further

$$(6)\quad \sum_{k=0}^{n}(-1)^k \sum_{(\alpha_1\cdots\alpha_k)\in(1\cdots n)} \lim_{m\to\infty} {}_{m'}\mathfrak{S}_1^{\infty}\{F(\phi_{\alpha_1},\cdots,\phi_{\alpha_k},\psi_{\alpha_{k+1}},\cdots,\psi_{\alpha_n})\} = \lim_{m\to\infty} {}_m\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}\{F(\theta_1,\cdots,\theta_n)\}.$$

We shall now state some applications of the above formulas.

Application to partitions. We denote by $p_n(m)$ the number of partitions of m into parts not exceeding n or into at most n parts. Let $\{\beta_1\cdots\beta_n\}$ be an ordered partition of m, namely,

$$m = \beta_1 + \beta_2 + \cdots + \beta_n, \qquad \beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_n.$$

We further denote by $p_{a_1\cdots a_n}^{b_1\cdots b_n}(m)$ the number of ordered partitions, $\{\beta_1\cdots\beta_n\}$'s, of m into exactly n parts which are restricted to $a_1 \leqq \beta_1 \leqq b_1$, $a_2 \leqq \beta_2 \leqq b_2$, $\cdots$, $a_n \leqq \beta_n \leqq b_n$. We have then

$$(7)\quad p_{a_1\cdots a_n}^{b_1\cdots b_n}(m) = \sum_{k=0}^{n}(-1)^k \sum_{(\alpha_1\cdots\alpha_k)\in(1\cdots n)} p_n(m - k - b_{\alpha_1} - \cdots - b_{\alpha_k} - a_{\alpha_{k+1}} - \cdots - a_{\alpha_n}),$$

where the $p_n(m)$'s can be evaluated by the generating function

$$(8)\quad G_n(x) = \frac{1}{(1-x)(1-x^2)\cdots(1-x^n)} = 1 + \sum_{m=1}^{\infty} p_n(m)x^m.$$

PROOF OF (7). Starting with (5) we define

$$\theta_i(x) = \begin{cases} 1 & \text{if } a_i \leqq x \leqq b_i, \\ 0 & \text{if } x < a_i \text{ or } b_i < x, \end{cases} \qquad (i = 1\cdots n),$$

$$F(\theta_1(x_1),\cdots,\theta_n(x_n)) = \begin{cases} \theta_1(x_1)\cdots\theta_n(x_n) & \text{if the order } x_1 \geqq x_2 \geqq \cdots \geqq x_n \text{ holds,} \\ 0 & \text{if the order } x_1 \geqq x_2 \geqq \cdots \geqq x_n \text{ does not hold.} \end{cases}$$

Thus we see that

$$ {}_{m}\mathfrak{S}_{a_1\cdots a_n}^{b_1\cdots b_n}\{F(\theta_1, \cdots, \theta_n)\} = p_{a_1\cdots a_n}^{b_1\cdots b_n}(m). $$

Similarly

$$ {}_{m'}\mathfrak{S}_{1}^{\infty}\{F(\phi_{\alpha_1}, \cdots, \phi_{\alpha_k}, \psi_{\alpha_{k+1}}, \cdots, \psi_{\alpha_n})\} $$
$$ = p_n(m - b_{\alpha_1} - \cdots - b_{\alpha_k} - a_{\alpha_{k+1}} - \cdots - a_{\alpha_n} - k). $$

Hence (7) is proved.

It may be noted that the formula (7) still holds when $p_{a_1\cdots a_n}^{b_1\cdots b_n}(m)$ denotes the number of partitions of m into n parts which are restricted to more conditions than

$$ \beta_1 \geqq \beta_2 \geqq \cdots \geqq \beta_n; \quad a_1 \leqq \beta_1 \leqq b_1, \cdots, a_n \leqq \beta_n \leqq b_n. $$

Application to number of solutions. Let $A_{a_1\cdots a_n}^{b_1\cdots b_n}(N)$ denote the number of integral (or prime) solutions of the equation

$$ x_1^k + x_2^k + \cdots + x_n^k = N $$

with $a_1 \leqq x_1 < b_1$, $a_2 \leqq x_2 < b_2$, $\cdots$, $a_n \leqq x_n < b_n$, then by (5) we have

$$ (9) \quad A_{a_1\cdots a_n}^{b_1\cdots b_n}(N) = \sum_{k=0}^{n} (-1)^k \sum_{(\nu_1 \cdots \nu_k) \in (1 \cdots n)} A_{b_{\nu_1}\cdots b_{\nu_k} a_{\nu_{k+1}} \cdots a_{\nu_n}}^{\infty \cdots \infty\ \infty \cdots \infty}(N), $$

where $(\nu_1 \cdots \nu_k \cdots \nu_n) = (1 \cdots n)$.

We shall now proceed to find an asymptotic formula concerning the number of integral solutions of a linear equation with integral coefficients.

Let $A(N)$ denote the number of integral solutions of the equation

$$ \sum_{k=1}^{n} c_k x_k = N \qquad (c_1 \geqq 1, \cdots, c_n \geqq 1) $$

under the restrictions $\alpha_k N < x_k < \beta_k N$ $(k = 1, \cdots, n)$, where $c_1, \cdots, c_n$ are relatively prime to each other, α_k, β_k are real values.

When $\alpha_k = 0$, $\beta_k = 1$ $(k = 1, \cdots, n)$ it is well known that[3]

$$ (10) \qquad A(N) = N^{n-1}/c_1 \cdots c_n \cdot (n-1)! + O(N^{n-2}). $$

Define

$$ \theta_i(x) = \begin{cases} 1 & \text{if } x \text{ is divisible by } c_i \text{ and the inequality } \alpha_i c_i N < x < \beta_i c_i N \text{ holds,} \\ & \qquad (1 \leqq i \leqq n) \\ 0 & \text{if } x \text{ is not divisible by } c_i \text{ or the inequality does not hold.} \end{cases} $$

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[3] The proof of (10) can be obtained easily by the method of partial fractions.

Thus by (4) and (10) we obtain the following consequence.[4]

Let $S_k = \sum_{(\nu_1 \cdots \nu_k) \in (1 \cdots n)} \phi_{\nu_1 \cdots \nu_k}$, where

$$\phi_{\nu_1 \cdots \nu_k} = \begin{cases} 0 & \text{for } (1 - \beta_{\nu_1} c_{\nu_1} - \cdots - \alpha_{\nu_n} c_{\nu_n}) \leqq 0, \\ (1 - \beta_{\nu_1} c_{\nu_1} - \cdots - \beta_{\nu_k} c_{\nu_k} - \alpha_{\nu_{k+1}} c_{\nu_{k+1}} - \cdots - \alpha_{\nu_n} c_{\nu_n})^{n-1} & \\ & \text{for } (1 - \beta_{\nu_1} c_{\nu_1} - \cdots - \alpha_{\nu_n} c_{\nu_n}) > 0. \end{cases}$$

Then

$$\frac{A(N)}{N^{n-1}} = \frac{S_0 - S_1 + \cdots \pm S_n}{c_1 \cdots c_n \cdot (n-1)!} + O\left(\frac{1}{N}\right). \tag{11}$$

Evidently (11) may be seen as a generalization of (10). If $c_1 = \cdots = c_n = 1$, $a_k \leqq x_k \leqq b_k$ $(k = 1, \cdots, n)$, we can express $A(N)$ more precisely as[5]

$$A(N) = \sum_{k=0}^{n} (-1)^k \sum_{(\nu_1 \cdots \nu_k) \in (1 \cdots n)} C_{N'(k) + (a_{\nu_1} + \cdots + a_{\nu_k}) - (b_{\nu_1} + \cdots + b_{\nu_k}), n-1} \tag{12}$$

where $N'(k) = N + n - k - 1 - (a_1 + \cdots + a_n)$.

Application to Dirichlet's integral. The following theorem is well known.

Let

$$I_{(0)} = \int\int_D \cdots \int x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \cdots x_n^{\alpha_n - 1} \cdot f(x_1 + x_2 + \cdots + x_n) dx_1 dx_2 \cdots dx_n,$$

where the variables $x_1, x_2, \cdots, x_n$ *are restricted to the region*

$$D\colon 0 \leqq k_1 \leqq x_1 + x_2 + \cdots + x_n \leqq k_2;\ 0 \leqq x_1, 0 \leqq x_2, \cdots, 0 \leqq x_n.$$

Then the integral $I_{(0)}$ *can be reduced to the form*

$$I_{(0)} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n)} \int_{k_1}^{k_2} u^{\alpha_1 + \alpha_2 + \cdots + \alpha_n - 1} f(u) du. \tag{13}$$

We shall now extend the formula (13). Let

$$I \begin{Bmatrix} b_1, \cdots, b_n \\ x_1, \cdots, x_n \\ a_1, \cdots, a_n \end{Bmatrix} = \int\int_R \cdots \int x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \cdots x_n^{\alpha_n - 1} \cdot f(x_1 + x_2 + \cdots + x_n) dx_1 dx_2 \cdots dx_n,$$

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[4] The detailed proof of this is omitted.

[5] The formula (12) can be obtained also by considering two generating functions.

where the region R is defined as the set of all points $(x_1 \cdots x_n)$ such that

$$0 \leqq a_i \leqq x_i \leqq b_i; \qquad x_1 + x_2 + \cdots + x_n \leqq k.$$

Since the Dirichlet integral $I_{(0)}$ can be considered as a limit of multiple summations with variable upper limits, by applying (6) we have

$$I\left\{\begin{matrix} b_1 \cdots b_n \\ x_1 \cdots x_n \\ a_1 \cdots a_n \end{matrix}\right\} = \sum_{j=0}^{n} (-1)^j \sum_{(\nu_1 \cdots \nu_j) \in (1 \cdots n)} I\left\{\begin{matrix} \infty \cdots \infty & \infty \cdots \infty \\ x_{\nu_1} \cdots x_{\nu_j} & x_{\nu_{j+1}} \cdots x_{\nu_n} \\ b_{\nu_1} \cdots b_{\nu_j} & a_{\nu_{j+1}} \cdots a_{\nu_n} \end{matrix}\right\}. \tag{14}$$

We have to establish a formula for

$$I\left\{\begin{matrix} \infty \cdots \infty \\ x_1 \cdots x_n \\ c_1 \cdots c_n \end{matrix}\right\} \qquad (c_1 \geqq 0, \cdots, c_n \geqq 0).$$

Let $I_{(1 \cdots s)}$ denote the integral

$$\frac{\Gamma(\alpha_{s+1}) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_{s+1} + \cdots + \alpha_n)} \int_0^{\min(c_1,k)} x_1^{\alpha_1 - 1} dx_1 \int_0^{\min(c_2, k - x_1)} x_2^{\alpha_2 - 1} dx_2$$
$$\cdots \int_0^{\min(c_s, k - x_1 - \cdots - x_{s-1})} x_s^{\alpha_s - 1} dx_s$$
$$\cdot \int_0^{k - x_1 - \cdots - x_s} u^{\alpha_{s+1} + \cdots + \alpha_n - 1} f(u + x_1 + \cdots + x_s) du.$$

Now, by (14) it can be shown that

$$I\left\{\begin{matrix} \infty \cdots \infty \\ x_1 \cdots x_n \\ c_1 \cdots c_n \end{matrix}\right\} = I_{(0)} - \sum_{(i) \in (1 \cdots n)}^{C_{n,1}} I_{(i)} + \sum_{(ij) \in (1 \cdots n)}^{C_{n,2}} I_{(ij)} - \cdots + (-1)^n I_{(1 \cdots n)}, \tag{15}$$

where

$$I_{(0)} = I\left\{\begin{matrix} \infty \cdots \infty \\ x_1 \cdots x_n \\ 0 \cdots 0 \end{matrix}\right\},$$

$$I_{(1 \cdots n)} = \int_0^{\min(c_1,k)} \cdots \int_0^{\min(c_n, k - x_1 - \cdots - x_{n-1})} x_1^{\alpha_1 - 1} \cdots x_n^{\alpha_n - 1} dx_1 \cdots dx_n.$$

To prove (15), the formal logic theorem is also applicable.

Consider a differential $\bar{x}_1^{\alpha_1-1} \cdots \bar{x}_n^{\alpha_n-1} f(\bar{x}_1 + \cdots + \bar{x}_n) d\bar{x}_1 \cdots d\bar{x}_n$. We may assume that $\bar{x}_{\nu_1} \leqq c_{\nu_1}, \cdots, \bar{x}_{\nu_t} \leqq c_{\nu_t}, \bar{x}_{\nu_{t+1}} > c_{\nu_{t+1}}, \cdots, \bar{x}_{\nu_n} > c_{\nu_n}$, $(\nu_1 \cdots \nu_n) = (1 \cdots n)$.

Since the integral is a limit of multiple summations and may be written as

$$I_{(1\cdots s)} = \int_0^{\min(c_1,k)} x_1^{\alpha_1-1} dx_1 \cdots \int_0^{\min(c_s, k-x_1-\cdots-x_{s-1})} x_s^{\alpha_s-1} dx_s$$

$$\cdot \int_{R_1} \cdots \int x_{s+1}^{\alpha_{s+1}-1} \cdots x_n^{\alpha_n-1} f(x_1 + \cdots + x_n) dx_{s+1} \cdots dx_n$$

$$= \int_0^{c_1} \cdots \int_0^{c_s} \int_{R_2(x_1 \cdots x_n)} \cdots \int x_1^{\alpha_1-1} \cdots x_s^{\alpha_s-1} x_{s+1}^{\alpha_s+1} \cdots x_n^{\alpha_n}$$

$$\cdot f(x_1 + \cdots + x_n) dx_1 \cdots dx_n,$$

where

$$R_1(x_{s+1} \cdots x_n):\quad 0 \leqq x_{s+1} + \cdots + x_n \leqq k - (x_1 + \cdots + x_s); \qquad x_{s+1}, \cdots, x_n \geqq 0,$$

$$R_2(x_1 \cdots x_n):\quad 0 \leqq x_1 \leqq c_1, \cdots, 0 \leqq x_s \leqq c_s; \quad 0 \leqq x_1 + \cdots + x_n \leqq k; \qquad x_{s+1}, \cdots, x_n \geqq 0,$$

we see that the differential $\bar{x}_1^{\alpha_1-1} \cdots \bar{x}_n^{\alpha_n-1} f(\bar{x}_1 + \cdots + \bar{x}_n) d\bar{x}_1 \cdots d\bar{x}_n$ appears exactly $C_{t,s}$ times in $\sum_{(\nu_1 \cdots \nu_s) \in (1 \cdots n)} I_{(\nu_1 \cdots \nu_s)}$. Therefore the number of occurrences of the given differential in the right-hand side of (15) is equal to

$$C_{t,0} - C_{t,1} + \cdots + (-1)^t C_{t,t} = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t > 0. \end{cases}$$

Hence the formula (15) is proved.

The integral $I_{(1 \cdots s)}$ may be calculated by dividing the limits of the integral and integrating it separately.

It is seen that the integral $I_{(1 \cdots s)}$ can be written also in the form:

$$\frac{\Gamma(\alpha_{s+1}) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_{s+1} + \cdots + \alpha_n)} \int_0^{c_1} x_1^{\alpha_1-1} dx_1 \int_0^{c_2} x_2^{\alpha_2-1} dx_2 \cdots \int_0^{c_s} x_s^{\alpha_s-1} dx_s$$

$$\cdot \int_0^{[(k-x_1-\cdots-x_s)+|k-x_1-\cdots-x_s|]/2} u^{\alpha_{s+1}+\cdots+\alpha_n-1} f(u + x_1 + \cdots + x_s) du.$$

Connecting (14) with (15), we see that it is a generalization of Dirichlet's integral $I'_{(0)}$ with $k_2 = 0$, $k_1 = k$ in (13).

It may be noted that the formula (14) is also called Liouville's extension and the integral regions D and R can be defined also by

$$D:\quad 0 \leqq k_1 \leqq a_1x_1^{p1} + \cdots + a_nx^{p_n} \leqq k_2, \qquad 0 \leqq x_i;$$

$$R:\quad 0 \leqq a_i \leqq x_i \leqq b_i, \qquad d_1x_1^{q1} + \cdots + dx_n^{q_n} \leqq k.$$

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TRANSFORMATIONS IN METRIC SPACES AND ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction. It is evident that the solutions of a differential equation $y' = f(t, y)$ passing through a point (τ, η) in the region of definition of $f(t, y)$ may be considered as invariant functions of the transformation $Ty(t) = \eta + \int_\tau^t f(s, y(s))ds$ when suitable restrictions are placed upon the functions $y(t)$ considered. That such invariant functions exist for continuous $f(t, y)$ can be made a consequence of Schauder's fixed point theorem for completely continuous transformations in bounded convex subsets of a Banach space.[1] For $f(t, y)$ satisfying a Lipschitz condition in y the existence and uniqueness of an invariant function can be made to follow from a fixed point theorem of Caccioppoli of an essentially simpler nature.[2] In the present paper we wish to show that the existence of invariant functions for continuous $f(t, y)$ as well as several other theorems concerning solutions of differential equations can be made to follow from some theorems concerning a particular class of transformations in a complete metric space. Although the existence theorem for fixed points given

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[1] J. Schauder, *Der Fixpunktsatz in Funktionalraumen*, Studia Mathematica vol. 2 (1930) pp. 171–180; also, *Zur Theorie stetiger Abbildungen in Funktionalräumen*, Math. Zeit. vol. 26 (1927) pp. 47–65 and 417–431.

[2] R. Caccioppoli, *Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale*, Rendiconti R. Accademia dei Lincei (6) vol. 11 (1930) pp. 794–799; for proofs and various applications of both Caccioppoli's and Schauder's theorems we refer also to Niemytsky, *Metod nepodvizhnykh Tochek v Analize*, Uspekhi Mat. Nauk vol. 1 (1936) pp. 141–174.

here is much more elementary than Schauder's theorem, it is no more difficult to apply to ordinary differential equations provided one is willing to admit that differential equations for which $f(t, y)$ is a polynomial in t and y have unique solutions.

2. **Transformations in metric spaces.** Let M be a complete metric space. A transformation T of M into itself will be called completely continuous if (1) it is continuous, (2) $T(M)$ is bounded, and (3) the image of any bounded set in M has a compact closure. If M is bounded, condition (2) is of course trivial. The "distance" between two such transformations may be defined by $\rho^*(T_1, T_2)=\sup_{x\in M} \rho(T_1x, T_2x)$; this is clearly finite from condition (2). If $\mathfrak{T}$ is any class of such transformations it is easy to see that this distance function may be used to convert $\mathfrak{T}$ into a metric space. It is a routine matter to prove the following theorem.

THEOREM. *If $\mathfrak{T}$ is the class of all completely continuous transformations defined on M and metrized as above, then $\mathfrak{T}$ is complete.*

In general, however, $\mathfrak{T}$ will not denote the class of all completely continuous transformations but merely some subclass. Moreover, it will be advantageous to alter the distance function in $\mathfrak{T}$ by substituting for $\rho^*(T_1, T_2)$ any distance function $\rho(T_1, T_2)$ which is at least as "strong" as $\rho^*(T_1, T_2)$, that is, one such that $\lim \rho(T_n, T)=0$ implies $\lim \rho^*(T_n, T)=0$. Topological properties of subsets of $\mathfrak{T}$ such as "closed," "dense," and so on are to be understood in terms of the topology defined by the function $\rho(T_1, T_2)$. There seems to be little danger of confusing this function ρ with that defined in M.

Let $\mathfrak{T}_0$ denote those elements of $\mathfrak{T}$ having no fixed points in M, $\mathfrak{T}_1$ those elements having exactly one fixed point in M, and $\mathfrak{T}_2$ those having more than one fixed point. For a given $T\in\mathfrak{T}$ let $F(T)$ denote the set of fixed points of T. This set is clearly closed and compact. For any $x\in M$, $\epsilon>0$, let $S(x, \epsilon)=E_{y\in M}[\rho(x, y)<\epsilon]$ and for any set $A\subset M$ let $U(A, \epsilon)$ be the union of all $S(x, \epsilon)$ for $x\in A$; analogous notation will be used for elements and subsets of $\mathfrak{T}$.

THEOREM 1. *There exists $\delta(T)>0$ defined for $T\in\mathfrak{T}_0$ such that $\rho(T', T)<\delta(T)$ implies $T'\in\mathfrak{T}_0$, that is, $\mathfrak{T}_0$ is open in $\mathfrak{T}$.*

For suppose there exist $T_n\in\mathfrak{T}_1+\mathfrak{T}_2$ and $T\in\mathfrak{T}_0$ such that $\rho(T_n, T)<1/n$. Let $x_n\in F(T_n)$. It is easily seen that the sequence $\{x_n\}$ is bounded. Since T is completely continuous there exist x_{n_i}, x_0 such that $\rho(Tx_{n_i}, x_0)\rightarrow 0$. But $\rho(x_{n_i}, x_0)\leqq\rho(T_{n_i}x_{n_i}, Tx_{n_i})+\rho(Tx_{n_i}, x_0)$ $\leqq\rho^*(T_{n_i}, T)+\rho(Tx_{n_i}, x_0)$ which approaches 0 as $i\rightarrow\infty$. But then, since T is continuous, $\rho(Tx_{n_i}, Tx_0)\to 0$ which implies that $Tx_0=x_0$, contradicting the choice of $T\in\mathfrak{T}_0$.

THEOREM 2. *There exists* $\delta(T, \epsilon)>0$ *defined for* $T\in\mathfrak{T}_1+\mathfrak{T}_2$ *and* $\epsilon>0$ *such that* $\rho(T', T)<\delta(T, \epsilon)$ *implies* $F(T')\subset U(F(T), \epsilon)$.

If $F(T')$ is empty the conclusion is trivially true. Suppose that for some $\epsilon>0$ and $T\in\mathfrak{T}_1+\mathfrak{T}_2$ there exist $T_n\in\mathfrak{T}_1+\mathfrak{T}_2$ and $x_n\in F(T_n)$ such that $\rho(T_n, T)<1/n$ but $x_n\notin U(F(T), \epsilon)$. Since T is completely continuous there exist x_{n_i}, x_0 such that $\rho(Tx_{n_i}, x_0)\to 0$. It now follows as in the previous theorem that $\rho(x_{n_i}, x_0)\to 0$ and $Tx_0=x_0$, that is, $x_0\in F(T)$ which contradicts the assumption that $x_n\notin U(F(T), \epsilon)$.

THEOREM 3. *If* $\mathfrak{T}_1+\mathfrak{T}_2$ *is dense in* $\mathfrak{T}$, $\mathfrak{T}_0$ *is empty.*

This is an immediate consequence of Theorem 1 since the complement of a dense set cannot be open unless it is empty.

THEOREM 4. *If* $\mathfrak{T}_1$ *is dense in* $\mathfrak{T}$, $\mathfrak{T}_0$ *is empty and* $\mathfrak{T}_2$ *is of the first category in* $\mathfrak{T}$. *If* $\mathfrak{T}$ *is complete,* $\mathfrak{T}_1$ *is of the second category in* $\mathfrak{T}$.

That $\mathfrak{T}_0$ is empty follows from Theorem 3. Let $d[A]$ denote the diameter of $A\subset M$. Then $T\in\mathfrak{T}_2$ is equivalent to $d[F(T)]\geqq 0$. Clearly, $\mathfrak{T}_2=\mathfrak{N}_1+\mathfrak{N}_2+\cdots+\mathfrak{N}_i+\cdots$ where $\mathfrak{N}_i=E[T\in\mathfrak{T}, d[F(T)]\geqq 1/i]$. If it is shown that $\mathfrak{N}_i$ is nowhere-dense, the statement concerning $\mathfrak{T}_2$ will be proved. Let $T_0\in\mathfrak{T}$, $\epsilon>0$ be arbitrary. Since $\mathfrak{T}_1$ is dense in $\mathfrak{T}$, there exists $T_1\in\mathfrak{T}_1$ such that $T_1\in S(T_0, \epsilon/2)$. But also there exists $\eta<\epsilon/2$ such that $S(T_1, \eta)\subset S(T_0, \epsilon)$ and such that $S(T_1, \eta)$ has an empty intersection with $\mathfrak{N}_i$. For, by Theorem 2, if $\eta<\min[\epsilon/2, \delta(T_1, 1/3i)]$ and $T\in S(T_1, \eta)$, then $F(T)\subset U(F(T_1), 1/3i)$ and consequently $d[F(T)]\leqq d[F(T_1)]+2/3i<1/i$, that is, $T\notin\mathfrak{N}_i$. This is just the condition that $\mathfrak{N}_i$ be nowhere-dense. The final statement follows from the facts that $\mathfrak{T}=\mathfrak{T}_1+\mathfrak{T}_2$ and that $\mathfrak{T}$, being complete, is of the second category in itself.

THEOREM 5. *If* $\mathfrak{T}_2$ *is dense in* $\mathfrak{T}$ *and* $\mathfrak{A}\subset\mathfrak{T}_1$ *is closed in* $\mathfrak{T}$, *then* $\mathfrak{A}$ *is nowhere-dense in* $\mathfrak{T}$.

For suppose $\mathfrak{A}$ is not nowhere-dense in $\mathfrak{T}$. Then there is some sphere in $\mathfrak{T}$ in which $\mathfrak{A}$ is dense and which consequently $\mathfrak{A}$ contains since it is closed. But this contradicts the hypothesis concerning $\mathfrak{T}_2$.

3. **Applications.** We wish now to apply these theorems to systems of real differential equations of the form $y_i'=f_i(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$. In order to do this we shall introduce certain metric spaces, in this case Banach spaces. Let R be the set of points

$(t, y_1, \cdots, y_n)$ in E_{n+1} determined by the inequalities $a \leqq t \leqq b$ and $c_i \leqq y_i \leqq d_i$, $i=1, \cdots, n$, where $a<b$, $c_i<d_i$ and any of the values c_i, d_i may be infinite. Let $C_n(a, b)$ be the Banach space of n-tuples of functions $Y(t)=(y_1(t), \cdots, y_n(t))$, each $y_i(t)$ being continuous on $[a, b]$, and with norm defined by $\|Y\|=\sum_{i=1}^n\|y_i\|=\sum_{i=1}^n \max_{a\leqq t\leqq b}|y_i(t)|$. Let $C_n(R)$ be the Banach space of n-tuples of functions $F(t, y_1, \cdots, y_n) =(f_1(t, y_1, \cdots, y_n), \cdots, f_n(t, y_1, \cdots, y_n))$, each f_i being continuous and bounded on R, with norm $\|F\|=\sum_{i=1}^n\|f_i\|=\sum_{i=1}^n \sup_R|f_i(t, y_1, \cdots, y_n)|$. Let M be the subset of $C_n(a, b)$ consisting of elements Y such that $c_i \leqq y_i(t) \leqq d_i$. M is a complete metric space.

For any $P(\tau, \eta_1, \cdots, \eta_n) \in R$ and $F \in C_n(R)$ we may define the following transformation in M: $TY=(T_1Y, \cdots, T_nY)$ where for $a \leqq t \leqq b$

$$T_iY(t) = \begin{cases} d_i & \text{if } \eta_i + \int_\tau^t f_i ds > d_i, \\ \eta_i + \int_\tau^t f_i(s, y_1(s), \cdots, y_n(s)) ds & \text{if } c_i \leqq \eta_i + \int_\tau^t f_i ds \leqq d_i, \\ c_i & \text{if } c_i > \eta_i + \int_\tau^t f_i ds. \end{cases}$$

Each transformation T is thus associated with a definite point $P \in R$ and a definite $F \in C_n(R)$. Clearly, any fixed point $Y(t)$ of this transformation gives a solution in R through the point P of the set of differential equations $y_i' = f_i(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$, after one has taken proper account of the modification of TY on the boundary of R. The actual solution of the differential equations will be a subcontinuum containing P of the curve representing $Y(t)$. This solution will not be a "local solution" but will extend in both directions[3] to the boundary of R. At the left and right end points of the curve representing the actual solution y_i' will of course be interpreted as the right and left derivative respectively.

The class $\mathfrak{T}$ which we shall consider consists of all transformations of the form above associated with some $P \in R$ and $F \in C_n(R)$. A metric in $\mathfrak{T}$ is defined by $\rho(T', T'') = |\tau' - \tau''| + \sum_{i=1}^n |\eta_i' - \eta_i''| + \|F' - F''\|$. Since R and $C_n(R)$ are complete $\mathfrak{T}$ is a complete metric space. It follows from the form of T that $TM \subset M$. That TM is bounded follows from the boundedness on R of the functions f_i. Continuity of T

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[3] It may happen, however, in case some of the values η_i are equal to c_i or d_i, that the curve representing the actual solution will reduce to the single point P. This will not happen if P is interior to R.

follows from the uniform continuity of the functions f_i on bounded closed subsets of R. That T takes bounded subsets of M into sets with compact closures follows from the equicontinuity of the functions $T_iY(t)$ for $Y\in M$ and Arzela's criterion for compactness in the space $C_n(a, b)$. One may conclude from the following inequalities that $\rho(T', T'')$ is at least as strong as $\rho^*(T', T'')$: $\rho^*(T', T'') = \sum_{i=1}^{n} \sup_{Y\in M} \|T_i' Y - T_i'' Y\| \leqq \sum_{i=1}^{n} \sup_{Y\in M} \max_{a\leqq t\leqq b} \left|\eta_i' - \eta_i'' + \int_{\tau'}^{t} f_i'(s, y_1(s), \cdots, y_n(s))ds - \int_{\tau''}^{t} f_i''(s, y_1(s), \cdots, y_n(s))ds\right| \leqq \sum_{i=1}^{n} |\eta_i' - \eta_i''| + \sum_{i=1}^{n} \sup_{Y\in M} \left|\int_{\tau'}^{\tau''} f_i' ds\right| + \sum_{i=1}^{n} \sup_{Y\in M} \int_a^b |f_i' - f_i''| ds$, which evidently becomes small as $\rho(T', T'')$ becomes small.

In order to apply Theorem 4 to $\mathfrak{T}$ we must still show that $\mathfrak{T}_1$ is dense in $\mathfrak{T}$. But this follows from the well known facts that the elements F whose components are polynomials are dense in $C_n(R)$ and that for such $F\in C_n(R)$ the corresponding set of differential equations $y_i' = f_i(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$, has a unique solution through every $P\in R$. From Theorem 4 we may now conclude that $\mathfrak{T}_0$ is empty and, consequently, that every set of differential equations of the kind considered has at least one solution through every point in R. Theorem 4 gives the additional information that those sets of differential equations, associated with some $P\in R$ and $F\in C_n(R)$, which have multiple solutions are of the first category in $\mathfrak{T}$ whereas those with unique solutions are of the second category. This theorem is closely related to the following theorem of Orlicz:[4] Let G be a domain in E_2 and $C(G)$ the Banach space of bounded continuous functions $f(t, y)$ defined on G. Then those functions $f(t, y)$ for which $y' = f(t, y)$ has a unique solution through every point of G are of the second category in $C(G)$ whereas the complementary set is of the first category. The chief difference between the theorems seems to be that Orlicz's theorem is concerned with a property of the element F while the theorem above deals with a property of the pair (P, F). Orlicz's theorem does not seem to follow readily from Theorem 4 but requires additional discussion.

Theorem 2, when applied to our special choice of M and $\mathfrak{T}$, gives the following result. If $P_k(\tau_k, \eta_{1k}, \cdots, \eta_{nk}) \to P(\tau, \eta_1, \cdots, \eta_n)$ where $c_i < \eta_i < d_i$ and $\|F_k - F\| \to 0$, then there exists an interval $[a', b']$ with $a \leqq a' \leqq \tau \leqq b' \leqq b$ and $a' < b'$ such that for any $\epsilon > 0$ there exists k_ϵ such that for $k > k_\epsilon$ each solution through P_k of $y_i' = f_{ik}(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$, is within norm distance ϵ of some solution through P of $y_i' = f_i(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$, when one restricts consideration of the solutions to $[a', b']$. In case $y_i' = f_i(t, y_1, \cdots, y_n)$,

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[4] W. Orlicz, *Zur Theorie der Differentialgleichung* $y' = f(x, y)$, Bulletin International de l'Académie Polonaise des Sciences et Lettres, Série A, Nos. 8, 9 (1932) pp. 221–228.

$i=1, \cdots, n$, has a unique solution through P, the solutions through P_k of $y_i' = f_{ik}(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$, will converge uniformly on $[a', b']$ to this solution.[5] It is evident that the indices k do not need to be integers but could be elements of any directed set, for example, a set of r parameters upon which the point P and the functions F depend continuously in the sense of our topology. That it may be necessary to consider the solutions only on some subinterval of $[a, b]$ is clear when one recalls the modification of the transformation T on the boundary of R.

In order to apply Theorem 5 we must first show that $\mathfrak{T}_2$ is dense in $\mathfrak{T}$. Let $P(\tau, \eta_1, \cdots, \eta_n)$ and $F \in C_n(R)$ be the point and functions corresponding to an arbitrary transformation $T \in \mathfrak{T}$. For any $\epsilon > 0$ select $\alpha > 0$ and $\beta_i > 0$ so that in the neighborhood $|t-\tau| < \alpha$, $|y_i - \eta_i| < \beta_i$ one has $|f_i(t, y_1, \cdots, y_n) - f_i(\tau, \eta_1, \cdots, \eta_n)| < \epsilon/2$, $i=1, \cdots, n$. In this neighborhood define $g_i(t, y_1, \cdots, y_n) = f_i(\tau, \eta_1, \cdots, \eta_n) + \gamma|y_i - \eta_i - (t-\tau)f_i(\tau, \eta_1, \cdots, \eta_n)|^{1/2}$ where γ is chosen so small that in the neighborhood one has $|g_i - f_i| < \epsilon$. Then extend g_i over R as a continuous function in such a manner that $|f_i - g_i| < \epsilon$ on R. The transformation T' associated with P and G will then be such that $\rho(T, T') < \epsilon$. But $T' \in \mathfrak{T}_2$, for the set of equations $y_i' = g_i(t, y_1, \cdots, y_n)$, $i=1, \cdots, n$, has more than one solution (in fact, continuum-many) through the point P. In the neighborhood defined above two of them are: $y_i(t) = \eta_i + (t-\tau)f_i(\tau, \eta_1, \cdots, \eta_n)$, $i=1, \cdots, n$, and $y_i(t) = \eta_i + (t-\tau)f_i(\tau, \eta_1, \cdots, \eta_n) + (\gamma/2)^2(t-\tau)^2 sgn(t-\tau)$, $i=1, \cdots, n$. These functions will then extend over R to two different solutions.[6] Since $\epsilon > 0$ was arbitrary, $\mathfrak{T}_2$ is dense in $\mathfrak{T}$. To see the significance of Theorem 5 we remark that the set of $F \in C_n(R)$ satisfying a uniform Lipschitz condition on R with a fixed constant $m > 0$, that is, those $(f_1, \cdots, f_n)$ with $|f_i(t, \bar{y}_1, \cdots, \bar{y}_n) - f_i(t, y_1, \cdots, y_n)| \leq m \cdot \sum_{i=1}^{n} |\bar{y}_i - y_i|$ on R, is closed in $C_n(R)$. Consequently, the set $\mathfrak{A}(m)$ of transformations associated with an arbitrary $P \in R$ and any F in the class above is closed in $\mathfrak{T}$. Moreover, as is well known, $\mathfrak{A}(m) \subset \mathfrak{T}_1$. But then, from Theorem 5, $\mathfrak{A}(m)$ is nowhere-dense in $\mathfrak{T}$. Since the class of transformations with the associated F satisfying a uniform Lipschitz condition on R for any $m > 0$ is given by $\sum_{m=1}^{\infty} \mathfrak{A}(m)$, this class is of the first category in $\mathfrak{T}$. This theorem can be weakened to the following: The class $\mathfrak{B}$ of transformations such that F satisfies a uniform Lipschitz condition in some neighborhood containing P is of the first category in $\mathfrak{T}$. This does not follow directly from Theorem 5 for this class is not contained in $\mathfrak{T}_1$. However, $\mathfrak{B}=\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\mathfrak{B}(m, r)$ where $\mathfrak{B}(m, r)$ is the class of transformations for which F satisfies a uniform Lipschitz condition with constant m in a neighborhood of radius $1/r$ about the point P associated with T. The class $\mathfrak{B}(m, r)$ may be shown to be closed. Inspection of the functions $(g_1, \cdots, g_n)$ defined above shows that the complement of $\mathfrak{B}(m, r)$ is dense in $\mathfrak{T}$ and, consequently, that $\mathfrak{B}(m, r)$ is nowhere-dense in $\mathfrak{T}$, and $\mathfrak{B}$ of the first category in $\mathfrak{T}$. Not only for uniform Lipschitz conditions, but also for more general uniqueness criteria is it true that the class of functions satisfying them is closed. Without going into details we mention the criterion given by Kamke[7] which, when satisfied at every point of R, gives a class of functions closed in $C_n(R)$ and consequently a subclass of $\mathfrak{T}_1$ nowhere-dense in $\mathfrak{T}$. These theorems are also closely related to theorems of Orlicz[8] where again the present theorems differ from Orlicz's in the way described above.

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[5] This theorem is given by Kamke, *Differentialgleichungen reeler Funktionen*, Leipzig, 1930, p. 149, Theorems 3 and 4.

[6] Kamke, loc. cit. p. 135, Theorem 2.

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[7] Kamke, loc. cit. p. 139, Theorem 3.

[8] Orlicz, loc. cit. pp. 226–228.

DISTRIBUTIVE PROPERTIES OF SET OPERATORS

WILLIAM WERNICK

A set operator, designated by small letters, $a, b, \cdots$, is one which takes subsets $A, B, \cdots$ of a given space S into subsets $aA, aB, \cdots \subset S$. A *property* of a set operator a is a constant relation between argument and image sets under a and is expressed in a statement of equation or inclusion, for example, $a(A+B)=aA\cdot aB$ or $A\subset bA$ where a, b may mean, for example, "complement" or "closure."

We investigate properties expressed by relations of the form:

$$a(Af_1B)\cdot\mathfrak{R}: aAf_2aB \tag{1}$$

where f_1, f_2 are either set sum: $+$, or set product: $\cdot$, and where $\mathfrak{R}$ is either $=$, $\supset$, or $\subset$. A property defined by such a relation (1) is a distributive property, but not all distributive properties can be defined by (1), for example $a(A+B)=A\cdot aB+B\cdot aA$, and so on.

When $f_1, f_2, \mathfrak{R}$ are given constant values, (1) becomes the statement of a specific distributive property of a. We now list them individually for reference. Properties of *monotonicity* and *inverse-monotonicity* (α_{13} and α_{14} below) are closely related to properties of distributivity so they are listed in the table also. (The arrow, →, is used for implication throughout this paper.)

TABLE I

α_1: $a(A+B)=aA+aB$
α_2: $a(A+B)\supset aA+aB$
α_3: $a(A+B)\subset aA+aB$
α_4: $a(A\cdot B)=aA\cdot aB$
α_5: $a(A\cdot B)\supset aA\cdot aB$
α_6: $a(A\cdot B)\subset aA\cdot aB$

α_7: $a(A+B)=aA\cdot aB$
α_8: $a(A+B)\supset aA\cdot aB$
α_9: $a(A+B)\subset aA\cdot aB$
α_{10}: $a(A\cdot B)=aA+aB$
α_{11}: $a(A\cdot B)\supset aA+aB$
α_{12}: $a(A\cdot B)\subset aA+aB$

α_{13}: $A\subset B\rightarrow aA\subset aB$
α_{14}: $A\subset B\rightarrow aA\supset aB$

To say that a has property α_1 (notation $a:\alpha_1$) means: "For every A, B, $a(A+B)=aA+aB$." These properties α_i are obviously not independent, for example, $a:\alpha_1\rightarrow a:\alpha_2, \alpha_3$ (which we may shorten, at our convenience, to $\alpha_1\rightarrow\alpha_2, \alpha_3$).

Our first main question is: if we hypothesize to a a single property α_i, what other properties *must* a have? This is completely answered by the following diagram of implications:

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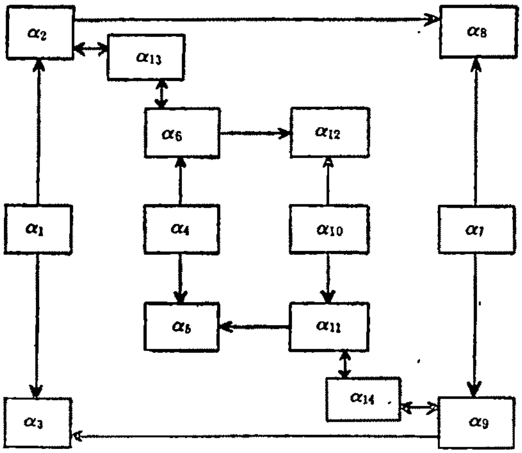
I



Since most of these implications are quite obvious, I present only two proofs which are models for the not too obvious implications. Introduction of the hypothesis is indicated by subscript *h*.

$$\alpha_2 \rightarrow \alpha_{13}\colon\ A \subset B \rightarrow B = A + B \rightarrow aB = a(A+B) \supset_h aA + aB$$
$$\supset aA \rightarrow aA \subset aB$$

$$\alpha_{14} \rightarrow \alpha_{11}\colon\ [A \cdot B \subset A \rightarrow a(A\cdot B) \supset_h aA;$$
$$A \cdot B \subset B \rightarrow a(A\cdot B) \supset_h aB] \rightarrow a(A\cdot B) \supset aA + aB.$$

Sometimes we know or wish to assume that a given operator a does *not* have a specific property α_i (notation $\overline{a\colon\alpha_i}$) and we wish to know the implications of this.

We define the property $\bar{\alpha}_i$ thus: $a\colon\bar{\alpha}_i$ if and only if $\overline{a\colon\alpha_i}$; and easily get the implications for $\bar{\alpha}_i$ from diagram I by replacing α_i by $\bar{\alpha}_i$ and reversing every arrow. The resulting diagram, which will not be drawn, can be called II.

Our second main question is: if $a\colon\alpha_i$ what other properties *may* be hypothesized for a (from now on, unless otherwise noted, "property," and "α_i" will include the properties $\bar{\alpha}_i$).

The *distributive character* of an operator a is (said to be) determined with respect to α_i (notation $a\colon i$) if it is known definitely that $a\colon\alpha_i$ or that $a\colon\bar{\alpha}_i$. Otherwise the distributive character of a is not determined with respect to α_i (notation $a\colon\bar{i}$) and we may then hypothesize $a\colon\alpha_i$ (or $a\colon\bar{\alpha}_i$), provided there is no subsequent contradiction between properties already hypothesized for a and the consequences of the joint hypothesis of α_i (or $\bar{\alpha}_i$) and these properties.[1]

All the implications from the joint hypothesis of any of the α_i are derivable from the implication diagrams I and II, already given, and III, which is indicated below:

III

$$\alpha_2,\ \alpha_3 \rightarrow \alpha_1; \qquad \alpha_5,\ \alpha_6 \rightarrow \alpha_4; \qquad \alpha_8,\ \alpha_9 \rightarrow \alpha_7; \qquad \alpha_{11},\ \alpha_{12} \rightarrow \alpha_{10}.$$

These implications are immediate consequences of the definitions in Table I and the usual meanings for set sum, product, and inclusion.

If $a:\alpha_i$, then from I and II we can easily determine all the j for which $a:\bar{j}$. If $a:\alpha_i$ and $a:\bar{j}$ then we test the self-consistency of $a:\alpha_i,\ \alpha_j$, and of $a:\alpha_i,\ \bar{\alpha}_j$, using I, II and III. If both of these are possible (that is, self-consistent) then α_j is independent of α_i. If, furthermore, $a:\bar{\alpha}_i,\ \alpha_j$ and $a:\bar{\alpha}_i,\ \bar{\alpha}_j$ are also possible, then α_i and α_j are completely independent of each other. We investigate the complete existential theory of all the α_i, following the procedure outlined above, so that if $a:\alpha_i,\ \alpha_j,\ \bar{k}$, we shall investigate $a:\alpha_i,\ \alpha_j,\ \alpha_k$, and $a:\alpha_i,\ \alpha_j,\ \bar{\alpha}_k$; and so on.

If, finally, $a:i$ for $i=1, \cdots, 14$, then we say that the distributive character of a is completely determined. It is easily seen that the distributive character of an operator can be completely determined in a variety of ways, for example, if $a:\alpha_1,\ \alpha_5$, then $a:i$ for every i; likewise if $a:\alpha_8,\ \alpha_{10}$. The selection of properties that completely determine the distributive character of an operator can be said to define a particular distributive *type*, thus two operators of the same type possess exactly the same selection of properties $\{\alpha_i\}$. The complete existential theory of these properties as well as all questions on their interdependence will be determined as soon as we find all possible distributive types.[2]

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[1] The logical distinction between "$(a:\alpha_i)$ or $(a:\bar{\alpha}_i)$" and "$a:(\alpha_i$ or $\bar{\alpha}_i)$" should be noted. To determine the truth of either "It is now snowing at the North Pole" or "It is now not snowing at the North Pole" would require considerable effort whereas I already know that "It is now either snowing or not snowing at the North Pole."

[2] I have made an attempt to list all distinct partially determined (p.d.) operators. With the assumption of one α_i, there are 20 different p.d. operators. With the assumption of two independent α_i, there are 5 completely determined (c.d.)and 99 p.d. operators. If we assume three independent α_i, there are 4 more c.d. and 185 more p.d. operators. If we assume four independent α_i, we know, from Table II below, that there are 8 more c.d. operators, but I have not determined the number of p.d. operators with 4 or more independent α_i assumed. The enumeration is simple and cumulative but tedious and not especially rewarding. The total number of p.d. operators is probably less than 500, but I see no importance in their exact number or constituency so they are not listed here.

The distributive types have been determined, and they are exhibited in Table II below. Each line represents a single type T_i. A "+" in row i, column j indicates that an operator of type T_i is assumed to have property α_j; a "−" indicates that the operator is assumed to have property $\bar{\alpha}_j$.

TABLE II

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	Defining Sets	
T_1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	1, 7	(4)
T_2	+	+	+	+	+	+	−	+	−	−	−	+	+	−	1, 4, $\bar{9}$	(1)
T_3	+	+	+	−	−	+	−	+	−	−	−	+	+	−	1, $\bar{5}$	(1)
T_4	−	+	−	+	+	+	−	+	−	−	−	+	+	−	$\bar{3}$, 4	(1)
T_5	−	+	−	−	−	+	−	+	−	−	−	+	+	−	2, $\bar{3}$, $\bar{5}$	(3)
T_6	−	−	+	−	+	−	+	+	+	+	+	+	−	+	$\bar{2}$, 7, 10	(3)
T_7	−	−	+	−	+	−	+	+	+	−	+	−	−	+	7, $\overline{12}$	(1)
T_8	−	−	+	−	+	−	−	−	+	+	+	+	−	+	$\bar{8}$, 10	(1)
T_9	−	−	+	−	+	−	−	−	+	−	+	−	−	+	$\bar{8}$, 9, $\overline{12}$	(3)
T_{10}	−	−	+	−	+	−	−	+	−	−	−	+	−	−	$\bar{2}$, 3, 5, 8, $\bar{9}$, 12	(9)
T_{11}	−	−	+	−	+	−	−	+	−	−	−	−	−	−	3, 5, 8, $\bar{9}$, $\overline{12}$	(3)
T_{12}	−	−	+	−	+	−	−	−	−	−	−	+	−	−	3, 5, $\bar{8}$, $\bar{9}$, 12	(3)
T_{13}	−	−	+	−	+	−	−	−	−	−	−	−	−	−	3, 5, $\bar{8}$, $\bar{9}$, $\overline{12}$	(3)
T_{14}	−	−	+	−	−	−	−	+	−	−	−	+	−	−	$\bar{2}$, 3, $\bar{5}$, 8, 12	(3)
T_{15}	−	−	+	−	−	−	−	+	−	−	−	−	−	−	3, $\bar{5}$, 8, $\bar{9}$, $\overline{12}$	(3)
T_{16}	−	−	+	−	−	−	−	−	−	−	−	+	−	−	3, $\bar{5}$, $\bar{8}$, 12	(1)
T_{17}	−	−	+	−	−	−	−	−	−	−	−	−	−	−	3, $\bar{5}$, $\bar{8}$, $\overline{12}$	(1)
T_{18}	−	−	−	−	+	−	−	+	−	−	−	+	−	−	$\bar{2}$, $\bar{3}$, 5, 8, 12	(3)
T_{19}	−	−	−	−	+	−	−	+	−	−	−	−	−	−	$\bar{3}$, 5, 8, $\overline{12}$	(1)
T_{20}	−	−	−	−	+	−	−	−	−	−	−	+	−	−	$\bar{3}$, 5, $\bar{8}$, 12	(1)
T_{21}	−	−	−	−	+	−	−	−	−	−	−	−	−	−	$\bar{3}$, 5, $\bar{8}$, $\overline{12}$	(1)
T_{22}	−	−	−	−	−	−	−	+	−	−	−	+	−	−	$\bar{2}$, $\bar{3}$, $\bar{5}$, 8, 12	(3)
T_{23}	−	−	−	−	−	−	−	+	−	−	−	−	−	−	$\bar{3}$, $\bar{5}$, 8, $\overline{12}$	(1)
T_{24}	−	−	−	−	−	−	−	−	−	−	−	+	−	−	$\bar{3}$, $\bar{5}$, $\bar{8}$, 12	(1)
T_{25}	−	−	−	−	−	−	−	−	−	−	−	−	−	−	$\bar{3}$, $\bar{5}$, $\bar{8}$, $\overline{12}$	(1)

Questions of redundancy, though not essential here, have a certain esthetic appeal, and also a later practical use. It is always possible, for each T_i, to make a selection of properties from among the 14 on that line, which have the following 2 properties collectively:

1. Sufficiency. All the other properties on that line can be deduced from those of this selection (that is, set) of properties.

2. Non-redundancy. No property of this set can be deduced from the other properties of this set.

Such a set will be called a "defining set" of properties for that type. All defining sets for each type have been determined. In the last column of Table II, a defining set is given for each type, and, in parentheses, the total number of such sets for that type. (For purposes of brevity, in this column only, we have replaced α_i by i). The equivalence (as shown in diagram I) of α_2, α_{13}, α_6 or of α_9, α_{14}, α_{11} is a frequent reason for the multiplicity of defining sets for a particular type.

From Table II we can quickly determine whether or not two properties are completely independent; for example, α_7 and α_{10}, both of which are enjoyed by the operator "complement," are completely independent, since, from T_6, T_7, α_{10} is independent of α_7, and from T_6, T_8, α_7 is independent of α_{10}. Similarly we could determine the complete independence for groups of properties.

We may show that a given set is a defining set by proving sufficiency and non-redundancy; for example, for T_7 we test $\{\alpha_7, \bar{\alpha}_{12}\}$. From I, $\alpha_7 \rightarrow \alpha_8$, α_9, α_3, α_{14}, α_{11}, α_5; from II, $\bar{\alpha}_{12} \rightarrow \bar{\alpha}_{13}$, $\bar{\alpha}_6$, $\bar{\alpha}_4$, $\bar{\alpha}_{13}$, $\bar{\alpha}_2$, $\bar{\alpha}_1$; therefore this set is sufficient. From Table II there exist types T_6, T_7, T_8, T_9, which show the complete independence of α_7, α_{12}, therefore this set is non-redundant and thus a defining set.

Our second main question: "If $a:\alpha_i$ what other properties *may* a have?" is obviously answered in Table II.

We shall not prove here that these are the only possible types (which could be done from our definitions) but shall show by examples that functions exist for each of the 25 types. We use mainly a Boolean algebra of four elements (for T_{13} and T_{22} we require eight elements) in which 1 is the universal, 0 is the null, and 2, 3 are the mutually exclusive, exhaustive elements. The tables for sum, product and inclusion are indicated below:

+	1	2	3	0
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
0	1	2	3	0

·	1	2	3	0
1	1	2	3	0
2	2	2	0	0
3	3	0	3	0
0	0	0	0	0

⊂	1	2	3	0
1	*Y*	*N*	*N*	*N*
2	*Y*	*Y*	*N*	*N*
3	*Y*	*N*	*Y*	*N*
0	*Y*	*Y*	*Y*	*Y*

An operator for each type will be given by listing the values for $a(1)$, $a(2)$, $a(3)$, $a(0)$ in that order. Thus, from the table below, an example of an operator of type T_1 is the operator for which $a(1)=1$ $a(2)=1$, $a(3)=1$, $a(0)=1$, and so on.[3]

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[3] The simplest way to check these is to use the defining sets given in Table II, since if an operator has the properties required in the defining set of T_k, then it must, from the definition of "defining set," have all the other properties required for an operator of type T_k.

T_1 : (1, 1, 1, 1)
T_2 : (1, 2, 3, 0)
T_3 : (1, 1, 1, 0)
T_4 : (1, 0, 0, 0)
T_5 : (1, 2, 2, 0)
T_6 : (0, 3, 2, 1)
T_7 : (0, 0, 0, 1)
T_8 : (0, 1, 1, 1)
T_9 : (0, 2, 2, 1)
T_{10}: (2, 1, 0, 0)
T_{11}: (2, 2, 0, 1)
T_{12}: (2, 1, 3, 1)
T_{13}: given below
T_{14}: (2, 2, 1, 3)
T_{15}: (2, 2, 2, 3)
T_{16}: (0, 1, 2, 0)
T_{17}: (0, 3, 3, 2)
T_{18}: (1, 0, 2, 2)
T_{19}: (1, 0, 0, 1)
T_{20}: (2, 3, 3, 3)
T_{21}: (2, 3, 3, 1)
T_{22}: given below
T_{23}: (1, 3, 3, 2)
T_{24}: (3, 2, 2, 0)
T_{25}: (3, 2, 2, 3)

The 4-element algebra is provably inadequate for an example of T_{13} and T_{22}, so we use 8 elements, 1, 2, 3, 4, 5, 6, 7, 0, where 1 is the universal, 0 is the null element, 2, 3, 4 are the mutually exclusive and exhaustive elements, and $5=2+3$, $6=2+4$, $7=3+4$. The tables for $+$, $\cdot$, and $\subset$ for the algebra can be filled in very easily from the above description, so they will not be given. The operator definitions are now given as before by listing, in order, the value of $a(1)$, $a(2)$, $\cdots$, $a(0)$.

T_{13}: (2, 2, 3, 4, 5, 6, 7, 7); T_{22}: (7, 2, 3, 2, 3, 6, 7, 2)

Proofs that these operators have the properties listed for their respective types will not be given since they are lengthy and present no special difficulties.[4]

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[4] See previous footnote for easiest method of proof.

SOME REMARKS ON ALMOST PERIODIC TRANSFORMATIONS

P. ERDÖS AND A. H. STONE

In a recent paper in this Bulletin (see [3][1]), W. H. Gottschalk has proved a number of interesting theorems on "recurrent" and "almost periodic" homeomorphisms of a space on itself. In the first part of the present note we give very simple proofs of some of Gottschalk's theorems in an even more general form. In the second half we consider "regular" transformations in more detail.

1. Recurrent and almost periodic transformations.

Notations. Let f be a continuous mapping (not necessarily a homeomorphism) of a topological space X in itself (that is, $f(X)\subset X$). We say that f is *recurrent* at a point $x\in X$, or that x is *recurrent* under f, if, given any neighbourhood $U(x)$ of x, there exist infinitely many positive integers n for which $f^n(x)\in U(x)$. (This definition is equivalent to Gottschalk's if X is a T_1 space.) Further, f is *almost periodic* at x if, given any $U(x)$, there exists an $N(x, U(x))>0$ such that for the (infinite) sequence $\{n_i\}$ of positive integers for which $f^{n_i}(x)\in U(x)$ we have $n_{i+1}-n_i\leqq N$.

THEOREM I. *If a continuous mapping f of a topological space X in itself is either* (a) *recurrent, or* (b) *almost periodic, at x, then so is f^k, for each positive integer k.*[2]

PROOF. Let N_r denote the class of positive integers congruent to r mod k. We may clearly assume that one at least of the classes $N_1, N_2, \cdots, N_{k-1}$, say N_r, satisfies: every neighbourhood $U(x)$ of x contains $f^n(x)$ for infinitely many values of $n\in N_r$; for otherwise each $U(x)$ will contain $f^n(x)$ for all large enough $n\in N_k$, and the theorem will follow trivially.

Now let U_0 be any given open set containing x. Choose $n_1\in N_r$ such that $f^{n_1}(x)\in U_0$. Since f^{n_1} is continuous, there exists an open set $U_1\ni x$ such that $U_1\subset U_0$ and $f^{n_1}(U_1)\subset U_0$. Choose $n_2\in N_r$ such that $f^2(x)\in U_1$; and so on. In this way, we define integers $n_1, \cdots, n_{k-1}\in N_r$ and open sets $U_1\supset U_2\supset\cdots\supset U_{k-1}\ni x$ such that $f^{n_i}(x)\in U_{i-1}$ and $f^{n_i}(U_i)\subset U_{i-1}$.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Theorem I(a) is Theorem 1 of [3], without the restriction that f be a homeomorphism. Theorem I(b) is Theorem 6 of [3], without the restrictions that f be a homeomorphism and that X be compact.

In case (a), we observe that for infinitely many integers $m\in N_r$ we have $f^m(x)\in U_{k-1}$. For each such m, consider $M=n_1+n_2+\cdots+n_{k-1}+m$; clearly $M\equiv 0 \bmod k$, and it is easy to see that $f^M(x)\in U_0$. Since there are infinitely many values of M, and U_0 was any open set containing x, this proves that f^k is recurrent at x.

In case (b), there will be an infinite sequence $\{m_i\}$ of positive integers (not necessarily in N_r) such that $f^{m_i}(x)\in U_{k-1}$ and $\sup(m_{i+1}-m_i)<\infty$. Now assume temporarily that k is prime. Then, for each m_i, one of the k incongruent integers $m_i, m_i+n_{k-1}, \cdots, m_i+n_{k-1}+n_{k-2}+\cdots+n_2+n_1$ will be congruent to 0 mod k; call this integer M_i. It is easy to see that $f^{M_i}(x)\in U_0$ and that $\sup(M_{i+1}-M_i)<\infty$. On rearranging the sequence $\{M_i\}$ in increasing order of magnitude, we see that the theorem is proved in this case—if k is prime. And the theorem follows in general, by induction over k.

THEOREM II. *Let f be a homeomorphism of a connected topological space X on itself ($f(X)=X$). If a point x, recurrent under f, separates two other recurrent points in X, then f is periodic at x.*[3]

PROOF. Let $X-(x)=A\cup B$, where A, B are mutually separated sets and A, B contain the recurrent points y, z respectively. Thus $\overline{A}=A\cup(x)$, $\overline{B}=B\cup(x)$, and the sets $\overline{A}$ and $\overline{B}$ are connected. If the theorem is false, then for every positive integer n we have $f^n(x)\neq x$; from this we shall derive a contradiction.

We first show that a positive integer k exists such that:

(1) $f^k(\overline{A})$ meets $\overline{A}$ and $f^k(\overline{B})$ meets $\overline{B}$.

For suppose not. Without loss of generality, we may assume that $f(x)\in A$. We assert that:

(2) $f^n(\overline{A})$ meets A, for every positive integer n.

For this is true when $n=1$, since $f(\overline{A})\ni f(x)$. Suppose (2) is true when $n=m$. Then, since (1) is false, $f^m(\overline{B})\cap\overline{B}=0$, and so $f^m(\overline{B})\subset A$. Hence, since f^m is a 1-1 mapping, $f^m(A)\supset\overline{B}$, so that $f^{m+1}(\overline{A})\supset f^{m+1}(A)\supset f(\overline{B})\ni f(x)\in A$. Thus (2) follows for $n=m+1$, and therefore holds for all n, by induction. Hence, since (1) is false, we have from (2) that $f^n(\overline{B})\cap\overline{B}=0$ for every positive integer n. This contradicts the fact that f is recurrent at $z\in B$; and so (1) is proved.

Again without loss of generality, we may assume now that $f^k(x)\in A$. Thus $f^k(\overline{B})$ meets A; but, from (1), $f^k(\overline{B})$ also meets $\overline{B}$, and therefore (being connected) contains x. Since f^k is 1-1, we thus have $x\notin f^k(A)$, and therefore $x\notin f^k(\overline{A})$. Therefore the connected set

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[3] Essentially [3, Theorem 2], without compactness restrictions. The proof as given assumes (x) is closed, but could easily be modified so as to dispense with this assumption.

$f^k(\overline{A})$, which meets A (in $f^k(x)$ at least), cannot meet $\overline{B}$, so that $f^k(\overline{A}) \subset A$. It readily follows that, for every positive integer n, $f^{nk}(\overline{A}) \subset f^k(\overline{A}) \subset A$. Since $f^k(\overline{A})$ is a closed set not containing x, this contradicts the fact that (by Theorem I) the mapping f^k is recurrent at x.

2. Regular and strongly almost periodic transformations. Now let X be a metric space. A mapping $f(X) \subset X$ is said to be *strongly almost periodic* (cf. [3]—or "almost periodic," in [1]) if, given $\epsilon > 0$, there exists an integer $L > 0$ such that every set of L consecutive positive integers contains an n satisfying $\rho(x, f^n(x)) < \epsilon$ for all $x \in X$.

THEOREM III. *If X is a totally bounded metric space, and if f is a homeomorphism of X in itself all of whose negative powers are equi-uniformly continuous, then f is strongly almost periodic.*[4]

PROOF. By hypothesis, given $\epsilon > 0$, there exists a $\delta > 0$ such that $\rho(f^{-m}(x), f^{-m}(y)) < \epsilon/2$ whenever m is a positive integer and x, y are points of $f^m(X)$ such that $\rho(x, y) < \delta$. We may clearly assume that $\delta < \epsilon/2$.

Let $A_1, \cdots, A_r$ be a finite covering of X by sets of diameter less than δ. For each positive integer m we define a square matrix $B(m)$ of order r by setting $b_{ij}(m) = 1$ if $f^m(A_i)$ meets A_j, 0 otherwise. Of all the matrices $B(1), B(2), \cdots$, only a finite number can be distinct; let $B(1), \cdots, B(L)$ include all the distinct matrices $B(m)$. We shall prove the theorem by showing that, given any positive integer M, an integer n exists such that: (a) $M \leqq n < M+L$; (b) $\rho(x, f^n(x)) < \epsilon$, for all $x \in X$.

In fact, $B(M+L) = B(m)$ for some $m \leqq L$. Define $n = M+L-m$; thus (a) is certainly true. To verify (b), let $x \in X$ be given, and suppose $x \in A_i$, $f^{M+L}(x) \in A_j$. Then $b_{ij}(M+L) = 1$, and so $b_{ij}(m) = 1$ also; thus $f^m(A_i)$ meets A_j. Let $y \in A_j \cap f^m(A_i)$. Then $\rho(x, f^n(x)) \leqq \rho(x, f^{-m}(y)) + \rho(f^{-m}(y), f^{M+L-m}(x)) \leqq \delta(A_i) + \delta(f^{-m}(A_j))$ (where $\delta(A)$ denotes the diameter of A) $< \delta + \epsilon/2 < \epsilon$. Q.E.D.

A *homeomorphism* f of a metric space in itself is said to be *regular* if all its powers (positive or negative) are equi-uniformly continuous. Thus every isometry is regular; and conversely it is easy to see that if a regular homeomorphism f maps the space *on* itself (as it must if the space is compact), then the space can be remetrized so that f becomes an isometry.

THEOREM IV. *Let f be a regular homeomorphism of a totally bounded*

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[4] This is closely related to Theorems 5 and 6 of [2, pp. 701, 702]. It would be desirable to weaken the equi-continuity assumption on the powers of f.

metric space in itself. There exists an increasing sequence $\{n_i\}$ *of positive integers such that* $f^{n_i}(x)\rightarrow x$ *and* $f^{(n_i)^2}(x)\rightarrow x$, *uniformly, for all* $x\in X$.[5]

Proof. Given $\epsilon>0$, we must show that arbitrarily large integers n exist such that, for all $x\in X$, $\rho(f^n(x), x)<\epsilon$ and $\rho(f^{n^2}(x), x)<\epsilon$. Now, from Theorem III, there exists a sequence $\{m'\}$ such that $f^{m'}(x)\rightarrow x$ uniformly. We shall first show that a sufficiently rapidly increasing subsequence $\{m_i\}$ of $\{m'\}$ will satisfy (for all $x\in X$, and every pair of integers r, s with $0<r<s$) both

$$(1) \qquad \rho(f^{(m_r+m_{r+1}+\cdots+m_s)^2}(x), \quad f^{m_r^2+m_{r+1}^2+\cdots+m_s^2}(x)) < \epsilon/2$$

and

$$(2) \qquad \rho(f^{(m_r+m_{r+1}+\cdots+m_s)}(x), x) < \epsilon.$$

For, by hypothesis, there is a positive function $\delta(\epsilon)$ of the positive variable ϵ such that $\rho(f^m(x), f^m(y))<\epsilon/2$ whenever $\rho(x, y)<\delta(\epsilon)$ and $f^m(x)$, $f^m(y)$ are defined ($m=0, \pm 1, \pm 2, \cdots$). Choose m_1 to be an m' so large that $\rho(f^{m_1}(x), x)<\epsilon/2$ for all $x\in X$. Then define m_s inductively to be an m' so large that: (a) $\rho(f^{m_s}(x), x)<\epsilon/2^s$, and (b) $\rho(f^{2(m_r+\cdots+m_{s-1})m_s}(x), x)<\delta(\epsilon/2^s)$, for all $x\in X$ and every r between 1 and $s-1$. Clearly (2) follows from (a); and (1) follows from (b), since (by the triangle law)

$$\begin{aligned}
&\rho(f^{(m_r+m_{r+1}+\cdots+m_s)^2}(x), f^{m_r^2+m_{r+1}^2+\cdots+m_s^2}(x)) \\
&\qquad \leqq \sum_{t=r}^{s-1} \rho\left\{f^{(m_r+\cdots+m_t)^2+m_{t+1}^2+\cdots+m_s^2}(x), \quad f^{(m_r+\cdots+m_{t+1})^2+\cdots+m_s^2}(x)\right\} \\
&\qquad < \sum \epsilon/2^{t+2} < \epsilon/2.
\end{aligned}$$

Now let $A_1, \cdots, A_k$ be a finite covering of X by sets of diameter less than $\min\{\delta(\delta(\epsilon)), \delta(\epsilon)/2\}$; and, as in the proof of Theorem III, let $B(n)$ denote the square matrix of order k formed by setting $b_{ij}(n)=1$ if $f^n(A_i)$ meets A_j, 0 otherwise. Consider the sequence of matrices

$$B(m_1^2),\ B(m_1^2+m_2^2), \cdots, B(m_1^2+m_2^2+\cdots+m_t^2), \cdots.$$

For some two integers r, s with $0<r<s$ we must have $B(m_1^2+\cdots+m_r^2)=B(m_1^2+\cdots+m_s^2)$. An easy calculation (cf. the end of the proof of Theorem III) now shows that, for all $x\in X$, $\rho(f^{m_1^2+\cdots+m_r^2}(x), f^{m_1^2+\cdots+m_s^2}(x))<\delta(\epsilon)$, so that $\rho(f^{m_{r+1}^2+\cdots+m_s^2}(x), x)<\epsilon/2$. Combining this with (1), and writing $n = m_{r+1} + \cdots + m_s$, we see that $\rho(f^{n^2}(x), x) < \epsilon$. Since we also have $\rho(f^n(x), x) < \epsilon$, from (2), the theorem is proved.

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[5] This is essentially a generalization of a theorem of Hardy and Littlewood on the denseness of the fractional parts of $\{n^2\alpha\}$, α irrational. See [4, p. 157]. It would be desirable to weaken the hypothesis of regularity.

In the same way (using induction over k) one could prove that, given any positive integer k, there exists an increasing sequence $\{n_i\}$ of positive integers for which, simultaneously, $f^{n_i}(x) \to x$, $f^{n_i^2}(x) \to x, \cdots, f^{n_i^k}(x) \to x$, uniformly, for all $x \in X$. In fact, substantially the same argument will show that this holds if each power n^t $(t = 1, \cdots, k)$ is replaced by any polynomial $\phi_t(n)$ of degree t, having integer coefficients, positive leading coefficient, and zero constant term.

These results provide partial answers to the question, raised by Theorem I, as to what can be said about the sequence of integers n for which $f^n(x)$ is in a given neighbourhood of x, f being recurrent at x. Thus, under the hypotheses of Theorem IV, this sequence contains infinitely many squares, and in fact infinitely many kth powers. Query: Will it (under "reasonable" hypotheses—for example, non-periodicity) contain infinitely many primes? (For the special case in which X is the real line mod 1, and $f(x) = x + \alpha$, where α is a fixed irrational number, the answer is affirmative, though the proof is difficult.[6])

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[6] This follows from results of Vinogradoff; cf. [6]. See also [5, p. 234].

ON THE LEAST PRIMITIVE ROOT OF A PRIME p

P. ERDÖS

Vinogradoff[1] proved that the least primitive root of a prime p is less than $2^m p^{1/2} \log p$ for sufficiently large p; m denotes the number of different prime factors of $p-1$. Later he improved this to $2^m p^{1/2} \log \log p$.[2] Hua[3] improved this to $2^{m+1}p^{1/2}$. In the present note we are going to prove that the least primitive root is less than $p^{1/2}(\log p)^{17}$ for large p. This result is better than Hua's if $p-1$ has "many" prime factors, but worse if it has "few" prime factors. We shall use Brun's method. It is very likely that the least primitive root is $o(p^\epsilon)$ but I can not even prove that it is less than $cp^{1/2}$.

Lemma 1. *Let* $x<p$, $k\,|\,p-1$. *Denote by* $f_k(x)$ *the number of kth power residues not exceeding* x. *Then* $f_k(x)=x/k+O(p^{1/2}\log p)$.

Denote by $\chi_0(a), \chi_1(a), \cdots, \chi_{k-1}(a)$ the characters for which $[\chi_i(a)]^k=1$. $\chi_0(a)$ is the principal character. Clearly $\sum_{i=0}^{k-1}[\chi_i(a)]=k$ if a is a kth power residue, and is 0 otherwise. Thus

$$\frac{1}{k}\sum_{i=0}^{k-1}\sum_{a=1}^{x}\chi_i(a) = f_k(x). \tag{1}$$

On the other hand by a well known result of Pólya[4]

$$\left|\sum_{a=1}^{x}\chi_i(a)\right| < p^{1/2}\log p \qquad (i \neq 0). \tag{2}$$

Thus combining (1) and (2) we obtain

$$f_k(x) = x/k + c(p^{1/2}\log p), \qquad |c| \leqq 1, \tag{3}$$

which proves the lemma.

Denote by $F(x)$ the number of primitive roots not exceeding x. We clearly have by the sieve of Eratosthenes $F(x)=\sum_{d|p-1}\mu(d)f_d(x)$. Hence clearly

$$F(x) > \Bigl(x - {\sum_{q|p-1}}' f_q(x) + {\sum_{q_1,q_2|p-1,\,q_1\neq q_2}}' f_{q_1,q_2}(x) - \cdots\Bigr) - \sum_{q|p-1,\,q>(\log p)^3} f_q(x) = \sum_1 - \sum_2,$$

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[1] Landau, *Vorlesungen über Zahlentheorie*, vol. 2, p. 178.

[2] C. R. (Doklady) Acad. Sci. URSS (1936) pp. 7–11.

[3] Bull. Amer. Math. Soc. vol. 48 (1942) pp. 726–730.

[4] Landau, ibid. p. 178.

where the q's are primes and the dashes in $\sum_1$ indicate that the summation is extended over only the $q < (\log p)^3$. By (3)

$$\sum_2 < \sum_{q|p-1,\, q>(\log p)^3} \frac{x}{q} + 2p^{1/2}(\log p)^2 < \frac{4x}{(\log p)^2} \tag{4}$$

for $x > p^{1/2}(\log p)^4$, since the number of different prime factors of $p-1$ is less than $2 \log p$.

Now we estimate $\sum_1$. Here we use Brun's method. If we replace $f_k(x)$ by x/k the error we make is less than $p^{1/2} \log p$ by (3). Thus following Brun's[5] reasoning (p. 23, equation 21) we obtain

$$\begin{aligned} \sum_1 &> x \cdot 0.3 \prod_{q|p-1,\, q<(\log p)^3} \left(1 - \frac{1}{q}\right) - c_1(\log p)^{15} p^{1/2} \log p \\ &> c_2 \frac{x}{\log \log p} - c_1 p^{1/2}(\log p)^{16}.\text{[6]} \end{aligned} \tag{5}$$

We do not give the details since the argument follows literally from Brun's original argument. We make only the following remark. Put

$$A = x - \sum_{q_1|p-1}' \left[\frac{x}{q_1}\right] + \sum_{q_1, q_2|p-1,\, q_1 \neq q_2}' \left[\frac{x}{q_1 q_2}\right] - \cdots .$$

Then Brun obtains (p. 23, equation 21)

$$A > x \cdot 0.3 \prod_{q|p-1,\, q<(\log p)^3} \left(1 - \frac{1}{q}\right) - c_1(\log p)^{15}.$$

The error term $c_1(\log p)^{15}$ comes from replacing in $(\log p)^{15}$ terms $[x/k]$ by x/k (see p. 22). Our error term $c_1 p^{1/2}(\log p)^{16}$ comes from replacing in $(\log p)^{15}$ terms $f_k(x)$ by x/k. Thus we obtain from (4) and (5)

$$F(x) > c_2 x/\log \log p - c_1 p^{1/2}(\log p)^{16} - 4x/(\log p)^2 > 0$$

for $x > p^{1/2}(\log p)^{17}$, and p sufficiently large, which completes the proof.

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[5] *Le crible d'Eratosthene et la theoreme de Goldbach*, Skrifter utgit av Videnskapsselskapets I. Christiania Matematich Naturvidenskabelig Klasse (1920) No. 3. See in particular p. 23, formula (21), and the preceding pages. The prime p_r in that formula is less than $(\log p)^3$ here.

[6] We have $\prod_{p<y}(1-1/p) > c/\log y$. See, for example, Hardy-Wright, *Theory of numbers*, p. 349.

ON A SPECIAL CLASS OF ABELIAN FUNCTIONS

O. F. G. SCHILLING

In the applications of the theory of Riemann matrices to algebraic geometry and arithmetic it is necessary to emphasize the rings of complex multiplications in preference to the rational algebra associated to a given Riemann matrix. In recent years A. A. Albert completed and amplified the algebraic theory of Riemann matrices that was inaugurated by G. Scorza, S. Lefschetz, and others.[1] We shall construct in this note the fields of abelian functions of genus n whose multiplication rings contain a given order of a totally real field of absolute degree n. For the proof we shall compare two types of Riemann matrices defined by S. Lefschetz and O. Blumenthal.[2] Applying equivalences to these Riemann matrices we find suitable matrices which show explicitly how a permissible ring of complex multiplications can be realized.

Let $k=R(\alpha)$ be a totally real field of degree n. Suppose that $\alpha=\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(n)}$ are the n conjugate roots of the defining irreducible equation $f(\alpha)=0$. Lefschetz[3] has determined the form of the most general Riemann matrix Ω_1 of genus n whose multiplication algebra $\mathfrak{A}(\Omega_1)$ contains an isomorphic image of the field k. The matrix Ω_1 is equal to

$$\left\|\begin{matrix} \tau_{11}, & \tau_{11}\alpha^{(1)}, & \cdots, & \tau_{11}\alpha^{(1)n-1}; & \tau_{12}, & \tau_{12}\alpha^{(1)}, & \cdots, & \tau_{12}\alpha^{(1)n-1} \\ \tau_{21}, & \tau_{21}\alpha^{(2)}, & \cdots, & \tau_{22}\alpha^{(2)n-1}; & \tau_{22}, & \tau_{22}\alpha^{(2)}, & \cdots, & \tau_{22}\alpha^{(2)n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \tau_{n1}, & \tau_{n1}\alpha^{(n)}, & \cdots, & \tau_{n1}\alpha^{(n)n-1}; & \tau_{n2}, & \tau_{n2}\alpha^{(n)}, & \cdots, & \tau_{n2}\alpha^{(n)n-1} \end{matrix}\right\|,$$

where $\tau_{11}, \cdots, \tau_{n1}, \tau_{12}, \cdots, \tau_{n2}$ are complex parameters subject to the restriction that Ω_1 satisfy the defining equations of a Riemann matrix.

Now suppose that $\alpha_1=\alpha_1^{(1)}, \cdots, \alpha_n=\alpha_n^{(1)}$ is a basis of k/R. Let $A_i^{(j)}$ be the minors of the determinant $\delta=\delta^{(1)}=|\alpha_i^{(j)}|$. Then there ex-

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[1] For references see Solomon Lefschetz, *Selected topics on algebraic geometry*, National Research Council Bulletin vol. 63 (1928) pp. 310–395, and A. A. Albert, *Structure of Algebras*, Amer. Math. Soc. Colloquium Publications, vol. 24, 1939.

[2] Solomon Lefschetz, *On certain numerical invariants of algebraic varieties with applications to abelian varieties*, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 327–482 (quoted as L); Otto Blumenthal, *Über Thetafunktionen und Modulfunktionen mehrerer Veränderlicher*, Jber. Deutschen Math. Verein. vol. 13 (1904) pp. 120–132 (quoted as B).

[3] L, loc. cit. p. 397.

ist infinitely many numbers $\zeta=\zeta^{(1)}$ of k so that $\zeta^{(1)}A_1^{(1)}/\delta^{(1)}=\beta_1^{(1)}, \cdots, \zeta^{(1)}A_n^{(1)}/\delta^{(1)}=\beta_n^{(1)}$ form the basis of an ideal $\mathfrak{b}=\mathfrak{b}^{(1)}$ of k.[4] The Riemann matrix defined by Blumenthal[5] can then be written in the form

$$\Omega_2 = \left\| \begin{matrix} \tau_{11}\alpha_1^{(1)}, & \tau_{11}\alpha_2^{(1)}, & \cdots, & \tau_{11}\alpha_n^{(1)}; & \tau_{12}\beta_1^{(1)}, & \tau_{12}\beta_2^{(1)}, & \cdots, & \tau_{12}\beta_n^{(1)} \\ \tau_{21}\alpha_1^{(2)}, & \tau_{21}\alpha_2^{(2)}, & \cdots, & \tau_{21}\alpha_n^{(2)}; & \tau_{22}\beta_1^{(2)}, & \tau_{22}\beta_2^{(2)}, & \cdots, & \tau_{22}\beta_n^{(2)} \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \tau_{n1}\alpha_1^{(n)}, & \tau_{n1}\alpha_2^{(n)}, & \cdots, & \tau_{n1}\alpha_n^{(n)}; & \tau_{n2}\beta_1^{(n)}, & \tau_{n2}\beta_2^{(n)}, & \cdots, & \tau_{n2}\beta_n^{(n)} \end{matrix} \right\|,$$

where τ_{j2}/τ_{j1}, $j=1, \cdots, n$, are restricted by the inequalities $R[(\tau_{j2}/\tau_{j1})/\zeta^{(j)}]>0$.

LEMMA 1. *The matrices Ω_1 and Ω_2 are isomorphic.*

PROOF. We first observe that $R(1, \alpha^{(1)}, \cdots, \alpha^{(1)n-1})=R(\alpha_1^{(1)}, \cdots, \alpha_n^{(1)})=R(\beta_1^{(1)}, \cdots, \beta_n^{(1)})$. Hence there exist two regular $n\times n$ matrices $\|a_{ij}\|$ and $\|b_{ij}\|$ in R for which $(1, \alpha^{(1)}, \cdots, \alpha^{(1)n-1})\|a_{ij}\|=(\alpha_1^{(1)}, \alpha_2^{(1)}, \cdots, \alpha_n^{(1)})$ and $(1, \alpha^{(1)}, \cdots, \alpha^{(1)n-1})\|b_{ij}\|=(\beta_1^{(1)}, \beta_2^{(1)}, \cdots, \beta_n^{(1)})$, respectively. Consequently

$$\Omega_1 \left\| \begin{matrix} \|a_{ij}\| & 0 \\ 0 & \|b_{ij}\| \end{matrix} \right\| = \Omega_2,$$

in other words, Ω_1 and Ω_2 are isomorphic Riemann matrices.[6]

We next describe the explicit realization of k as a field of multiplications of the matrix Ω_2. Let $\gamma^{(1)}=\gamma$ be an element of k and set $(\alpha_1^{(j)}, \alpha_2^{(j)}, \cdots, \alpha_n^{(j)})=\mathfrak{a}^{(j)}$, $(\beta_1^{(j)}, \beta_2^{(j)}, \cdots, \beta_n^{(j)})=\mathfrak{b}^{(j)}$, $j=1, \cdots, n$. Then $\gamma^{(j)}\mathfrak{a}^{(j)}=\mathfrak{a}^{(j)}C(\mathfrak{a}, \gamma)$ and $\gamma^{(j)}\mathfrak{b}^{(j)}=\mathfrak{b}^{(j)}C(\mathfrak{b}, \gamma)$ where $C(\mathfrak{a}, \gamma)$ and $C(\mathfrak{b}, \gamma)$ are regular $n\times n$ matrices of R. Consequently

$$\operatorname{diag}(\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(n)})\Omega_2 = \Omega_2 \left\| \begin{matrix} C(\mathfrak{a}, \gamma) & 0 \\ 0 & C(\mathfrak{b}, \gamma) \end{matrix} \right\|\text{[7]}.$$

The full algebra of multiplications $\mathfrak{A}(\Omega_2)$ is simple. In case $\mathfrak{A}(\Omega_2)$ contains elements satisfying an irreducible equation of degree $2n$ it must be a totally imaginary extension $K=k(\mu^{1/2})$, $\mu\ll 0$ in k.[8]

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[4] Georg Landsberg, *Ueber das Analogon des Riemann-Rach'schen Satzes in der Theorie der Algebraischen Zahlen*, Math. Ann. vol. 50 (1897–1898) pp. 577–582.

[5] B, loc. cit. p. 122.

[6] L, loc. cit. p. 365.

[7] We use diag $(\gamma^{(1)}, \cdots, \gamma^{(n)})$ for the $n\times n$ matrix whose diagonal elements are $\gamma^{(1)}, \cdots, \gamma^{(n)}$.

[8] $\Delta\ll 0$ means that Δ and all its conjugates are negative.

By the general theory of Riemann matrices the algebra $\mathfrak{A}(\Omega_2)$ has a representation by $2n\times 2n$ matrices in R.[9] Hence each element of $\mathfrak{A}(\Omega_2)$ is the root of an equation of degree $2n$ over R. Since $\mathfrak{A}(\Omega_2)\supseteq k$ there are multiplications satisfying an irreducible equation of degree n. The algebra $\mathfrak{A}(\Omega_2)$ certainly cannot contain a real multiplication satisfying an irreducible equation of degree $2n$ for the roots of the characteristic equation must be double in case they are real.[10] Since Ω_2 admits a "general projectivity" $\alpha\in k$ the general characteristic equation cannot be the product of relative prime factors.[11] Hence the algebra $\mathfrak{A}(\Omega_2)$ cannot be semi-simple, that is, the direct sum of at least two simple rational algebras. Consequently Ω_2 must be isomorphic to a matrix diag $(\omega, \cdots, \omega)$. The number of diagonal elements can at most be equal to two for otherwise the field k would have a rational representation of degree less than n. In case that

$$\Omega_2 \cong \begin{Vmatrix} \omega & 0 \\ 0 & \omega \end{Vmatrix}$$

the algebra $\mathfrak{A}(\Omega_2)$ would be the algebra of square matrices over k. Then $\mathfrak{A}(\Omega_2)$ would contain all quadratic extensions over k, in particular a totally real extension of degree $2n$ over R. This is impossible as was observed at the beginning of the proof. Thus there remains only $\mathfrak{A}(\Omega_2)\cong k(\mu^{1/2})$, $\mu\ll 0$.

In order to realize k as the multiplication algebra of a Riemann matrix (real case) we let ζ be equal to the absolute value of the discriminant[12] δ^2, select $\tau_{j1}=1$, $j=1, \cdots, n$, and take for τ_{j2}, $j=1, \cdots, n$, a set of transcendental numbers with positive imaginary parts. In case $\mathfrak{A}(\Omega_2)=K$ (imaginary case) let $\tau_{j1}=1$, $j=1, \cdots, n$, and $\tau_{j2}={}_{+}(\mu^{(j)})^{1/2}=\mathrm{M}^{(j)}$, $\mu\ll 0$ in k. Then $\Omega_2=\Omega_2(\mu)$ has the form

$$\begin{Vmatrix}
\alpha_1^{(1)}, & \alpha_2^{(1)}, & \cdots, & \alpha_n^{(1)}; & \mathrm{M}^{(1)}\beta_1^{(1)}, & \mathrm{M}^{(1)}\beta_2^{(1)}, & \cdots, & \mathrm{M}^{(1)}\beta_n^{(1)} \\
\alpha_1^{(2)}, & \alpha_2^{(2)}, & \cdots, & \alpha_n^{(2)}; & \mathrm{M}^{(2)}\beta_1^{(2)}, & \mathrm{M}^{(2)}\beta_2^{(2)}, & \cdots, & \mathrm{M}^{(2)}\beta_n^{(2)} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\alpha_1^{(n)}, & \alpha_2^{(n)}, & \cdots, & \alpha_n^{(n)}; & \mathrm{M}^{(n)}\beta_1^{(n)}, & \mathrm{M}^{(n)}\beta_2^{(n)}, & \cdots, & \mathrm{M}^{(n)}\beta_n^{(n)}
\end{Vmatrix}.$$

Observe now that the first row of $\Omega_2(\mu)$ represents a basis of the field $K=k(\mathrm{M})$. Hence $\mathrm{M}(\alpha_1^{(1)}, \cdots, \alpha_n^{(1)}; \mathrm{M}^{(1)}\beta_1^{(1)}, \cdots, \mathrm{M}^{(1)}\beta_n^{(1)})$

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[9] L, loc. cit. p. 392.

[10] L, loc. cit. p. 392.

[11] L, loc. cit. pp. 395, 397.

[12] B, loc. cit. p. 122.

$(\alpha_1^{(1)}, \cdots, \alpha_n^{(1)}; \mathrm{M}^{(1)}\beta_1^{(1)}, \cdots, \mathrm{M}^{(1)}\beta_n^{(1)}) \, \|m_{ij}\|$ where the matrix $\|m_{ij}\|$ is a regular representation of M in R. Consequently diag $(\mathrm{M}^{(1)}, \cdots, \mathrm{M}^{(n)}) \ \Omega_2(\mu) = \Omega_2(\mu) \ \|m_{ij}\|$, that is, $\Omega_2(\mu)$ admits $K = k(\mathrm{M})$ for its multiplication algebra.

THEOREM. *In the real case there exists for each order* $\mathfrak{o}$ *of maximal rank in k at least one Riemann matrix which admits for its ring of multiplications the order* $\mathfrak{o}$. *In the imaginary case there exists for each order* $\mathfrak{D}$ *of maximal rank in* $K = k(\mathrm{M})$ *at least one Riemann matrix which admits for its ring of multiplications the order* $\mathfrak{D}$.

Proof. Let Ω_2 be a Riemann matrix which has $\mathfrak{A}(\Omega_2) \cong k$. Suppose that $\omega_1, \cdots, \omega_n$ is a basis of $\mathfrak{o}$ over the ring of integers. Then there exist rational regular $n \times n$ matrices S_1 and S_2 so that $(\omega_1, \cdots, \omega_2)S_1 = \mathfrak{a}$ and $(\omega_1, \cdots, \omega_2)S_2 = \mathfrak{b}$. Consider the matrix

$$\Omega_2 \left\| \begin{matrix} S_1 & 0 \\ 0 & S_2 \end{matrix} \right\|,$$

it is isomorphic to Ω_2 and admits exactly $\mathfrak{o}$ for its ring of multiplications. For this observe that $\phi(\omega_1, \cdots, \omega_n) = (\omega_1, \cdots, \omega_n) \, \|f_{ij}\|$ with $\phi \in k$, $\phi \notin \mathfrak{o}$ gives rise to a matrix $\|f_{ij}\|$ whose elements do not all lie in the ring of ordinary integers since $(\omega_1, \cdots, \omega_n)$ is a basis of $\mathfrak{D}$.

Now let $\Omega_2(\mu)$ be a Riemann matrix admitting $K = k(\mathrm{M})$ for its algebra of multiplications. We suppose that $\Omega_2(\mu)$ is given as in the proof of Lemma 2. Then the first row of $\Omega_2(\mu)$ is a basis of K over R. Let $(\omega_1, \cdots, \omega_{2n})$ be a basis of the given order $\mathfrak{D}$ in K. Then there exists a regular rational matrix S so that $\Omega_2(\mu) \ S$ has $(\omega_1, \cdots, \omega_{2n})$ for its first row. Again $\Omega_2(\mu)S$ is isomorphic to $\Omega_2(\mu)$ and admits exactly $\mathfrak{D}$ for its ring of multiplications, that is, $\Phi\Omega_2(\mu)S = \Omega_2(\mu)S\|F_{ij}\|$ gives rise to a non-integral matrix $\|F_{ij}\|$ in case $\Phi \in K$, $\Phi \notin \mathfrak{D}$.

COROLLARY. *In both the real and imaginary case there exist fields of abelian functions A of genus n which have models whose rings of multiplications can be prescribed.*

For the proof we refer to the construction of the fields A for a given matrix of periods.[13]

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[13] A. Krazer, *Lehrbuch der Thetafunktionen*, Leipzig, 1903 and L, loc. cit. pp. 367–370.

CONGRUENCES CONNECTED WITH THE SOLUTION OF A CERTAIN DIOPHANTINE EQUATION

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Introduction. In this paper we shall show that, with the application of certain parts of the theory of functions, it is possible to derive some congruences connected with the solution of a certain diophantine equation. The use of analysis to derive certain arithmetical facts is not new, a typical such paper is that of Rademacher,[1] on the derivation of the Dedekind reciprocal formula. The methods of analysis have been used repeatedly in analytic number theory. Rademacher in an invitation address said:[2] "It would, however, be a misplacement of emphasis if we were to look upon analysis, which here means function theory, only as a tool applied to investigation of number theory. It is more the inner harmony of a system which we wish to depict, · · · ." Here, by analysis, we shall derive general congruences from which the previously known congruences will appear as special cases. Heretofore these special congruences have been developed not as a single entity, but by a gradual sharpening of the methods used, which were primarily algebraic in character.

1. **Historical résumé.** The solution of the diophantine equation

$$x^p + y^p + z^p = 0, \qquad p = \text{odd prime}, \tag{1.1}$$

in terms of the integers x, y, z prime to p, is connected with the congruences

$$B_n f_{p-2n}(t) \equiv 0 \pmod{p}, \qquad f_{p-1}(t) \equiv 0 \pmod{p}, \tag{1.2}$$

where $-t$ may be any of the quantities

$$x/y,\ y/x,\ x/z,\ z/x,\ y/z,\ z/y \pmod{p},$$

and where

$$f_r(t) = \sum_{i=0}^{p-1} i^{r-1} t^i \qquad (r > 1; n = 1, 2, \cdots, (p-3)/2),$$

and $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, and so on, are the Bernoulli num-

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[1] *Über a Reziprozitäts formel aus der Theorie der Modulfunktionen*, Matematikai es Physikai Lapok vol. 40 (1933) pp. 24–31 (in German), Zusammenfassung, pp. 32–34 (Hungarian).

[2] *Fourier expansions of modular forms and problems of partition*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 59–73.

bers. The congruences involving the B's are generally referred to as the Kummer criteria for the solution of (1.1).[3]

Vandiver[4] in studying the equation (1.1) has obtained certain generalizations of (1.2). For this purpose he introduced the polynomial

$$\sum_{s=0}^{p-1} (ks + l)^{n-1} x^s = h_n^{(k,l)}(x), \tag{1.3}$$

where $p, n, k = 1, 2, \cdots; l = 1, 2, \cdots, k-1$; x is arbitrary. Then, Vandiver has shown that, if (1.1) is satisfied by integers x, y, z, prime to p,

$$\sum_{l=0}^{k-1} h_n^{(k,l)}(t) h_{p-n}^{(k,k-l)}(t) \equiv 0 \pmod{p}, \tag{1.4}$$

$n = 1, 2, \cdots, p-1$; k being any positive integer and $-t$ the ratio of any pair of x, y, z modulus p. Relation (1.4) will be referred to as the general congruences of Vandiver. For $k = 1$, (1.4) reduces to the form

$$f_n(t) f_{p-n}(t) \equiv 0 \pmod{p}. \tag{1.5}$$

The relation (1.5) will be referred to as the Mirimanoff congruences.[5]

For $k = 2$, after using (1.5) we have

$$h_n^{(2,1)}(t) h_{p-n}^{(2,1)}(t) \equiv 0 \pmod{p}. \tag{1.6}$$

This latter relation will be referred to as the special congruences of Vandiver.[6]

2. **Analytic approach.** The foundation of the analytic approach to our problem is based upon the use of a function studied by Maier[7] and the addition-theorems obtained by Maier for this function. The function, denoted by $f(x, u)$,[8] was defined by the series

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[3] See Kummer's memoir, Abhandlungen der Akademie der Wissenschaften zu Berlin (1857) pp. 41–77. See also the papers of Vandiver under the title *On Kummer's memoir of* 1857 *concerning Fermat's Last Theorem*, Proc. Nat. Acad. Sci. U.S.A. vol. 6 (1920) pp. 266–268; and Bull. Amer. Math. Soc. vol. 28 (1922) pp. 400–407. For a brief history of this problem, read Mordell, *Fermat's Last Theorem*, Cambridge Press, 1921.

[4] *Transformations of the Kummer criteria in connection with Fermat's Last Theorem*, Ann. of Math. vol. 27 (1926) pp. 171–176.

[5] *L'Equation indeterminée* $x^l+y^l+z^l=0$ *et le criterium de Kummer*, J. Reine Angew. Math. vol. 128 (1905) pp. 45–68.

[6] *Note on some results concerning Fermat's Last Theorem*, Bull. Amer. Math. Soc. vol. 28 (1922) pp. 258–260.

[7] *Zur Theorie der elliptischen Funktionen*, Math. Ann. vol. 104 (1931) pp. 745–769.

[8] The choice of the function $f(x, u)$, which can be thought of as a generating function for the Bernoulli polynomials, or a function related to it, appears reasonable when we recall the intimate connection between the Bernoulli numbers and many of the congruences connected with the attempted solutions of Fermat's Last Theorem.

$$f(x, u) = \sum_{k=-\infty}^{+\infty} \frac{e^{2\pi i x k}}{u + k}, \tag{2.1}$$

for $0 < x < 1$, and $u \not\equiv 0 \pmod 1$. Further, it has been shown, by considering a special contour integral,[9] that

$$f(x, u) = 2\pi i \frac{e^{-2\pi i x u}}{1 - e^{-2\pi i u}}, \tag{2.2}$$

$u \not\equiv 0 \pmod 1$. In this form the function $f(x, u)$ is defined for all values of x. One of the addition-laws obtained by Maier for $f(x, u)$ is:

$$\begin{aligned} f(x, u)f(\xi, v) = {} & f(\xi, u + v)f(x - \xi, u) \\ & - f(x, u + v)f(x - \xi, -v), \end{aligned} \tag{2.3}$$

$0 < \xi < x < 1$; $u, v, (u+v) \not\equiv 0 \pmod 1$.

It was necessary, for this work, to introduce a new function, denoted by $f_p(x, u)$, and defined as

$$f_p(x, u) = f(x + p, u) - f(x, u), \qquad p \equiv 0 \pmod 1. \tag{2.4}$$

Now, on using (2.2), we obtain the following relations for the functions $f(x, u)$ and $f_p(x, u)$, which will prove useful for our immediate purpose.

$$f(x, u) + f(1 - x, -u) = 0. \tag{2.5}$$

$$f(x, u) = \frac{f_p(x, u)}{e^{-2\pi i p u} - 1}. \tag{2.6}$$

$$f_p(x, u) = -2\pi i \sum_{k=0}^{p-1} e^{-2\pi i (x+k) u}. \tag{2.7}$$

$$f_p(x, u) + f_{-p}(1 - x, -u) = 0. \tag{2.8}$$

$$f_p(x + p, u) = e^{-2\pi i p u} f_p(x, u). \tag{2.9}$$

$$f_{-p}(x, u) = -f_p(x - p, u) = -e^{2\pi i p u} f_p(x, u). \tag{2.10}$$

Next, from the definition (2.4) and the above relations, we derive an addition-theorem for the function $f_p(x, u)$, which for our needs takes the form:

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[9] Namely, the integral $I_n = \int_{C_n} (e^{(2\mu-1)z\pi i}/\sin \pi z) \cdot (dz/(z+u))$, where $0 < \mu < 1$, $u \not\equiv 0 \pmod 1$ and where C_n is a rectangle with sides $x = \pm(2n+1/2)$ and $y = \pm m$, where m is an arbitrarily large positive number and n an arbitrarily large positive integer so chosen that $z = -u$ lies in the interior of the rectangle C_n. Finally it is shown that $\lim_{m,n\to\infty} I_n = 0$.

$$
\begin{aligned}
f_p(x+p,\,u+\alpha)f_p(\xi+p,\,v+\beta) &- f_p(x,\,u+\alpha)f_p(\xi,\,v+\beta)\\
&= f_p(\xi+p,\,u+v+\alpha+\beta)f_p(x-\xi-p,\,u+\alpha)\\
&\quad - f_p(\xi,\,u+v+\alpha+\beta)f_p(x-\xi,\,u+\alpha)\\
&\quad + f_p(x+p,\,u+v+\alpha+\beta)f_p(x-\xi,\,-v-\beta)\\
&\quad - f_p(x,\,u+v+\alpha+\beta)f_p(x-\xi-p,\,-v-\beta),
\end{aligned}
\tag{2.11}
$$

where $u+\alpha$, $v+\beta$, $u+v+\alpha+\beta\not\equiv 0 \pmod 1$ and $p\equiv 0 \pmod 1$.

Next, in the analytical development, we consider some special contour integrals involving the function $f_p(x,\,u)$. In order to illustrate our procedure, let us consider some sample integrals. First,

$$\int_C \frac{du}{u^k} f_p(x,\,u+\alpha).$$

Using (2.7) and the calculus of residues, this integral becomes

$$\int_C \frac{du}{u^k} f_p(x,\,u+\alpha) = \frac{(-1)^k(2\pi i)^{k+1}}{(k-1)!}\sum_{h=0}^{p-1} e^{-2\pi i(x+h)\alpha}(x+h)^{k-1}, \tag{2.12}$$

where $k=1,\,2,\,\cdots,\,p-1$; $p\equiv 0 \pmod 1$; $(u+\alpha)\not\equiv 0 \pmod 1$, the path of integration C being a simple closed curve about the origin. Similarly, we obtain the expansion

$$
\begin{aligned}
\int_C \frac{dv}{v^s} f_p(\xi+p,\,u+v+\alpha+\beta)&\\
&= \frac{(-1)^s(2\pi i)^{s+1}}{(s-1)!}\cdot\sum_{r=0}^{p-1}(\xi+r+p)^{s-1}\\
&\quad\times (e^{-2\pi i\beta})^{\xi+r+p}(e^{-2\pi i(u+\alpha)})^{\xi+r+p},
\end{aligned}
\tag{2.13}
$$

where $s=1,\,2,\,\cdots,\,p-1$; $p\equiv 0 \pmod 1$; $u+\alpha$, $v+\beta$, $u+v+\alpha+\beta\not\equiv 0$ (mod 1), the path of integration being similar to that used in (2.12). Finally we obtain

$$
\begin{aligned}
\int_C\int_{C'} \frac{du}{u^k}\,\frac{dv}{v^s} f_p(x-\xi-p,\,u+\alpha)f_p(\xi+p,\,u+v+\alpha+\beta)&\\
&= \int_C \frac{du}{u^k} f_p(x-\xi-p,\,u+\alpha)\int_{C'} \frac{dv}{v^s} f_p(\xi+p,\,u+v+\alpha+\beta)\\
&= \frac{(-1)^{k+s}(2\pi i)^{k+s+2}}{(s-1)!(k-1)!}\cdot\sum_{r=0}^{p-1}\Big\{(\xi+r+p)^{s-1}(e^{-2\pi i\beta})^{\xi+r+p}\\
&\quad\times\sum_{h=0}^{p-1}(x+r+h)^{k-1}(e^{-2\pi i\alpha})^{x+r+h}\Big\},
\end{aligned}
\tag{2.14}
$$

where k, $s<p$; k, s, $p\equiv 0 \pmod 1$; $(u+\alpha)$, $(v+\beta)$, $(u+v+\alpha+\beta)\not\equiv 0 \pmod 1$.

The method applied to the special contour integrals gives a device for expressing these and similar integrals as finite sums of terms involving all of the variables and parameters x, ξ, p, α, and β. The integral (like (2.14)) of both members of the equation (2.1) can be written in the form:

$$\begin{aligned}
&\sum_{h=0}^{p-1}(x+p+h)^{k-1}(e^{-2\pi i\alpha})^{x+p+h}\cdot\sum_{r=0}^{p-1}(\xi+p+r)^{s-1}(e^{-2\pi i\beta})^{\xi+p+r}\\
&\qquad-\sum_{h=0}^{p-1}(x+h)^{k-1}(e^{-2\pi i\alpha})^{x+h}\cdot\sum_{r=0}^{p-1}(\xi+r)^{s-1}(e^{-2\pi i\beta})^{\xi+r}\\
&=\sum_{r=0}^{p-1}\Big\{(\xi+p+r)^{s-1}(e^{-2\pi i\beta})^{\xi+p+r}\\
&\qquad\qquad\cdot\sum_{h=0}^{p-1}(x+r+h)^{k-1}(e^{-2\pi i\alpha})^{x+r+h}\Big\}\\
&\qquad-\sum_{r=0}^{p-1}\Big\{(\xi+r)^{s-1}(e^{-2\pi i\beta})^{\xi+r}\\
&\qquad\qquad\cdot\sum_{h=0}^{p-1}(x+r+h)^{k-1}(e^{-2\pi i\alpha})^{x+r+h}\Big\}\\
&\qquad+\sum_{h=0}^{p-1}\Big\{(x+p+h)^{k-1}(e^{-2\pi i\alpha})^{x+p+h}\\
&\qquad\qquad\cdot\sum_{r=0}^{p-1}(\xi+p+h-r)^{s-1}(e^{-2\pi i\beta})^{\xi+p+h-r}\Big\}\\
&\qquad-\sum_{h=0}^{p-1}\Big\{(x+h)^{k-1}(e^{-2\pi i\alpha})^{x+h}\\
&\qquad\qquad\cdot\sum_{r=0}^{p-1}(\xi+p+h-r)^{s-1}(e^{-2\pi i\beta})^{\xi+p+h-r}\Big\}.
\end{aligned}\tag{2.15}$$

In order to write this equation, (2.15), in a more symmetric form we shall put $\lambda=r+h$ in the first two terms on the right side of (2.15), while in the last two terms on the right-hand side we shall put $\mu=h-r$. Next, we shall interchange the summation indices h and r in the first two terms on the right-hand side, and in the last two terms on the right we replace μ by λ, thus obtaining the expression:

$$
\begin{aligned}
&\sum_{h=0}^{p-1}(x+p+h)^{k-1}(e^{-2\pi i\alpha})^{x+p+h}\sum_{r=0}^{p-1}(\xi+p+r)^{s-1}(e^{-2\pi i\beta})^{\xi+p+r}\\
&\qquad-\sum_{h=0}^{p-1}(x+h)^{k-1}(e^{-2\pi i\alpha})^{x+h}\sum_{r=0}^{p-1}(\xi+r)^{s-1}(e^{-2\pi i\beta})^{\xi+r}\\
&\quad=\sum_{h=0}^{p-1}\Bigg[\left\{(\xi+p+h)^{s-1}(e^{-2\pi i\beta})^{\xi+p+h}(-\xi+h)^{s-1}(e^{-2\pi i\beta})^{\xi+h}\right\}\\
&\qquad\qquad\cdot\sum_{\lambda=h}^{p-1+h}(x+\lambda)^{k-1}(e^{-2\pi i\alpha})^{x+\lambda}\Bigg]\\
&\qquad+\sum_{h=0}^{p-1}\Bigg[\left\{(x+p+h)^{k-1}(e^{-2\pi i\alpha})^{x+p+h}-(x+h)^{k-1}(e^{-2\pi i\alpha})^{x+h}\right\}\\
&\qquad\qquad\cdot\sum_{\lambda=h}^{p-1+h}(\xi+p+\lambda)^{s-1}(e^{-2\pi i\beta})^{\xi+p+\lambda}\Bigg].
\end{aligned}
\tag{2.16}
$$

From the equation (2.16) we shall obtain congruences connected with the solution of the diophantine equation (1.1).

3. **Vandiver polynomial. A generalized polynomial.** In the previous section we developed a rather complicated equality involving finite sums. We shall now connect our work with some of the attempts that have been made to solve or extend the known range of validity of "Fermat's Last Theorem." Vandiver, in order to obtain a transformation of the Kummer criteria (1.2), introduced a polynomial, (1.3). These polynomials, (1.3), shall be referred to as the Vandiver polynomials. Then, following the successful lead of Vandiver, we shall introduce the polynomial

$$
\sum_{s=-r}^{p-1-r}(s+z)^{n-1}t^{s+z}=h_n(-r;z,t),
\tag{3.1}
$$

for p, n, $r\equiv 0 \pmod 1$; $r<p$; and any t and z. We shall refer to these as the generalized polynomials of Vandiver or for brevity as the generalized polynomials.

These new polynomials (3.1) are related to the polynomials of Vandiver (1.3) in the following manner:

$$
\begin{aligned}
h_n(0;c/d,t)&=\sum_{s=0}^{p-1}(s+c/d)^{n-1}t^{s+c/d}\\
&=d^{-(n-1)}t^{c/d}\sum_{s=0}^{p-1}(ds+c)^{n-1}t^{s}\\
&=d^{-(n-1)}t^{c/d}h_n^{(d,c)}(t),
\end{aligned}
\tag{3.2}
$$

where p, n, c, $d\equiv 0 \pmod 1$; $c<d$; and t arbitrary.

Before making use of the generalized polynomials we shall list some of the special forms:

$$h_n(0;\,0,\,t)=\sum_{s=0}^{p-1} s^{n-1}t^{s}=f_n(t), \tag{3.3}$$

$$h_n(0;\,1/2,\,t)=2^{-(n-1)}t^{1/2}h_n^{(2,1)}(t), \tag{3.4}$$

$$h_n(0;\,(d-c)/d,\,t)=d^{-(n-1)}t^{(d-c)/d}h_n^{(d,d-c)}(t), \tag{3.5}$$

where p, n, c, $d\equiv 0 \pmod 1$; $c<d$; $n<p$; t arbitrary. The function $f_n(t)$, in (3.3), was used by Mirimanoff in his transformation (1.5) of the Kummer criteria. The function $h_n^{(2,1)}(t)$ in (3.4) was used by Vandiver in obtaining a transformation of the Kummer criteria, namely (1.6). Finally the functions $h_n^{(d,d-c)}(t)$ from (3.5) and $h_n^{(d,c)}(t)$ from (3.2) were used by Vandiver in his general transformation, (1.4), of the Kummer criteria.

Next, let us consider the sum

$$\begin{aligned}\sum_{\lambda=h}^{h-p+1}(\xi+p+\lambda)^{n-1}\tau^{\xi+p+\lambda}&=\sum_{\nu=h}^{h+p-1}(\xi+\nu+1)^{n-1}\tau^{\xi+\nu+1}\\&=h_n(h;\,\xi+1,\,\tau).\end{aligned} \tag{3.6}$$

We shall develop, in the following, other properties of the generalized polynomials (2.12).

The equation (2.16), which we obtained as an expansion of certain contour integrals similar to (2.14), on replacing $e^{-2\pi i\alpha}$ by t and $e^{-2\pi i\beta}$ by τ, can be written in terms of the generalized Vandiver polynomials, $h_n(-r;\,z,\,t)$. This equation, (2.16), after applying (3.6), can be written as

$$\begin{aligned}&h_k(0;\,x+p,\,t)h_s(0;\,\xi+p,\,\tau)-h_k(0;\,x,\,t)h_s(0;\,\xi,\,\tau)\\&\quad=\sum_{h=0}^{p-1}\left[h_k(h;\,x,\,t)\left\{(\xi+p+h)^{s-1}\tau^{\xi+p+h}-(\xi+h)^{s-1}\tau^{\xi+h}\right\}\right]\\&\qquad+\sum_{h=0}^{p-1}\left[h_s(h;\,\xi+1,\,\tau)\left\{(x+p+h)^{k-1}t^{x+p+h}\right.\right.\\&\qquad\qquad\left.\left.-(x+h)^{k-1}t^{x+h}\right\}\right].\end{aligned} \tag{3.7}$$

This equation will be used to obtain a variety of congruences connected with the solution of equation (1.1).

4. Some special congruences. In this section we shall derive some special congruences which are necessary for the final derivation of our general congruences.

The definition of the generalized polynomials gives us

$$\begin{aligned}
&h_n(0;\, ap,\, t)h_{p-n}(0;\, bp,\, t)\\
&\quad = \left(\sum_{h=0}^{p-1} (h+ap)^{n-1}t^{h+ap}\right)\left(\sum_{s=0}^{p-1} (s+bp)^{p-n-1}t^{s+bp}\right)\\
&\quad = t^{(a+b)p}\left(\sum_{h=0}^{p-1} (h+ap)^{n-1}t^{h}\right)\left(\sum_{s=0}^{p-1} (s+bp)^{p-n-1}t^{s}\right)\\
&\quad \equiv t^{(a+b)p}\left(\sum_{h=0}^{p-1} h^{n-1}t^{h}\right)\left(\sum_{s=0}^{p-1} s^{p-n-1}t^{s}\right) \pmod{p}\\
&\quad = t^{(a+b)p}h_n(0;\, 0,\, t)h_{p-n}(0;\, 0,\, t)\\
&\quad = t^{(a+b)p}f_n(t)f_{p-n}(t).
\end{aligned}$$

Hence we obtain the following variation of the Mirimanoff congruences (1.5), namely[10]

$$(4.1)\qquad t^{-(a+b)p}h_n(0;\, ap,\, t)h_{p-n}(0;\, bp,\, t) \equiv 0 \pmod{p},$$

where $a, b=0, \pm 1, \pm 2, \cdots$. A particularly simple congruence occurs when $a=-b$, namely

$$(4.2)\qquad h_n(0;\, ap,\, t)h_{p-n}(0;\, -ap,\, t) \equiv 0 \pmod{p}.$$

If we put $a=b=0$ in (4.1) this reduces to the original Mirimanoff congruences (1.5).

Let us next consider the product $h_n(0;\, 1/2,\, t)h_{p-n}(0;\, 1/2,\, t)$ which reduces to $t2^{-(p-2)}h_n^{(2,1)}(t)h_{p-n}^{(2,1)}(t)$. Thus from the special congruences of Vandiver, (1.6), we have

$$(4.3)\qquad t^{-1}2^{p-2}h_n(0;\, 1/2,\, t)h_{p-n}(0;\, 1/2,\, t) \equiv 0 \pmod{p}.$$

In the same manner we obtain the congruences

$$(4.4)\quad 2^{p-2}t^{-1-(a+b)p}h_n(0;\, 1/2+ap,\, t)h_{p-n}(0;\, 1/2+bp,\, t) \equiv 0 \pmod{p},$$

where $a, b=0, \pm 1, \pm 2, \cdots$.

Following much the same procedure as before we shall derive some transformations of the general congruences of Vandiver, (1.4). For this purpose let us consider first

$$\sum_{c=0}^{d-1} h_n(0;\, c/d,\, t)h_{p-n}(0;\, (d-c)/d,\, t),$$

---

[10] In all the congruences given in this paper the following statements are understood. If the equation $x^p+y^p+z^p=0$, $p=$odd prime, is satisfied by integers x, y, z prime to p, where $-t$ may be any of the quantities x/y, y/x, x/z, z/x, y/z, z/y (mod p), then the congruences are satisfied.

where $p, n, c, d \equiv 0 \pmod 1$; $c < d$. This sum can be written in the form

$$\sum_{c=0}^{d-1}\left[\left(\sum_{h=0}^{p-1}(h + c/d)^{n-1}t^{h+c/d}\right)\left(\sum_{s=0}^{p-1}(s + (d-c)/d)^{p-n-1}t^{s+(d-c)/d}\right)\right]$$

$$= td^{-(p-2)}\sum_{c=0}^{d-1}\left[\left(\sum_{h=0}^{p-1}(hd + c)^{n-1}t^{h}\right)\left(\sum_{s=0}^{p-1}(sd + d - c)^{p-n-1}t^{s}\right)\right]$$

$$= td^{-(p-2)}\sum_{c=0}^{d-1} h_n^{(d,c)}(t)\, h_{p-n}^{(d,d-c)}(t).$$

Hence the general congruences of Vandiver, (1.4), are replaced by the relation

$$(4.5)\quad d^{(p-2)}t^{-1}\sum_{c=0}^{d-1} h_n(0; c/d, t)\, h_{p-n}(0; (d-c)/d, t) \equiv 0 \pmod p.$$

Next, if we consider the sum

$$\sum_{c=0}^{d-1} h_n(0; c/d + ap, t)\, h_{p-n}(0; (d-c)/d + bp, t)$$

we obtain another transformation of (1.4), namely

$$(4.6)\quad \begin{aligned} d^{(p-2)}t^{-1-(a+b)p}\sum_{c=0}^{d-1} h_n(0; c/d + ap, t)\, h_{p-n}(0; (d-c)/d + bp, t) \\ \equiv 0 \pmod p; \end{aligned}$$

where $p, n, c, d \equiv 0 \pmod 1$; $c < d$; and $a, b = 0, \pm 1, \pm 2, \cdots$.

5. **Some general congruences.** In §3 we derived an addition-theorem (3.7) for the generalized polynomials of Vandiver, (3.1). Since (3.7) is an identity in x, ξ, t, τ, k, and s we can substitute particular values for the variables and parameters without destroying the equality. Among a variety of possible substitutions we shall introduce the following:

(A) $\quad x = \xi = 0,$

(B) $\quad x = mp, \qquad \xi = qp,$

(C) $\quad x = 1/2 + mp, \qquad \xi = 1/2 + qp,$

(D) $\quad x = c/d + mp, \qquad \xi = (d-c)/d + qp,$

and in each case we put

$$\tau = t, \qquad k = n, \qquad s = p - n,$$

where $p, n, c, d \equiv 0 \pmod 1$; $c < d$; and $m, q = 0, \pm 1, \pm 2, \cdots$.

Now, on applying, successively, these substitutions to the equation (3.7) we shall obtain several different quadratic functional equations for $h_n(-r; x, t)$.

The congruences resulting from applying (A) to the equation (3.7) and by using a proper choice of a and b in (4.1) give us a direct transformation of the Mirimanoff congruences, (1.5).

On applying (B) to (3.7) and using (4.1) we obtain the congruence

$$\begin{aligned}
&\sum_{h=0}^{p-1} \big[h_n(h; mp, t)\big\{((q+1)p+h)^{p-n-1}t^{-(m+1)p+h} \\
&\qquad\qquad - (qp+h)^{p-n-1}t^{-(m+2)p+h}\big\}\big] \\
&\quad + \sum_{h=0}^{p-1} \big[h_{p-n}(h; qp+1, t)\big\{((m+1)p+h)^{n-1}t^{-(q+1)p+h} \\
&\qquad\qquad - (mp+h)^{n-1}t^{-(q+2)p+h}\big\}\big] \\
&\quad + t^{-(m+q+2)p}h_n(0; mp, t)h_{p-n}(0; qp, t) \equiv 0 \pmod{p}.
\end{aligned} \tag{5.1}$$

If we put $m=q=0$ in (5.1) we obtain the same congruence which resulted from applying (A) to (3.7).

Next on applying (C) and (D) respectively to (3.7) and using appropriate forms of (4.4), (4.5) and (4.6) we obtain the following congruences

$$\begin{aligned}
&2^{p-2}\sum_{h=0}^{p-1} \big[h_n(h; 1/2+mp, t)\big\{(1/2+(q+1)p+h)^{p-n-1} \\
&\qquad t^{-1/2-(m+1)p+h} - (1/2+qp+h)^{p-n-1}t^{-1/2-(m+2)p+h}\big\}\big] \\
&\quad + 2^{p-2}\sum_{h=0}^{p-1} \big[h_{p-n}(h; 3/2+qp, t)\big\{(1/2+(m+1)p+h)^{n-1} \\
&\qquad t^{-1/2-(q+1)p+h} - (1/2+mp+h)^{n-1}t^{-1/2-(q+2)p+h}\big\}\big] \\
&\quad + 2^{p-2}t^{-1-(m+q+2)p}h_n(0; 1/2+mp, t)h_{p-n}(0; 1/2+qp, t) \\
&\qquad \equiv 0 \pmod{p},
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
&d^{p-2}\sum_{c=0}^{d-1}\sum_{h=0}^{p-1} \big[h_n(h; c/d+mp, t)\big\{((d-c)/d+(q+1)p+h)^{p-n-1} \\
&\qquad \cdot t^{-c/d-(m+1)p+h} - ((d-c)/d+qp+h)^{p-n-1}t^{-c/d-(m+2)p+h}\big\}\big] \\
&\quad + d^{p-2} \sum_{c=0}^{d-1}\sum_{h=0}^{p-1}\big[h_{p-n}(h; (2d-c)/d+qp, t)\big\{(c/d+(m+1)p+h)^{n-1} \\
&\qquad \cdot t^{-(d-c)/d-(q+1)p+h} - (c/d+mp+h)^{n-1}t^{-(d-c)/d-(q+2)p+h}\big\}\big] \\
&\quad + d^{p-2}t^{-1-(m+q+2)p}\sum_{c=0}^{d-1} h_n(0; c/d+mp, t) \\
&\qquad \cdot h_{p-n}(0; (d-c)/d+qp, t) \equiv 0 \pmod{p},
\end{aligned} \tag{5.3}$$

where $p, n, d \equiv 0 \pmod 1$; $n<p$; $c=0, 1, 2, \cdots, d-1$; and $m, q=0, \pm 1, \pm 2, \cdots$.

Thus we have obtained transformations of the Mirimanoff and Vandiver congruences connected with the solution of equation (1.1). Other, and in some cases more symmetric, transformations of these congruences are possible by using one of the other permissible forms for the quadratic functional equation (2.11).

PURDUE UNIVERSITY

DEVELOPMENT OF CERTAIN QUADRATIC FUNCTIONAL EQUATIONS

HAROLD F. S. JONAH

1. **Introduction.** In some work devoted to the derivation of certain congruences connected with the solution of Fermat's Last Theorem, it was found necessary to develop several quadratic functional equations of a particular function which we shall define later. This note will deal with the derivation of these functional equations. Maier[1] derived two such quadratic functional equations for a generating function of the Bernoulli polynomials. This work of Maier serves as the basis for our developments.

2. **The Maier results.** The function, $f(x, u)$, used by Maier was defined by the infinite series

$$\sum_{r=-\infty}^{+\infty} \frac{e^{2\pi i x r}}{u+r}, \tag{2.1}$$

where x is a real variable satisfying the inequality $0<x<1$, and where u is a real variable subject to the restriction $u \not\equiv 0 \pmod 1$. Maier, then, showed that if u, v, x, ξ are such that $u, v, (u+v) \not\equiv 0 \pmod 1$ and $0<\xi<x<1$, that the function $f(x, u)$ is a solution of the functional equation

$$\begin{aligned} f(x, u)f(\xi, v) = {} & f(\xi, u+v)f(x-\xi, u) \\ & - f(x, u+v)f(x-\xi, -v). \end{aligned} \tag{2.2}$$

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[1] W. Maier, *Zur Theorie der elliptischen Funktionen*, Math. Ann. vol. 104 (1930) pp. 745–769.

If the restriction on x and ξ is replaced by the condition $0<\xi<1-x<1$ (since we must have $0<x+\xi<1$), Maier obtains the functional equation

$$f(x, u)f(\xi, v) = f(x, u - v)f(x + \xi, v) + f(\xi, v - u)f(x + \xi, u). \tag{2.3}$$

Returning to the definition of the function $f(x, u)$, (2.1), we note that by considering a special contour integral it has been shown[2] that

$$f(x, u) = 2\pi i e^{-2\pi i x u}/(1 - e^{-2\pi i u}), \qquad u \not\equiv 0 \pmod 1. \tag{2.4}$$

In this form the function $f(x, u)$ is defined for all values of x, and we shall now use this form of the function $f(x, u)$. Further we note that the addition-theorems (2.2) and (2.3) must remain satisfied by the function $f(x, u)$ when the form (2.4) is used.

3. **The function $f_p(x, u)$.** For our purposes it was necessary to introduce a new function, which we denote by $f_p(x, u)$ and which will be defined in terms of values of the function $f(x, u)$. We define

$$f_p(x, u) = f(x + p, u) - f(x, u), \qquad p \equiv 0 \pmod 1. \tag{3.1}$$

Hence by means of (2.4), we have

$$f_p(x, u) = [e^{-2\pi i p u} - 1]f(x, u). \tag{3.2}$$

Thus

$$f(x, u) = f_p(x, u)/(e^{-2\pi i p u} - 1). \tag{3.3}$$

By means of (3.2) we can show that

$$f_p(x, u) + f_{-p}(1 - x, - u) \equiv 0. \tag{3.4}$$

Other formulae involving the function $f_p(x, u)$ which shall prove to be useful for our immediate purpose are

$$f_p(x + p, u) = e^{-2\pi i p u}f_p(x, u), \tag{3.5}$$

$$f_{-p}(x, u) = - f_p(x - p, u) = e^{2\pi i p u}f_p(x, u). \tag{3.6}$$

In the above formulae for $f_p(x, u)$ we can replace u by $u+\alpha$, where the condition $u\not\equiv 0$ (mod 1) now becomes $(u+\alpha)\not\equiv 0$ (mod 1), and everything else remains valid.

---

[2] Namely, the integral $I_n=\int_{C^n}(e^{(2\mu-1)\pi z i}/\sin \pi z)\cdot(dz/(z+u))$, where $0<\mu<1$, $u\not\equiv 0$ (mod 1) and where C_n is a rectangle with the sides $x=\pm(2n+1/2)$ and $y=\pm m$ with m an arbitrarily large positive number and n as an arbitrarily large positive integer so chosen that $z=-u$ lies in the interior of the rectangle C_n. Finally it has been shown that $\lim_{m,n\to\infty}I_n=0$.

4. Addition-theorems for the function $f_p(x, u)$. In §2 we gave the two addition-theorems for the function $f(x, u)$. In this paragraph we shall derive similar addition-theorems for the function $f_p(x, u)$.

Let us write the addition-theorem (2.2) in the form

$$\begin{aligned} f(x, u+\alpha)f(\xi, v+\beta) &= f(\xi, u+v+\alpha+\beta)f(x-\xi, u+\alpha) \\ &\quad - f(x, u+v+\alpha+\beta)f(x-\xi, -v-\beta), \end{aligned} \tag{4.1}$$

where $(u+\alpha)$, $(v+\beta)$, $(u+v+\alpha+\beta)\not\equiv 0 \pmod 1$.

On applying (3.3) to the equation (4.1) we have

$$\begin{aligned} &\frac{f_p(x, u+\alpha)f_p(\xi, v+\beta)}{\left[e^{-2\pi ip(u+\alpha)}-1\right]\left[e^{-2\pi ip(v+\beta)}-1\right]} \\ &\qquad = \frac{f_p(\xi, u+v+\alpha+\beta)f_p(x-\xi, u+\alpha)}{\left[e^{-2\pi ip(u+v+\alpha+\beta)}-1\right]\left[e^{-2\pi ip(u+\alpha)}-1\right]} \\ &\qquad\quad - \frac{f_p(x, u+v+\alpha+\beta)f_p(x-\xi, -v-\beta)}{\left[e^{-2\pi ip(u+v+\alpha+\beta)}-1\right]\left[e^{2\pi ip(v+\beta)}-1\right]}. \end{aligned} \tag{4.2}$$

This reduces to

$$\begin{aligned} &\left[e^{-2\pi ip(u+v+\alpha+\beta)}-1\right]f_p(x, u+\alpha)f_p(\xi, v+\beta) \\ &\qquad = \left[e^{-2\pi ip(v+\beta)}-1\right]f_p(\xi, u+v+\alpha+\beta)f_p(x-\xi, u+\alpha) \\ &\qquad\quad - \left[e^{-2\pi ip(u+\alpha)}-1\right]\frac{\left[e^{-2\pi ip(v+\beta)}-1\right]}{\left[e^{2\pi ip(v+\beta)}-1\right]} \\ &\qquad\quad \cdot f_p(x, u+v+\alpha+\beta)f_p(x-\xi, -v-\beta). \end{aligned} \tag{4.3}$$

In order to further simplify the above equation, we write $\left[e^{-2\pi ip(v+\beta)}-1\right]=\left[e^{-2\pi ip(u+v+\alpha+\beta)}e^{2\pi ip(u+\alpha)}-1\right]$, and similarly

$$\begin{aligned} &\left[e^{-2\pi ip(u+\alpha)}-1\right]\frac{\left[e^{-2\pi ip(v+\beta)}-1\right]}{\left[e^{2\pi ip(v+\beta)}-1\right]} \\ &\quad = \left[e^{-2\pi ip(u+\alpha)}-1\right]\frac{\left[e^{-2\pi ip(v+\beta)}-1\right]}{\left[1-e^{-2\pi ip(v+\beta)}\right]}e^{-2\pi ip(v+\beta)} \\ &\quad = -\left[e^{-2\pi ip(u+\alpha)}-1\right]e^{-2\pi ip(v+\beta)} = -\left[e^{-2\pi ip(u+v+\alpha+\beta)}-e^{-2\pi ip(v+\beta)}\right]. \end{aligned}$$

Then (4.3) reduces to

$$\begin{aligned} &\left[e^{-2\pi ip(u+v+\alpha+\beta)}-1\right]f_p(x, u+\alpha)f_p(\xi, v+\beta) \\ &\qquad = \left[e^{-2\pi ip(u+v+\alpha+\beta)}e^{2\pi ip(v+\beta)}-1\right] \\ &\qquad\quad \cdot f_p(\xi, u+v+\alpha+\beta)f_p(x-\xi, u+\alpha) \\ &\qquad\quad + \left[e^{-2\pi ip(u+v+\alpha+\beta)}-e^{-2\pi ip(v+\beta)}\right] \\ &\qquad\quad \times f_p(x, u+v+\alpha+\beta)f_p(x-\xi, -v-\beta). \end{aligned} \tag{4.4}$$

Then on making use of (3.5) we obtain a quadratic functional equation of the function $f_p(x, u)$, namely

$$\begin{aligned}
f_p(x + p, u + \alpha)f_p(\xi + p, v + \beta) - f_p(x, u + \alpha)f_p(\xi, v + \beta)& \\
= f_p(\xi + p, u + v + \alpha + \beta)f_p(x - \xi - p, u + \alpha)& \\
- f_p(\xi, u + v + \alpha + \beta)f_p(x - \xi, u + \alpha)& \\
+ f_p(x + p, u + v + \alpha + \beta)f_p(x - \xi, - v - \beta)& \\
- f_p(x, u + v + \alpha + \beta)f_p(x - \xi - p, - v - \beta),&
\end{aligned} \tag{4.5}$$

where $(u+\alpha)$, $(v+\beta)$, $(u+v+\alpha+\beta) \not\equiv 0 \pmod 1$, $p \equiv 0 \pmod 1$.

Next, on using (3.4) with (4.5) we obtain for the latter equation the following modified form:

$$\begin{aligned}
- f_{-p}(1 - x - p, - u - \alpha)f_p(\xi + p, v + \beta)& \\
+ f_{-p}(1 - x, - u - \alpha)f_p(\xi, v + \beta)& \\
= - f_p(\xi + p, u + v + \alpha + \beta)f_{-p}(1 - x + \xi + p, - u - \alpha)& \\
+ f_p(\xi, u + v + \alpha + \beta)f_{-p}(1 - x + \xi, - u - \alpha)& \\
+ f_{-p}(1 - x - p, - u - v - \alpha - \beta)f_{-p}(1 - x + \xi, v + \beta)& \\
- f_{-p}(1 - x, - u - v - \alpha - \beta)f_{-p}(1 - x + \xi + p, v + \beta).&
\end{aligned} \tag{4.6}$$

Consequently, on replacing $1-x$ by x and $-u-\alpha$ by $u+\alpha$, (4.6) becomes

$$\begin{aligned}
f_{-p}(x - p, u + \alpha)f_p(\xi + p, v + \beta) - f_{-p}(x, u + \alpha)f_p(\xi, v + \beta)& \\
= f_p(\xi + p, v - u + \beta - \alpha)f_{-p}(x + \xi + p, u + \alpha)& \\
- f_p(\xi, v - u + \beta - \alpha)f_{-p}(x + \xi, u + \alpha)& \\
+ f_{-p}(x, u - v + \alpha - \beta)f_{-p}(x + \xi + p, v + \beta)& \\
- f_{-p}(x - p, u - v + \alpha - \beta)f_{-p}(x + \xi, v + \beta).&
\end{aligned} \tag{4.7}$$

Introducing (3.6) in equation (4.7) we obtain the following functional equation for the function $f_p(x, u)$,

$$\begin{aligned}
f_p(x - p, u + \alpha)f_p(\xi, v + \beta) - f_p(x - 2p, u + \alpha)f_p(\xi + p, v + \beta)& \\
= f_p(\xi, v - u + \beta - \alpha)f_p(x + \xi + p, u + \alpha)& \\
- f_p(\xi + p, v - u + \beta - \alpha)f_p(x + \xi, u + \alpha)& \\
+ f_p(x - p, u - v + \alpha - \beta)f_p(x + \xi, v + \beta)& \\
- f_p(x - 2p, u - v + \alpha - \beta)f_p(x + \xi + p, v + \beta),&
\end{aligned} \tag{4.8}$$

where $(u+\alpha)$, $(v+\beta)$, $(u-v+\alpha-\beta) \not\equiv 0 \pmod 1$.

The addition-theorem (2.3) for the function $f(x, u)$ may be written in the form

$$\begin{aligned} f(x, u+\alpha)f(\xi, v+\beta) = {} & f(x, u-v+\alpha-\beta)f(x+\xi, v+\beta) \\ & - f(\xi, v-u+\beta-\alpha)f(x+\xi, u+\alpha), \end{aligned} \tag{4.9}$$

where $(u+\alpha)$, $(v+\beta)$, $(u-v+\alpha-\beta) \not\equiv 0 \pmod 1$.

Then on applying (3.3) to the equation (4.9) and simplifying, we obtain the following addition-law

$$\begin{aligned} f_p(x+p, u+\alpha)f_p(\xi-p, v+\beta) & - f_p(x, u+\alpha)f_p(\xi, v+\beta) \\ = {} & f_p(x+p, u-v+\alpha-\beta)f_p(x+\xi+p, v+\beta) \\ & - f_p(x, u-v+\alpha-\beta)f_p(x+\xi, v+\beta) \\ & + f_p(\xi-p, v-u+\beta-\alpha)f_p(x+\xi, u+\alpha) \\ & - f_p(\xi, v-u+\beta-\alpha)f_p(x+\xi+p, u+\alpha), \end{aligned} \tag{4.10}$$

where $p \equiv 0 \pmod 1$ and $(u+\alpha)$, $(v+\beta)$, $(u-v+\alpha-\beta) \not\equiv 0 \pmod 1$.

If we apply (3.4) to (4.10), then replace $1-x$ by x and $-u-\alpha$ by $u+\alpha$, and finally apply (3.6), we obtain another functional equation

$$\begin{aligned} f_p(x-2p, u+\alpha)f_p(\xi-p, v+\beta) & - f_p(x-p, u+\alpha)f_p(\xi, v+\beta) \\ = {} & f_p(x-2p, u+v+\alpha+\beta)f_p(x-\xi-2p, -v-\beta) \\ & - f_p(x-p, u+v+\alpha+\beta)f_p(x-\xi-p, -v-\beta) \\ & + f_p(\xi-p, u+v+\alpha+\beta)f_p(x-\xi-p, u+\alpha) \\ & - f_p(\xi, u+v+\alpha+\beta)f_p(x-\xi-2p, u+\alpha), \end{aligned} \tag{4.11}$$

where $(u+\alpha)$, $(v+\beta)$, $(u+v+\alpha+\beta) \not\equiv 0 \pmod 1$ and $p \equiv 0 \pmod 1$.

Thus we have derived four addition-theorems for the function $f_p(x, u)$. These functional equations were used in connection with the solution of a certain diophantine equation.

PURDUE UNIVERSITY

ARITHMETIC UPON AN ALGEBRAIC SURFACE[1]

B. SEGRE

The title of my lecture is, I am afraid, probably misleading and certainly too ambitious. For, on the one hand, the connection between arithmetic and geometry suggested by it is not the modern development in divisors theory, but an application of algebraic geometry for arithmetical purposes. On the other hand, I shall confine the subject of this lecture to cubic surfaces in ordinary space, considered in the rational domain, so that a proper title would be for instance *The geometry of ternary cubic Diophantine equations.*[2] I prefer, however, the more ambitious and inaccurate one, as suggesting the possibility of similar investigations for other surfaces, possibly considered in more general arithmetical fields.

The short time at my disposal does not allow me to dwell on such extensions. I mention only that I have already completed an extensive arithmetical research on quartic surfaces; and that the whole subject—arithmetic upon an algebraic surface—seems to me to be so wide in scope, that I can envisage the possibility of further important developments.

Let us consider an ordinary space, where coordinates (x, y, z) are introduced and points at infinity are defined in the usual way. I call *rational* an algebraic surface, or curve, or point set of this space when it can be represented by one or more algebraic equations with rational coefficients,

$$F(x, y, z) = 0 \tag{1}$$

say. Moreover, I call *rational* any such equation, and also any polynomial such as $F(x, y, z)$.

The problem of finding the rational solutions in x, y, z of equation (1) can then be stated as that of determining the *rational points lying*

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[1] A lecture given at the London Mathematical Society, on December 16, 1943.

[2] On this subject cf. B. Segre, *A note on arithmetical properties of cubic surfaces*, J. London Math. Soc. vol. 18 (1943) p. 24; *On a parametric solution of the equation* $x^3+y^3+az^3=b$, *and on ternary forms representing every rational number*, ibid. p. 31; *On ternary nonhomogeneous cubic equations with more than one rational solution*, ibid. p. 88; *A parametric solution of the indeterminate cubic equation* $z^2=f(x, y)$, ibid. p. 226; *A complete parametric solution of certain homogeneous Diophantine equations, of degree* n *in* $n+1$ *variables*, ibid. vol. 19 (1944) p. 46. Another paper of mine, *On arithmetical properties of singular cubic surfaces*, will appear shortly in the same journal.

The present lecture sums up rather sketchily my previous results, to which it adds a few more, as for example the geometric construction of p. 156. Full details, together with further results, will be found in a forthcoming extensive and systematic work.

on the rational surface F, represented by (1). Moreover, a parametric solution

$$(2) \qquad x = \theta_1(\lambda)/\theta_4(\lambda), \quad y = \theta_2(\lambda)/\theta_4(\lambda), \quad z = \theta_3(\lambda)/\theta_4(\lambda)$$

of (1), where the θ's are relatively prime rational polynomials in λ, gives a *rational unicursal curve lying on F*. Likewise, a two-parameter solution

$$(3) \quad x = \theta_1(\lambda, \mu)/\theta_4(\lambda, \mu), \quad y = \theta_2(\lambda, \mu)/\theta_4(\lambda, \mu), \quad z = \theta_3(\lambda, \mu)/\theta_4(\lambda, \mu)$$

of (1), where the θ's are now relatively prime rational polynomials in λ and μ, gives a *rational representation of F upon a plane* α, on which (λ, μ) are coordinates of a variable point. The *degree* of a parametric solution (2) or (3) is defined as the greatest of the degrees of the polynomials θ in (2) or (3) respectively.

When the equations (3) induce a (1, 1)-correspondence between F and α, *all* the rational solutions of (1) are given by (3) either for rational values of the parameters λ, μ, or as limits, when the parameters tend to certain sets of rational values. Hence I shall then say that (3) is a *complete* two-parameter solution of (1).

From now on, I suppose F to be a rational cubic surface, represented by the equation (1). This can be written in the form

$$(4) \qquad \phi_0 + \phi_1(x, y, z) + \phi_2(x, y, z) + \phi_3(x, y, z) = 0,$$

where ϕ_0 is a rational number and ϕ_1, ϕ_2, ϕ_3 are homogeneous rational polynomials in x, y, z, of degrees 1, 2, 3 respectively, the two first of which may possibly vanish identically. F is said to be *singular*, if there are some points $P(x, y, z)$ (not necessarily rational, and possibly at infinity) which satisfy the equations (1) and

$$(5) \qquad \partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 0.$$

Such a point P is called a *double point* of F if some of the partial derivatives of the second order of F do not vanish at P; otherwise P is called a *triple point* of F. A point $P(x, y, z)$ lying on F, that is, which satisfies (1), is called a *simple point* of F if it does not satisfy (5).

I now investigate the Diophantine equation (1) or (4), on supposing that F is irreducible and that a particular rational solution (x_0, y_0, z_0) is known. I shall show that, apart from a single trivial exception, *it is then possible to deduce an infinity of rational solutions*. By choice of the rational coordinates (x, y, z), the rational point $P(x_0, y_0, z_0)$ can be taken at the origin, so that, from (4), $\phi_0 = 0$. I distinguish three cases, according as P is a triple point, or a double point, or a simple point of F.

If P is a *triple point* of F, the equation (4) is

$$\phi_3(x, y, z) = 0,$$

and can therefore be written in the form

$$\phi_3(X, Y, 1) = 0,$$

on putting $X=x/z, Y=y/z$. The problem of finding the rational points of F is now that of determining the rational points of a plane cubic curve, and lies outside the scope of the present lecture. Hence I suppose that F has no rational triple points, and then it is easily seen that F has no irrational triple point, so that its only singular points (if any) are double points.

If the origin P is a *double point* of F, the equation (4) becomes

$$\phi_2(x, y, z) + \phi_3(x, y, z) = 0, \tag{6}$$

where ϕ_2 and ϕ_3 are two relatively prime polynomials, neither of which vanishes identically. There is now a (1,1)-correspondence between the rational points Q of F and the rational lines PQ containing P. This gives the following *complete* two-parameter solution of (6), of degree three:

$$x = -\lambda\phi_2(\lambda, \mu, 1)/\phi_3(\lambda, \mu, 1), \qquad y = -\mu\phi_2(\lambda, \mu, 1)/\phi_3(\lambda, \mu, 1),$$
$$z = -\phi_2(\lambda, \mu, 1)/\phi_3(\lambda, \mu, 1).$$

If P is a *simple point* of F, then in (4) we have $\phi_0=0$ but $\phi_1(x, y, z)$ does not vanish identically, and the equation $\phi_1(x, y, z)=0$ represents the plane π touching F at P. On taking this plane as xy-plane, the equation of F becomes of the form

$$z + \phi_2(x, y, z) + \phi_3(x, y, z) = 0, \tag{7}$$

and I further distinguish two cases, according as $\phi_2(x, y, 0)$ vanishes identically or not.

When $\phi_2(x, y, 0)\neq 0$, there is a (1,1)-correspondence between the rational points Q of F lying on π and the rational lines PQ of π containing P. Such ∞^1 points Q are said to be obtained by means of the *tangent plane process* applied to F and P, and are all given by the following one-parameter solution of (7):

$$x = -\lambda\phi_2(\lambda, 1, 0)/\phi_3(\lambda, 1, 0), \quad y = -\phi_2(\lambda, 1, 0)/\phi_3(\lambda, 1, 0), \quad z = 0.$$

When $\phi_2(x, y, 0)=0$ the tangent plane process cannot be applied, and the equation (7) has the form

$$z[1 + \psi_1(x, y, z)] + \phi_3(x, y, z) = 0,$$

where ψ_1 is a rational linear form in x, y, z. Hence, on putting

$$X = x/z, \qquad Y = y/z, \qquad Z = [1 + 2^{-1}\psi_1(x, y, z)]/z,$$

this equation reduces to the form

$$Z^2 = f(X, Y), \tag{8}$$

where f is a rational cubic polynomial in X, Y, which cannot be written as a polynomial in a single linear function of X and Y.

I have recently *solved parametrically the general equation* (8), by first reducing it to a convenient canonical form, and then using a method which has been already employed by Professor Mordell in a particular case. Here I solve the equation in another particular case, which is perhaps the simplest one in which the method can be applied. I add that I have previously solved parametrically some cubic Diophantine equations included in (8), by considerations of algebraic geometry. Professor Mordell then devised his purely algebraic method, giving an independent proof and also an extension of these results of mine. I give now also a new algebraic-geometric process for solving parametrically the equation of any rational cubic surface of which a rational simple point is known. This process can be applied to (8) in particular, and has the advantage of making intuitive the solvability of this equation.

I solve first the following particular case of (8):

$$z^2 - ry^2 = x^3 + px + q, \tag{9}$$

where p, q, r are rational and $r \neq 0$. For this purpose, I remark that, on denoting by θ, ϕ, ψ the three roots of the cubic equation $x^3+px+q=0$ and putting

$$\begin{aligned} z + r^{1/2}y &= \prod_{\theta,\phi,\psi} [\xi + \lambda\theta^2 + r^{1/2}(\eta + \mu\theta)], \\ z - r^{1/2}y &= \prod_{\theta,\phi,\psi} [\xi + \lambda\theta^2 - r^{1/2}(\eta + \mu\theta)], \end{aligned} \tag{10}$$

the equation (9) is satisfied if

$$x - \theta = (\xi + \lambda\theta^2)^2 - r(\eta + \mu\theta)^2 \tag{11}$$

holds, together with two other similar relations in ϕ and ψ. Writing, in (11), $-p\theta^2 - q\theta$ for θ^4 and comparing coefficients of θ^0, θ^1, θ^2, we obtain

$$\begin{aligned} x &= \xi^2 - r\eta^2, \\ 1 &= q\lambda^2 + 2r\mu\eta, \\ 0 &= 2\lambda\xi - p\lambda^2 - r\mu^2. \end{aligned} \tag{12}$$

The last two equations give

$$\xi = (p\lambda^2 + r\mu^2)/2\lambda, \qquad \eta = (1 - q\lambda^2)/2r\mu.$$

Hence, on substituting these expressions for ξ, η in (12), and in the two relations obtained by subtracting and adding the equations (10) together, we deduce x, y, z as rational functions of λ, μ, with rational coefficients, satisfying (9).

I consider next a rational cubic surface F, of which a rational simple point P is known, and give the following *geometric construction* leading always to *an infinity of rational points of F.*

A rational line r passing through P meets F again in a rational pair of points, H, K say. This means that H, K are either rational or quadratic conjugate; and we can suppose, by choice of r, that H, K, and P are distinct. I fix then another rational line s in a general position, and consider the twisted cubic curve, Γ say, uniquely defined by the conditions of osculating at H, K the sections of F with the planes Hs, Ks respectively. It is easily seen that Γ is rational and irreducible and does not lie on F. Hence Γ and F meet at $3 \cdot 3 = 9$ points, of which three are absorbed by H and three are absorbed by K, so that the remaining intersections constitute a rational triplet, M, N, O say. These three points are noncollinear and lie in a plane which does not contain P.

Denoting by π the (obviously rational) tangent plane of F at P, I finally consider the quadrics touching π at P and containing M, N, O. These quadrics constitute a homaloidal system. It can be proved by means of the Cremonian transformation defined by them that it is possible to choose (in an infinity of ways) two such quadrics, so that their intersection is an irreducible rational quartic curve, Δ say, touching F at M, N, and O. Δ does not lie on F, and so meets F at $3 \cdot 4 = 12$ points, of which four are absorbed by P and two are absorbed by each of the points M, N, O. The remaining intersections are

$$12 - 4 - 2 \cdot 3 = 2$$

in number, and form a rational pair, S, T say. Clearly the line ST is rational and does not lie on F; moreover ST does not contain P, since Δ has no trisecants. Hence ST meets F again at a point which is distinct from P and uniquely defined, and therefore rational. This rational point assumes an infinity of positions on F, on varying the lines r, s and the quartic Δ considered above.

It is easily seen that the irreducible rational cubic surfaces F which are *singular*, but not of the types already investigated, are of the following three kinds.

(i) F has four nonrational and noncoplanar double points.

(ii) F has three nonrational and noncollinear double points.
(iii) F has two nonrational double points.

I state now the following results concerning the corresponding Diophantine equations.

In case (i), *it is possible to obtain a complete rational two-parameter solution of the 3rd degree.*

In case (ii), *there may be no rational solutions. When any particular rational solution is known, it is possible to deduce a complete rational two-parameter solution of the 6th degree. No complete two-parameter solution of degree less than six exists, except when F satisfies certain arithmetical conditions, in which case F has a complete rational two-parameter solution of the 3rd degree.*

In case (iii), *F can always be solved parametrically, but in general there are no complete two-parameter solutions.*

In case (i), there is an infinity of rational twisted cubic curves containing the four double points of F, say A, B, C, D. In general one of these cubics is irreducible and does not lie on F, and so it has $3\cdot 3=9$ intersections with F, of which $2\cdot 4=8$ are absorbed by A, B, C, D. Hence the remaining intersection is rational. It is possible to construct in this way *all* the rational points of F. A *complete* two-parameter solution of the 3rd degree can be obtained, on transforming F into a rational plane by means of the homaloidal system formed by the ∞^3 cubic surfaces having four double points at A, B, C, D. Similar geometric arguments can be applied in case (ii).

These indications may suffice, and I add only the following applications concerning two particular cubic Diophantine equations of types (i) and (ii) respectively.

First put

$$\Phi = (ky^2 - 2kxz + t^2)^2 - k(x^2 + kz^2 - 2yt)^2,$$
$$\Psi = (k\mu^2 - 2k\lambda\nu + \omega^2)^2 - k(\lambda^2 + k\nu^2 - 2\mu\omega)^2,$$

where k denotes any nonzero rational number, and consider the cubic homogeneous Diophantine equation in x, y, z, t:

$$a\partial\Phi/\partial x + b\partial\Phi/\partial y + c\partial\Phi/\partial z + d\partial\Phi/\partial t = 0, \tag{13}$$

where a, b, c, d are any four rational numbers not all zero. A *complete* two-parameter solution of (13) is then

$$x:y:z:t = \frac{\partial\Psi}{\partial\nu}:\frac{\partial\Psi}{\partial\mu}:\frac{\partial\Psi}{\partial\lambda}:k\,\frac{\partial\Psi}{\partial\omega}, \tag{14}$$

where the parameters are subject to the linear equation

$$k(c\lambda + b\mu + a\nu) + d\omega = 0. \tag{15}$$

More precisely, *all* the rational solutions of (13) are given by (14) for rational values of $\lambda:\mu:\nu:\omega$ subject to (15), if k is not the square of a rational number. If $k=\rho^2$, with ρ rational, the equation (13) has the additional solutions

$$x + \rho z = t + \rho y = 0 \quad \text{and} \quad x - \rho z = t - \rho y = 0.$$

Secondly consider the rational cubic equation

$$a\theta^3 + b\theta^2 + c\theta + d = 0 \qquad (a \neq 0), \tag{16}$$

and suppose that it has three nonrational (and therefore distinct) roots θ, ϕ, ψ. Hence (16) has a nonzero discriminant

$$D = -27a^2d^2 - 4ac^3 + 18abcd - 4db^3 + b^2c^2.$$

It is easily seen that the rational cubic equation

$$\prod_{\theta,\phi,\psi} (x + y\theta + z\theta^2) = 1 \tag{17}$$

is of type (ii). This equation has been solved parametrically by Lagrange and Euler by taking

$$x + y\theta + z\theta^2 = (\lambda + \mu\theta + \nu\theta^2)^3 / \prod_{\theta,\phi,\psi} (\lambda + \mu\theta + \nu\theta^2) \tag{18}$$

and so on, where λ, μ, ν are parameters of which only the ratios are significant. The rational two-parameter solution of (17) given by (18) is of the 3rd degree, and, in agreement with a previously quoted result, is *noncomplete.* With the method indicated, however, it is possible, from the particular solution

$$x = 1, \qquad y = 0, \qquad z = 0$$

of (17), to deduce a *complete rational two-parameter solution of the 6th degree.* Moreover, it can be proved that *the equation* (17) *has a complete rational two-parameter solution of degree less than six if and only if the discriminant D of* (16) *is the square of a rational number.* When this condition is satisfied, (17) has in fact the *complete rational two-parameter solution of the 3rd degree* defined by the equation

$$x + y\theta + z\theta^2 = (\lambda + \mu\phi + \nu\phi^2)(\lambda + \mu\psi + \nu\psi^2)^2 / \prod_{\theta,\phi,\psi} (\lambda + \mu\theta + \nu\theta^2)$$

and by those obtained from it on permuting θ, ϕ, ψ cyclically.

I consider now the cubic surfaces which are *nonsingular.*[3] It is well known that any such surface F contains 27 lines, each of which may of course be rational or not. I first remark that *F may contain no rational points*, as it is shown by the examples

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[3] For a new approach to these surfaces, cf. B. Segre, *The non-singular cubic surfaces*, Oxford, 1942.

$$x^3 + 2y^3 + 4z^3 = 9 \quad \text{and} \quad x^3 + 2y^3 + 7z^3 = 14,$$

which can be easily generalized.

The determination of necessary and sufficient conditions for the existence of rational points on F is an important and difficult question. I have not succeeded in solving this problem, but the considera- of certain arithmetical properties for the lines of F leads to some sufficient conditions.

It must be noticed that *F cannot contain one rational point without consequently having an infinity of rational points.* For, since F is nonsingular, a *rational parametric solution* can always be deduced from a single rational solution, as said before (p. 156). The number of parameters involved is not essential, and can be made arbitrarily large by a repeated application of the tangent plane process. In particular, it follows easily that:

A nonsingular cubic surface F can be rationally represented upon a plane α if and only if F contains a rational point.

Such a representation does not give in general a (1, 1)-*correspondence* between F and α. In fact, for the existence of a representation having this property, further conditions are required from F. I have studied these conditions, as well as those for the existence on F of *nontrivial rational curves*, that is, of rational curves which are not the complete intersection of F with another rational surface. The results are unexpectedly simple, but I have no time now to dwell on them. I confine myself to pointing out the remarkable fact that these arithmetical conditions affect the 27 lines of F only.

The 27 lines of the cubic surface

$$ax^3 + by^3 + cz^3 = 1 \qquad (abc \neq 0) \tag{19}$$

can be obtained easily, and the general theory leads now to particularly simple results. On altering, in (19), x, y, z by arbitrary nonzero rational factors, and permuting arbitrarily x, y, z, we obtain an infinity of equations of the same type as (19), any two of which will be called *equivalent*.

I first consider the equations

$$ax^3 + acy^3 + cz^3 = 1 \qquad (ac \neq 0), \tag{20}$$

$$a(x^3 + y^3) + z^3 = 1 \qquad (a \neq 0), \tag{21}$$

$$x^3 + y^3 + z^3 = 1, \tag{22}$$

which obviously are successive particularizations of (19). I can then state the result:

A cubic surface (19) *contains some nontrivial rational curves, if and only if* (19) *is equivalent to an equation of the form* (20). *This condition is also necessary, but not sufficient, for* (19) *to have a complete rational two-parameter solution. An equation* (19) *has a rational two-parameter solution of degree three or four, possibly noncomplete, if and only if* (19) *is equivalent to an equation of the form* (22) *or* (21) *respectively.*

Such parametric solutions of (22) and (21) have already been obtained by Euler and Hermite respectively, and these are both *complete.* I add that there are *equations* (20) *having no rational solutions,* as for example, the one obtained from (20) for $a=1/2$, $c=1/7$. *When a single rational solution of* (20) *is known, it is possible to determine a complete rational two-parameter solution of* (20), *of degree six.*

I consider next the following particularizations of (19):

$$x^3 + y^3 + kz^3 + k = 0, \tag{23}$$

$$x^3 + y^3 + kz^3 + 2 = 0, \tag{24}$$

$$x^3 + y^3 + 2z^3 + 2 = 0, \tag{25}$$

where k is a nonzero rational number which is not twice the cube of a rational number, and state the following theorem.

The equation (19) *has solutions with* x, y, z *rational polynomials in a parameter* λ, *of degree four or less, if and only if* (19) *is equivalent to one of the equations* (23), (24), (25). *By a proper choice of* λ, *the solutions are as follows. The equation* (23) *has only the solutions*

$$x = \lambda, \qquad y = -\lambda, \qquad z = -1$$

and

$$x = -(9/k)\lambda^4 + 3\lambda, \qquad y = (9/k)\lambda^4, \qquad z = (9/k)\lambda^3 - 1.$$

The equation (24) *has only the solution*

$$x = (6/k)\lambda^3 - 1, \qquad y = -(6/k)\lambda^3 - 1, \qquad z = (6/k)\lambda^2.$$

The equation (25), *in addition to the three solutions given for* $k=2$ *by the previous expressions, has only the further solutions*

$$\begin{aligned} x&=4\lambda^2-6\lambda+1 & & x=(2/27)(4\lambda^4-4\lambda^3-6\lambda^2+17\lambda-2),\\ y&=4\lambda^2-2\lambda-1 & \textit{and} \quad & y=(4/27)(2\lambda^4-8\lambda^3+6\lambda^2+4\lambda-13),\\ z&=-4\lambda^2+4\lambda-1 & & z=(1/27)(-8\lambda^4+20\lambda^3-24\lambda^2-16\lambda+37).\end{aligned}$$

The particular result following from this theorem on supposing

$a=b=c$ in (19) has previously been obtained by Professor Mordell, in an entirely different way.

In conclusion, I regret I have had to dwell more on my results than on my methods. The latter are purely algebraic-geometric in character and, as I said at the beginning, can be extended and employed in other directions.

It may be noticed that my work does for cubic surfaces what Poincaré did for plane cubic curves more than 40 years ago. In my work, however, the use of geometry is more essential and far-reaching, since there are several properties of cubic surfaces which are arithmetically significant, and have no counterpart for plane cubic curves. For instance, the general cubic surface F is homaloidal, that is, F can be related in the complex domain to a plane by a (1,1) algebraic correspondence. A similar result does not hold for the general plane cubic curve, since this is not unicursal, but elliptic. Another important property having no analogue for plane cubic curves is that F contains 27 lines, and that every algebraic curve lying on F can be constructed as partial intersection of F with another algebraic surface, the residual intersection being a set of lines of F taken with proper multiplicities.

It ought to be mentioned that a few interesting, but isolated and comparatively elementary, examples of application of geometric ideas in the study of cubic ternary Diophantine equations have been given by Libri, Euler, Hermite and, in recent times, by H. W. Richmond.

I am glad to have the opportunity of acknowledging that I owe the first stimulus for my research to Professor Mordell, who, a little more than a year ago, pointed out to me the geometric results of the authors just named, as well as his own arithmetical results on sums of three cubes.

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THE PSEUDO-ANGLE IN SPACE OF $2n$ DIMENSIONS

JOHN DeCICCO

1. **Introduction.** The theory of functions of a single complex variable is essentially identical with the conformal geometry of the real (or complex) plane. However, this is not the case in the theory of functions of *two or more* complex variables. Any set of $n \geqq 2$ functions of n complex variables with nonvanishing jacobian induces a correspondence between the points of a real (or complex) $2n$-dimensional euclidean space R_{2n}. The infinite group G of all such correspondences is obviously *not* the conformal group of R_{2n}, which is merely the inversive group of $(n+1)(2n+1)$ parameters.[1] Poincaré in his fundamental paper in Palermo Rendiconti (1907) has called G the group of *regular* transformations. However, in an abstract presented before the American Mathematical Society, 1908, Kasner found it more appropriate to term it the *pseudo-conformal group* G. This name is now standard.

In his work of 1908, which he later published in full in 1940, Kasner investigated the possibility of characterizing the pseudo-conformal group G of four dimensions (the case $n=2$ complex variables) in a purely geometric way.[2] His principal result is as follows: *A transformation of R_4 is pseudo-conformal if and only if it preserves the pseudo-angle between any curve and a three-dimensional hypersurface at their point of intersection.* This theorem demonstrates how the pseudo-angle may be used to characterize G within the group of arbitrary point transformations of R_4.

We shall show in this paper how Kasner's pseudo-angle theorem can be carried over to $2n$ dimensions almost without any change. The pseudo-angle is important also because all other differential invariants of the first order under the pseudo-conformal group are really combinations of this pseudo-angle.[3]

2. **The minimal coordinates.** Let $(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n)$

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[1] The conformal group of a euclidean space R_m of any dimension $m>2$, odd or even, is the inversive group of $(m+1)(m+2)/2$ parameters (Liouville's theorem). Fialkow has studied the conformal geometry of any curve not only in a euclidean space R_m but also in any riemann space V_m. See his paper, *Conformal geometry of curves*, Trans. Amer. Math. Soc. vol. 51 (1942).

[2] Kasner, *Conformality in connection with functions of two complex variables*, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 50–62.

[3] See Kasner and DeCicco, *Pseudo-conformal geometry: Functions of two complex variables*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 317–328.

$= (x_\alpha, y_\alpha)$ denote the cartesian coordinates of a real or complex euclidean $2n$-dimensional space R_{2n}. We shall find it convenient to introduce the minimal coordinates $(u_1, u_2, \cdots, u_n; v_1, v_2, \cdots, v_n) = (u_\alpha, v_\alpha)$ defined by

$$(1) \qquad u_\alpha = x_\alpha + iy_\alpha, \qquad v_\alpha = x_\alpha - iy_\alpha,$$

for $\alpha = 1, 2, \cdots, n$. The inverse of this correspondence is

$$(2) \qquad x_\alpha = (u_\alpha + v_\alpha)/2, \qquad y_\alpha = (u_\alpha - v_\alpha)/2i.$$

The following relations are noted between the partial derivatives in minimal coordinates and cartesian coordinates:

$$(3) \qquad \frac{\partial}{\partial u_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}\right), \qquad \frac{\partial}{\partial v_\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}\right).$$

The operators $\partial/\partial u_\alpha$ may be called the *mean derivatives*; and the operators $\partial/\partial v_\alpha$ may be termed the *phase derivatives*.[4]

In minimal coordinates, the square of the linear element ds is

$$(4) \qquad ds^2 = \sum_{\alpha=1}^{n} du_\alpha dv_\alpha.$$

The angle θ between any two curve elements through a common point is

$$(5) \qquad \cos\theta = \frac{\sum_{\alpha=1}^{n}\left[\overset{(1)}{du}_\alpha \overset{(2)}{dv}_\alpha + \overset{(2)}{du}_\alpha \overset{(1)}{dv}_\alpha\right]}{2\left[\left(\sum_{\alpha=1}^{n}\overset{(1)}{du}_\alpha \overset{(1)}{dv}_\alpha\right)\left(\sum_{\alpha=1}^{n}\overset{(2)}{du}_\alpha \overset{(2)}{dv}_\alpha\right)\right]^{1/2}}.$$

3. **The pseudo-conformal group.** This is given in minimal coordinates by

$$(6) \qquad U_\alpha = U_\alpha(u_1, u_2, \cdots, u_n), \qquad V_\alpha = V_\alpha(v_1, v_2, \cdots, v_n),$$

for $\alpha = 1, 2, \cdots, n$, where the jacobians $|\partial U_\alpha/\partial u_\beta|$ and $|\partial V_\alpha/\partial v_\beta|$ are each not zero. Our problem is to inaugurate the study of this group in detail.

In what follows, we shall omit from consideration the special minimal n-flats $u_\alpha = \text{const.}$ and $v_\alpha = \text{const.}$ Our pseudo-conformal group may be defined as the direct part of the total mixed group preserving these $2\infty^n$ special minimal n-flats.

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[4] Kasner, *The second derivative of a polygenic function*, Trans. Amer. Math. Soc. vol. 30 (1928). Also Kasner and DeCicco, *The derivative circular congruence-representation of a polygenic function*, Amer. J. Math. vol. 61 (1939) pp. 995–1003.

4. The pseudo-conformal geometry of differential elements of first order. We shall be interested mainly in the geometry of the curve elements at a fixed point of R_{2n}. These form a $(2n-1)$-dimensional manifold $\Sigma_{(2n-1)}$.

Let $(\phi_1, \phi_2, \cdots, \phi_n; \psi_1, \psi_2, \cdots, \psi_n) = (\phi_\alpha, \psi_\alpha)$ be any set of numbers (not all zero) proportional to the differentials $(du_1, du_2, \cdots, du_n; dv_1, dv_2, \cdots, dv_n) = (du_\alpha, dv_\alpha)$ respectively so that $\phi_\alpha = \rho du_\alpha$, $\psi_\alpha = \rho dv_\alpha$. Then any such set and only those sets proportional to it define, in homogeneous coordinates, any curve element e of $\Sigma_{(2n-1)}$.

The pseudo-conformal group $'G$ induces the $(2n^2-1)$-parameter group $G_{(2n^2-1)}$ among the curve elements of $\Sigma_{(2n-1)}$ defined as follows:

$$\rho\Phi_\alpha = \sum_{\beta=1}^{n} a_{\alpha\beta}\phi_\beta, \qquad \rho\Psi_\alpha = \sum_{\beta=1}^{n} b_{\alpha\beta}\Psi_\beta, \tag{7}$$

for $\alpha = 1, 2, \cdots, n$, where the determinants $|a_{\alpha\beta}|$ and $|b_{\alpha\beta}|$ are each not zero.

5. The isoclines. An isocline γ_{2r} of $2r$ dimensions where $0 < r < n$ is defined by the system of $(2n-2r)$ linear equations of the special forms

$$\sum_{\beta=1}^{n} \lambda_{\alpha\beta}\phi_\beta = 0, \qquad \sum_{\beta=1}^{n} \mu_{\alpha\beta}\psi_\beta = 0, \tag{8}$$

for $\alpha = 1. 2, \cdots, n-r$, where each of the matrices $(\lambda_{\alpha\beta})$ and $(\mu_{\alpha\beta})$ is of rank $(n-r)$.

A consideration of these equations will show that there are $\infty^{2r(n-r)}$ isoclines γ_{2r} in $\Sigma_{(2n-1)}$. Also r curve elements which do not lie in a lower dimensional isocline determine a unique $2r$-dimensional isocline. Of course, two distinct isoclines will intersect in an isocline whose greatest possible dimension is equal to the lowest of the two given isoclines or else they will have no common curve elements.

THEOREM 1. *Under the induced pseudo-conformal group $G_{(2n^2-1)}$, any two $2r$-dimensional isoclines are equivalent. Any isocline γ_{2r} is a pseudo-conformal manifold $\Sigma_{(2r-1)}$ contained in the larger pseudo-conformal manifold $\Sigma_{(2n-1)}$.*

The proof of Theorem 1 is as follows. By applying (7) to (8), any $2r$-dimensional isocline becomes a $2r$-dimensional isocline under $G_{(2n^2-1)}$. Any isocline γ_{2r} may be carried into the canonical isocline $\gamma_{2r}(0)$

$$\phi_{r+1} = \phi_{r+2} = \cdots = \phi_n = 0, \quad \psi_{r+1} = \psi_{r+2} = \cdots = \psi_n = 0. \tag{9}$$

By the preceding, we may note that $(\phi_1, \phi_2, \cdots, \phi_r; \psi_1, \psi_2, \cdots, \psi_r)$ may be used as homogeneous coordinates of any curve element of the canonical isocline $\gamma_{2r}(0)$. By this remark and by determining the subgroup of $G_{(2n^2-1)}$ preserving $\gamma_{2r}(0)$, it is seen that the proof of our Theorem 1 is complete.

6. **The pseudo-conformal geometry of two curve elements.** In the first place, it is observed that the most general transformation of $G_{(2n^2-1)}$ which will carry the curve element $(1, 0, \cdots, 0; 1, 0, \cdots, 0)$ into the curve element $(\rho_1\Phi_\alpha^{(1)}, \rho_1\Psi_\alpha^{(1)})$ is of the form

$$(10)\qquad \rho\Phi_\alpha = \rho_1\Phi_\alpha^{(1)}\phi_1 + \sum_{\beta=2}^{n} a_{\alpha\beta}\phi_\beta, \qquad \rho\Psi_\alpha = \rho_1\Psi_\alpha^{(1)}\psi_1 + \sum_{\beta=2}^{n} b_{\alpha\beta}\Psi_\beta.$$

Any curve element which lies in the isocline of dimension two determined by $(1, 0, \cdots, 0; 1, 0, \cdots, 0)$ must be of the form $(\phi_1^{(2)}, 0, \cdots, 0; \psi_1^{(2)}, 0, \cdots, 0)$. The transform of this is

$$(11)\qquad \rho\Phi_\alpha^{(2)} = \rho_1\Phi_\alpha^{(1)}\phi_1^{(2)}, \qquad \rho\Psi_\alpha^{(2)} = \rho_1\Psi_\alpha^{(1)}\psi_1^{(2)}.$$

These immediately guarantee that the transformed element $(\Phi_\alpha^{(2)}, \Psi_\alpha^{(2)})$ is in the isocline determined by $(\Phi_\alpha^{(1)}, \Psi_\alpha^{(1)})$. Moreover the expression

$$(12)\qquad (\Phi_\alpha^{(2)}/\Psi_\alpha^{(2)})\cdot(\Psi_\alpha^{(1)}/\Phi_\alpha^{(1)}),$$

which is the same for $\alpha=1, 2, \cdots, n$, is invariant. By taking the logarithm of this invariant, and then multiplying the result by $1/2i$, it is found by (5) that the resulting invariant represents the angle between the two given curve elements.

THEOREM 2. *Two curves elements e_1 and e_2 which lie in the same two-dimensional isocline γ_2 possess the unique invariant*

$$(13)\qquad \theta = \frac{1}{2i}\log\frac{\phi_\alpha^{(2)}}{\psi_\alpha^{(2)}}\cdot\frac{\psi_\alpha^{(1)}}{\phi_\alpha^{(1)}},$$

which is the same for all $\alpha=1, 2, \cdots, n$. It actually is the angle between the two curve elements e_1 and e_2.

It may be proved that any pair of curve elements not in the same two-dimensional isocline may be carried into any other such pair. Thus two curve elements will possess a differential invariant of the first order if and only if they lie in the same two-dimensional isocline γ_2. In that case, they have a unique invariant which is actually the angle between them.

7. **Kasner's pseudo-angle.** Any $(2n-1)$-dimensional hypersurface

element $S_{(2n-1)}$ of $\Sigma_{(2n-1)}$ may be given by the equation

$$\sum_{\alpha=1}^{n} (k_\alpha \phi_\alpha + l_\alpha \psi_\alpha) = 0. \tag{14}$$

The homogeneous coordinates of any hypersurface element of $(2n-1)$ dimensions are $(k_1, k_2, \cdots, k_n; l_1, l_2, \cdots, l_n) = (k_\alpha, l_\alpha)$.

Since we wish to omit from consideration those hypersurface elements which contain the special minimal n-flat elements, neither all the k's nor all the l's are zero.

The transformation formulas between the $(2n-1)$-dimensional hypersurface elements $S_{(2n-1)}$ in $\Sigma_{(2n-1)}$ are

$$\sigma k_\beta = \sum_{\alpha=1}^{n} a_{\alpha\beta} K_\alpha, \qquad \sigma l_\beta = \sum_{\alpha=1}^{n} b_{\alpha\beta} L_\alpha, \tag{15}$$

for $\beta = 1, 2, \cdots, n$.

Let $e(\phi_\alpha, \psi_\alpha)$ be any curve element of $\Sigma_{(2n-1)}$. The two-dimensional isocline determined by e is given parametrically by

$$\Phi_\alpha = r\phi_\alpha, \qquad \Psi_\alpha = s\psi_\alpha, \tag{16}$$

where r and s are the variable parameters. This isocline intersects the hypersurface element $S_{(2n-1)}$ given by equation (14) in the curve element $e^*(\Phi_\alpha, \Psi_\alpha)$ given by

$$\frac{\Phi_\beta}{\phi_\beta} = -\lambda \sum_{\alpha=1}^{n} l_\alpha \psi_\alpha, \qquad \frac{\Psi_\beta}{\psi_\beta} = \lambda \sum_{\alpha=1}^{n} k_\alpha \phi_\alpha. \tag{17}$$

By Theorem 2, the angle between the two curve elements e and e^* is invariant. Therefore it is an invariant between e and $S_{(2n-1)}$.

We shall now show that the angle obtained above is the unique invariant. Any curve element e may be carried into the canonical curve element $e^{(0)}(1, 0, \cdots, 0; 1, 0, \cdots. 0)$. By (10), it is seen that the group preserving this canonical curve element $e^{(0)}$ must satisfy the conditions $a_{11} = b_{11} = \rho_1$, $a_{\alpha 1} = b_{\alpha 1} = 0$, for $\alpha = 2, 3, \cdots, n$. Hence (15) may be written in the form

$$\begin{aligned} \sigma k_1 &= \rho_1 K_1, & \sigma l_1 &= \rho_1 L_1, \\ \sigma k_\beta &= \sum_{\alpha=1}^{n} a_{\alpha\beta} K_\alpha, & \sigma l_\beta &= \sum_{\alpha=1}^{n} b_{\alpha\beta} L_\alpha, \end{aligned} \tag{18}$$

for $\beta = 2, 3, \cdots, n$.

By choosing $a_{1\beta} = \sigma k_\beta / K_1$ for $\beta = 2, 3, \cdots, n$, the above transformation carries the hypersurface element (k_α, l_α) into the hypersurface element $(K_1, 0, \cdots, 0; L_1, 0, \cdots, 0)$. This clearly proves that a curve element e and a hypersurface element S_{2n-1} possess only one invariant.

THEOREM 3. *A curve element* $e(\phi_\alpha, \psi_\alpha)$ *and a* $(2n-1)$*-dimensional hypersurface element* $S_{2n-1}(k_\alpha, l_\alpha)$ *possess only the single invariant*

$$\theta = \frac{1}{2i} \log \left[- \sum_{\alpha=1}^{n} l_\alpha v_\alpha \Big/ \sum_{\alpha=1}^{n} k_\alpha u_\alpha \right]. \tag{19}$$

This is Kasner's pseudo-angle between e and S_{2n-1}. *It represents the actual angle* θ *between e and the curve element* e^* *in* S_{2n-1} *such that e and* e^* *are on the same two-dimensional isocline.*

In the next and final section, we shall show that the pseudo-angle characterizes the pseudo-conformal group G. It is remarked that this is a direct generalization of the fact that the group of functions of a single complex variable is identical with the conformal group of the plane.

8. Characterization of the pseudo-conformal group G **by the pseudo-angle.** We shall prove the following fundamental theorem which is essentially Kasner's characterization in $2n$ dimensions.

THEOREM 4. *A transformation of* $2n$*-dimensional euclidean space* R_{2n} *is pseudo-conformal if and only if it preserves the pseudo-angle defined (in cartesian coordinates) by*

$$\theta = \arc\tan \frac{\sum_{\alpha=1}^{n} (F_{x_\alpha} dx_\alpha + F_{y_\alpha} dy_\alpha)}{\sum_{\alpha=1}^{n} (F_{x_\alpha} dy_\alpha - F_{y_\alpha} dx_\alpha)}, \tag{20}$$

between any curve $C: x_\alpha = x_\alpha(t)$, $y_\alpha = y_\alpha(t)$, *and any* $(2n-1)$*-dimensional hypersurface* $S_{(2n-1)}: F(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_n) = 0$ *at their common point of intersection.*

Any arbitrary transformation T with nonvanishing jacobian induces at any given point a general projectivity in $\Sigma_{(2n-1)}$ which in minimal curve element coordinates may be written as

$$\rho\Phi_\alpha = \sum_{\beta=1}^{n} (a_{\alpha\beta}\phi_\beta + b'_{\alpha\beta}\psi_\beta), \qquad \rho\Psi_\alpha = \sum_{\beta=1}^{n} (b_{\alpha\beta}\psi_\beta + a'_{\alpha\beta}\phi_\beta). \tag{21}$$

Also this projectivity in hypersurface element coordinates may be written as

$$\sigma k_\beta = \sum_{\alpha=1}^{n} (a_{\alpha\beta}K_\alpha + a'_{\alpha\beta}L_\alpha), \qquad \sigma l_\beta = \sum_{\alpha=1}^{n} (b_{\alpha\beta}L_\alpha + b'_{\alpha\beta}K_\alpha), \tag{22}$$

where the determinants of the coefficients are not zero.

Now let T preserve the pseudo-angle (20). In minimal coordinates, this may be written in the form (19). Then the fraction of (19) is equal to the same fraction with the small letters replaced by the capital letters. Upon eliminating Φ_α, Ψ_α, k_β, l_β by means of (21) and (22) in this resulting identity, we find

$$\frac{\sum_{\alpha,\beta=1}^{n} L_\alpha(b_{\alpha\beta}\psi_\beta + a'_{\alpha\beta}\phi_\beta)}{\sum_{\alpha,\beta=1}^{n} K_\alpha(a_{\alpha\beta}\phi_\beta + b'_{\alpha\beta}\psi_\beta)} = \frac{\sum_{\gamma,\delta=1}^{n} \psi_\delta(b_{\gamma\delta}L_\gamma + b'_{\gamma\delta}K_\gamma)}{\sum_{\gamma,\delta=1}^{n} \phi_\delta(a_{\gamma\delta}K_\gamma + a'_{\gamma\delta}L_\gamma)}. \tag{23}$$

Now this must be an identity for all $(\phi_\alpha, \psi_\alpha, K_\alpha, L_\alpha)$. Placing the coefficients of ϕ_β^2 and ψ_β^2 equal to zero, we find

$$\begin{aligned} \left[\sum_{\alpha=1}^{n} a'_{\alpha\beta}L_\alpha\right]\left[\sum_{\gamma=1}^{n} (a_{\gamma\beta}K_\gamma + a'_{\gamma\beta}L_\gamma)\right] &= 0, \\ \left[\sum_{\alpha=1}^{n} b'_{\alpha\beta}K_\alpha\right]\left[\sum_{\gamma=1}^{n} (b_{\gamma\beta}L_\gamma + b'_{\gamma\beta}K_\gamma)\right] &= 0. \end{aligned} \tag{24}$$

These are identities in K and L. Now the second factors of each of these equations cannot identically vanish for then the determinant of (21) or (22) is zero. Therefore the preceding equations will be identities if and only if the first factors are identically zero. Therefore $a'_{\alpha\beta} = b'_{\alpha\beta} = 0$ for all $\alpha, \beta = 1, 2, \cdots, n$.

Since $a'_{\alpha\beta} = b'_{\alpha\beta} = 0$, our transformation (21) or (22) is induced pseudo-conformal. Our Theorem 4 is therefore proved, for this result is valid at any fixed point.

ILLINOIS INSTITUTE OF TECHNOLOGY

BI-ISOTHERMAL SYSTEMS

EDWARD KASNER AND JOHN DeCICCO

1. **Introduction.** A pair of functions of two complex variables with nonvanishing jacobian induces a correspondence between the points of a real (or complex) four-dimensional euclidean space R_4. The infinite group G of all such correspondences is not the conformal group of R_4, which is merely the inversive group of fifteen parameters. In 1908, the senior author termed G the *pseudo-conformal group*. This is now standard terminology. (See papers by Cartan, Carathéodory, and S. Bergmann.) It will be recalled that Poincaré, in his fundamental Palermo memoir of 1907, tentatively called G the regular group.

In 1908, Kasner showed that the pseudo-conformal group G is characterized by the preservation of the pseudo-angle between any curve and a hypersurface at their common point of intersection.[1] This is a proper generalization of the well known result that the group of functions of a single complex variable is identical with the (direct) conformal group of the plane.

In R_4, there is a class of surfaces which is transformed into itself under the infinite pseudo-conformal group G such that the induced correspondence between any corresponding pair of such surfaces is (direct) conformal. Any such surface is said to be a *conformal surface*. The term *analytic surface* is also in use.

Upon projecting orthogonally (by means of absolutely perpendicular planes) a conformal surface upon a pair of selected coordinate planes, the three induced correspondences (surface on each of the two planes, and plane on plane) are each conformal. Thus any conformal surface may be defined by means of a conformality between the selected pair of coordinate planes.

We shall say that a *bi-isothermal system of* ∞^3 *curves* in R_4 is any system which is pseudo-conformally equivalent to a parallel pencil of ∞^3 straight lines in R_4. Any such system consists of ∞^2 isothermal families of ∞^1 curves, each such family lying upon a conformal surface.

We define a *bi-isothermal system of* ∞^1 *hypersurfaces* in R_4 as any system which is pseudo-conformally equivalent to a parallel pencil of ∞^1 hyperplanes in R_4. Any bi-isothermal system of hypersurfaces may be defined by placing a biharmonic function equal to an arbi-

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[1] Kasner, *Conformality in connection with functions of two complex variables*, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 50–62.

trary constant. In that case, the constant may be called the *bi-isothermal parameter.* Of course, it is seen that, in general, a bi-isothermal system of ∞^1 hypersurfaces may be defined by placing a function of a biharmonic function equal to an arbitrary constant.

We find the following theorems on bi-isothermals fundamental in the geometry of functions of two complex variables.

1. *The pseudo-angle between any bi-isothermal system of hypersurfaces and any bi-isothermal system of curves is a biharmonic function.* This may be considered to be an extension of a theorem of Lie concerning isothermal systems in the plane. (See also a generalization to surfaces to appear in Proc. Nat. Acad. Sci. U.S.A., February, 1945.)

2. Any bi-isothermal system of hypersurfaces is intersected by a conformal surface in an isothermal system of curves. 3. If a surface is intersected by every bi-isothermal system of hypersurfaces in an isothermal system of curves, then the surface is conformal. 4. If a given system of ∞^1 hypersurfaces is intersected by every conformal surface in an isothermal system of curves, then the given system of ∞^1 hypersurfaces must be bi-isothermal.[2]

2. **Minimal coordinates.** Let $(x_1, x_2; y_1, y_2)$ define cartesian coordinates in the euclidean space R_4. We shall find it convenient to introduce the minimal coordinates defined by

$$
\begin{aligned}
u_1 &= x_1 + iy_1, & v_1 &= x_1 - iy_1, \\
u_2 &= x_2 + iy_2, & v_2 &= x_2 - iy_2.
\end{aligned} \tag{1}
$$

It is noted that the relations between the partial derivative operators in minimal and cartesian coordinates are

$$
\begin{aligned}
\frac{\partial}{\partial u_1} &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial y_1}\right), & \frac{\partial}{\partial v_1} &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial y_1}\right), \\
\frac{\partial}{\partial u_2} &= \frac{1}{2}\left(\frac{\partial}{\partial x_2} - i\frac{\partial}{\partial y_2}\right), & \frac{\partial}{\partial v_2} &= \frac{1}{2}\left(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial y_2}\right).
\end{aligned} \tag{2}
$$

The operators $\partial/\partial u_1$ and $\partial/\partial u_2$ are called the *mean derivatives* and the operators $\partial/\partial v_1$ and $\partial/\partial v_2$ are termed the *phase derivatives*:[3]

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[2] These theorems may be considered to be extensions of various theorems given in the paper of Kasner, *Biharmonic functions and certain generalizations,* Amer. J. Math. vol. 58 (1936) pp. 377–390.

[3] Kasner, *The second derivative of a polygenic function,* Trans. Amer. Math. Soc. vol. 30 (1928) pp. 803–818. Also Kasner and DeCicco, *The derivative circular congruence-representation of a polygenic function,* Amer. J. Math. vol. 61 (1939) pp. 995–1003. Also a forthcoming paper in Scripta Mathematica, 1945.

In minimal coordinates, the square of the linear element ds is

$$ds^2 = du_1dv_1 + du_2dv_2. \tag{3}$$

The cosine of the angle θ between any two curve elements through a common point is

$$\cos\theta = \frac{du_1^{(1)}dv_1^{(2)} + du_1^{(2)}dv_1^{(1)} + du_2^{(1)}dv_2^{(2)} + du_2^{(2)}dv_2^{(1)}}{2\left[(du_1^{(1)}dv_1^{(1)} + du_2^{(1)}dv_2^{(1)})(du_1^{(2)}dv_1^{(2)} + du_2^{(2)}dv_2^{(2)})\right]^{1/2}}. \tag{4}$$

3. **The infinite continuous pseudo-conformal group** G. This is defined by

$$\begin{aligned} U_1 &= U_1(u_1, u_2), & V_1 &= V_1(v_1, v_2),\\ U_2 &= U_2(u_1, u_2), & V_2 &= V_2(v_1, v_2), \end{aligned} \tag{5}$$

where the jacobians $\partial(U_1, U_2)/\partial(u_1, u_2)$ and $\partial(V_1, V_2)/\partial(v_1, v_2)$ are each not zero.

We have already found all the possible differential invariants of the first order of this infinite continuous pseudo-conformal group G.[4]

4. **Kasner's pseudo-angle.** A hypersurface $S_3\colon F(u_1, u_2; v_1, v_2)=0$ and a curve $C\colon u_1=u_1(t),\ u_2=u_2(t);\ v_1=v_1(t),\ v_2=v_2(t)$ possess the fundamental differential invariant of first order

$$\theta = \frac{1}{2i}\log\left[-\frac{F_{v_1}dv_1 + F_{v_2}dv_2}{F_{u_1}du_1 + F_{u_2}du_2}\right]. \tag{6}$$

This represents the actual angle θ between the given curve C and the curve of intersection C' between the hypersurface S_3 and the unique conformal surface determined by the curve C.

5. **Bi-isothermal systems of ∞^3 curves.** A system of ∞^3 curves is *bi-isothermal* if it is pseudo-conformally equivalent to a parallel pencil of straight lines in R_4. By means of (5), it is seen that any bi-isothermal system of ∞^3 curves may be given by equations of the forms

$$\begin{gathered} \lambda(u_1, u_2) = \text{const.}, \qquad \mu(v_1, v_2) = \text{const.},\\ \nu(u_1, u_2) + \omega(v_1, v_2) = \text{const.} \end{gathered} \tag{7}$$

It therefore follows that any such system of ∞^3 curves may be defined by a system of three differential equations of the forms

$$\begin{gathered} du_2/du_1 = \alpha(u_1, u_2), \qquad dv_2/dv_1 = \beta(v_1, v_2),\\ dv_1/du_1 = \gamma(v_1, v_2)/\delta(u_1, u_2). \end{gathered} \tag{8}$$

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[4] Kasner and DeCicco, *Pseudo-conformal geometry: Functions of two complex variables*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 317–328.

Conversely, the ∞^3 integral curves of any system of differential equations reducible to the form (8) must be a bi-isothermal system.

Since any conformal surface is given by equations of the form $l(u_1, u_2)=0$, $m(v_1, v_2)=0$, it follows by (7) that any bi-isothermal system of ∞^3 curves consists of ∞^2 isothermal families of ∞^1 curves, each such family lying on a conformal surface.

6. **Bi-isothermal systems of ∞^1 hypersurfaces.** Any system of ∞^1 hypersurfaces which is pseudo-conformally equivalent to a parallel pencil of ∞^1 hyperplanes is said to be bi-isothermal. Clearly the system

$$f(u_1, u_2) + g(v_1, v_2) = \text{const.} \tag{9}$$

is bi-isothermal. In this form, the constant is said to be a *bi-isothermal parameter.*

Conversely let $F(u_1, u_2; v_1, v_2)=\text{const.}$ define a bi-isothermal system of ∞^1 hypersurfaces. There must exist a function $\phi(F)$ which is a biharmonic function of $(u_1, u_2; v_1, v_2)$. Therefore, in consequence of the fundamental Poincaré conditions, we find

$$\Delta = \frac{F_{u_1v_1}}{F_{u_1}F_{v_1}} = \frac{F_{u_2v_2}}{F_{u_2}F_{v_2}} = \frac{F_{u_1v_2}}{F_{u_1}F_{v_2}} = \frac{F_{u_2v_1}}{F_{u_2}F_{v_1}} \tag{10}$$

must be a function of F only. Thus Δ must satisfy the additional third-order differential equations

$$\frac{\Delta_{u_1}}{F_{u_1}} = \frac{\Delta_{u_2}}{F_{u_2}} = \frac{\Delta_{v_1}}{F_{v_1}} = \frac{\Delta_{v_2}}{F_{v_2}}. \tag{11}$$

Therefore, we have proved the following result.

THEOREM 1. *The necessary and sufficient conditions that* $F(u_1, u_2; v_1, v_2)=\text{const.}$ *define a bi-isothermal system of* ∞^1 *hypersurfaces is that* F *satisfy the system of six partial differential equations of third order, given by* (10) *and* (11).

Upon substituting (8) and (9) into (6), we discover the following proposition.

THEOREM 2. *The pseudo-angle between any bi-isothermal system of* ∞^1 *hypersurfaces and any bi-isothermal system of* ∞^3 *curves is a biharmonic function of* $(u_1, u_2; v_1, v_2)$, *that is,* θ *is given by an equation of the form* $\theta=h(u_1, u_2)+k(v_1, v_2)$.

In the next and final section, we shall give certain theorems which may be considered to be generalizations of certain theorems of Kasner which characterize biharmonic functions.

7. Bi-isothermal systems of ∞^1 hypersurfaces and conformal surfaces. A conformal surface S_2 induces a conformal correspondence between the (x_1, y_1) and the (x_2, y_2) coordinate planes. Thus S_2 is defined by the equations $u_2 = u_2(u_1)$, $v_2 = v_2(v_1)$. By the induced conformal correspondences, it follows that an isothermal system on the (x_1, y_1) coordinate plane corresponds to an isothermal system on the (x_2, y_2) plane, and also to an isothermal system on the given conformal surface S_2.

THEOREM 3. *Any bi-isothermal system of ∞^1 hypersurfaces is intersected by a conformal surface in an isothermal system.*

This result is obtained by substituting the equations of a conformal surface into the equation (9), defining a bi-isothermal system of hypersurfaces.

THEOREM 4. *If a surface is intersected by every bi-isothermal system of ∞^1 hypersurfaces in an isothermal system of curves, then the surface is a conformal surface.*

Any surface S_2 may be given by the equations $u_2 = u_2(u_1, v_1)$, $v_2 = v_2(u_1, v_1)$. Upon substituting this into the equation (9) defining any bi-isothermal system, the resulting system of ∞^1 curves must be isothermal. Hence

$$\frac{\partial^2}{\partial u_1 \partial v_1} \log \left[\frac{f_{u_1} + f_{u_2} \partial u_2/\partial u_1 + g_{v_2} \partial v_2/\partial u_1}{g_{v_1} + g_{v_2} \partial v_2/\partial v_1 + f_{u_2} \partial u_2/\partial v_1} \right] = 0. \tag{12}$$

This must be an identity for all values of the partial derivatives of f and g.

The above equation involves partial derivatives of the third order in f and g. Upon setting the coefficients of $f_{u_1 u_1 u_2}$ and $g_{v_1 v_1 v_2}$ equal to zero, we discover that $\partial u_2/\partial v_1 = 0$ and $\partial v_2/\partial u_1 = 0$. This proves that the surface S_2 is conformal. Theorem 4 is completely proved.

THEOREM 5. *If a given system of ∞^1 hypersurfaces is intersected by every conformal surface in an isothermal family of curves, then the system of ∞^1 hypersurfaces is bi-isothermal.*

Let the given system of ∞^1 hypersurfaces be $F(u_1, u_2; v_1, v_2) = \text{const.}$ Substituting the equations of any conformal surface $u_2 = u_2(u_1)$, $v_2 = v_2(v_1)$ into it, the resulting family must be isothermal. Hence we have

$$\frac{\partial^2}{\partial u_1 \partial v_1} \log \left[\frac{F_{u_1} + F_{u_2} du_2/du_1}{F_{v_1} + F_{v_2} dv_2/dv_1} \right] = 0. \tag{13}$$

This is a differential equation of the second order in the total derivatives of $u_2 = u_2(u_1)$, $v_2 = v_2(v_1)$ which must be identically zero. Upon setting the coefficients of d^2u_2/du_1^2 and d^2v_2/dv_1^2 equal to zero, we find

$$\begin{aligned} &(F_{u_1}F_{u_2v_1} - F_{u_2}F_{u_1v_1}) + (F_{u_1}F_{u_2v_2} - F_{u_2}F_{u_1v_2})dv_2/dv_1 = 0, \\ &(F_{v_1}F_{u_1v_2} - F_{v_2}F_{u_1v_1}) + (F_{v_1}F_{u_2v_2} - F_{v_2}F_{u_2v_1})du_2/du_1 = 0. \end{aligned} \tag{14}$$

These identities yield the equations (10).

Substituting these into the equation (13), we find

$$\frac{\partial}{\partial u_1}\left[F_{v_1} + F_{v_2}\frac{dv_2}{dv_1}\right]\Delta = \frac{\partial}{\partial v_1}\left[F_{u_1} + F_{u_2}\frac{du_2}{du_1}\right]\Delta. \tag{15}$$

Upon setting the various coefficients of du_2/du_1 and dv_2/dv_1 of the above equation equal to zero, we obtain the equations (11).

Thus the hypothesis of Theorem 5 has led us to the conditions (10) and (11) for a bi-isothermal system of ∞^1 hypersurfaces. This completes our proof.

COLUMBIA UNIVERSITY AND
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PROJECTIVE DESCRIPTION OF SOME PLANE SEXTIC CURVES DERIVED FROM CONICS AS BASE CURVES

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Introduction. Comparatively few of the plane sextic curves are known. None of the papers written about them gives a complete treatment or a classification of these curves. The sextics appear more or less isolated in the different treatises. Apparently the methods employed produce only a very limited number of sextics.

The projective method of deriving higher plane curves described by Dr. H. P. Pettit in his *Projective description of some higher plane curves*[1] allows a more systematical study of sextics and especially a direct construction point by point.

It is the purpose of this paper to give a projective description of some of the sextics that can be derived by this method, if conics are the base curves. The number of types of these curves which could be generated by this method is here limited to 354 by giving the triangle of reference special positions with respect to the base conics.

Discussion of the general curve derived from conics as base curves. The method described by Dr. Pettit in his *Projective description of some higher plane curves* is as follows:

Let there be given two curves, C_1 of order n and C_2 of order m, two projective pencils A_1, A_3 and one connecting pencil A_2 in the plane. Any line on A_1 cuts C_1 in n points and the n lines joining these points to A_2 cut C_2 in mn points, which determine mn lines on A_3 cutting the line on A_1 in points of the generated curve.

For the present purpose let A_1, A_2, A_3 be the vertices of the triangle of reference and let the equation of the pencil on A_1 be

$$x_2 - \lambda x_3 = 0, \tag{1}$$

that of the pencil on A_2 be

$$x_1 - \mu x_3 = 0, \tag{2}$$

and that of the pencil on A_3 be

$$x_1 - \nu x_2 = 0. \tag{3}$$

Let the equations of the base conics be

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[1] Dr. H. P. Pettit, *Projective description of some higher plane curves*, Tôhoku Math. J. vol. 28 (1927–1928) pp. 72–79.

$$(4)\qquad u_0x_1^2 + u_1x_1 + u_2 = 0,$$

$$(5)\qquad v_2 + v_1x_3 + v_0x_3^2 = 0,$$

where

$$u_0 = a, \qquad v_0 = c',$$
$$u_1 = 2(gx_3 + hx_2), \qquad v_1 = 2(g'x_1 + f'x_2),$$
$$u_2 = bx_2^2 + cx_3^2 + 2fx_2x_3, \qquad v_2 = a'x_1^2 + b'x_2^2 + 2h'x_1x_2.$$

If we eliminate x_1, x_2, x_3 from (1), (2) and (4), (4) takes the form $f(\mu, \lambda, 1) = 0$, and if we eliminate x_1, x_2, x_3 from (2), (3) and (5), (5) takes the form $\phi(\mu\nu, \mu, \nu) = 0$. Then eliminating μ from $f(\mu, \lambda, 1) = 0$ and $\phi(\mu\nu, \mu, \nu) = 0$ and expressing λ and ν again in terms of x_1, x_2, x_3 from (1) and (3) we obtain the equation of the generated curve

$$(6)\qquad (u_0v_0x_1^2x_3^2 - u_2v_2)^2 - x_1x_3(u_0v_1x_1x_3 - u_1v_2)(u_1v_0x_1x_3 - u_2v_1) = 0.$$

Hence the general curve derived by this method, if conics are the base curves, is an octic curve. It follows from the equation (6) that the octic passes through the intersections of the quartic

$$Q: \qquad u_0v_0x_1^2x_3^2 - u_2v_2 = 0$$

with the cubic

$$C_1: \qquad u_0v_1x_1x_3 - u_1v_2 = 0$$

and the cubic

$$C_2: \qquad u_1v_0x_1x_3 - u_2v_1 = 0.$$

The quartic Q counted twice is generated by the same method, if A_1 is the pole of $x_1 = 0$ with respect to the first base conic (4) and A_3 is the pole of $x_3 = 0$ with respect to the second base conic (5). The base curves for the generation of the quartic are then

$$(a)\qquad u_0x_1^2 + u_2 = 0,$$

$$(b)\qquad v_2 + v_0x_3^2 = 0.$$

The conic (a) passes through the intersections of $x_1 = 0$, $u_1 = 0$ with the conic (4) and it is tangent to the lines $u_2 = 0$ at the intersections of $x_1 = 0$ with (4). The conic (b) passes through the intersections of $x_3 = 0$, $v_1 = 0$ with (5) and it is tangent to the lines $v_2 = 0$ at the intersections of $x_3 = 0$ with the conic (5).

By the same method the cubic C_1 is generated, if

$$\text{(a)} \qquad u_0x_1 + u_1 = 0,$$

$$\text{(b)} \qquad v_2 + v_1x_3 = 0$$

are the base curves. The line (a) passes through the intersection of $x_1=0$ with $u_1=0$. It cuts (4) in the points in which $u_2=0$ cuts (4) but not in the points where $x_1=0$ cuts (4) at the same time. The conic (b) has contact with (5) at both the intersections of $x_3=0$ with (5). It passes through A_3.

The cubic C_2 is generated, if

$$\text{(a)} \qquad u_1x_1 + u_2 = 0,$$

$$\text{(b)} \qquad v_1 + v_0x_3 = 0$$

are the base curves. The conic (a) passes through A_1 and has contact with (4) at both the intersections of $x_1=0$ with (4). The line (b) passes through the intersection of $x_3=0$ with $v_1=0$. It cuts (5) where $v_2=0$ cuts (5) but not at the point where $x_3=0$ cuts (5).

Thus the quartic and the cubics can be constructed easily and the intersections of these curves can be found geometrically. Algebraically the intersections of the quartic with the cubic C_1 are given by

$$\text{(7)} \quad x_2^2v_2\{[b(2gv_2 - ax_1v_1) - 2fhv_2]^2 - 4h^2v_2[(f^2 - bc)v_2 + abc'x_1^2]\} = 0.$$

Since a quartic and a cubic intersect in twelve points, four of the intersections are at A_3. Two intersections are at A_1, two given by $v_2=0$ are where $x_3=0$ cuts the conic (5) and the other four given by the expression in the braces are outside of the triangle of reference.

The intersections of Q with C_2 are given by

$$\text{(7}'\text{)} \quad x_1^2x_2^4\{[b(2gv_2 - ax_1v_1) - 2fhv_2]^2 - 4h^2v_2[(f^2 - bc)v_2 + abc'x_1^2]\} = 0.$$

Hence two of them are at A_3, four at A_1, two where $x_1=0$ cuts the conic (4) and four given by

$$\text{(8)} \quad [b(2gv_2 - ax_1v_1) - 2fhv_2]^2 - 4h^2v_2[(f^2 - bc)v_2 + abc'x_1^2] = 0$$

are outside of the triangle of reference. Comparing (7) with (7′), we see that the quartic cuts the cubics in the same points outside of the triangle of reference.

The intersections at A_1 and A_3 are quadruple points of the octic, for $x_2=0$ cuts (6) in $x_1^4x_3^4=0$. The line $x_1=0$ cuts (6) in the double points given by $u_2^2=0$, and $x_3=0$ cuts the octic in the double points given by $v_2^2=0$. These four double points are where $x_1=0$ cuts the conic (4) and where $x_3=0$ cuts the conic (5). The four intersections outside of the triangle of reference are also double points of the octic, for any line satisfying (8) cuts the octic in points given by

$$(k_1x_1 + k_2x_3)^2(K_1^2 - x_1x_3K_2K_3) = 0,$$

where K_1 is an expression of the third degree in x_1, x_3 and K_2, K_3 are of the second degree in x_1, x_3. $k_1x_1+k_2x_3=0$ gives a point of intersection of a line given by (8) with the octic or with the quartic and the cubics. But this point is a double point of the octic as follows from the above equation. Thus we have the following theorem.

THEOREM I. *If the base curves are conics, the curve generated by this method is generally an octic curve with two quadruple points and eight double points.*

This octic curve will degenerate into sextic curves, if (a) A_1 is on the first base conic (4) or what is the same if A_3 is on the second base conic (5); (b) if A_2 is on one of the base conics; (c) if the line A_1A_3 passes through two intersections of the base conics.[2] These degenerations of the octic produce only sextics with deficiencies zero and one. The singularities occurring are quadruple points and double points. If by this method sextics with a higher deficiency can be generated when conics are the base curves has not yet been determined. Likewise it remains a question whether or not there exists the possibility of obtaining by this method sextics with triple points when conics are the base curves.

Sextics with one quadruple point and three double points. Let A^1 be on the first base conic (4). Then the equation (6) becomes:

$$v_2[u_2^2v_2 + u_1x_1x_3(u_1v_0x_1x_3 - u_2v_1)] = 0.$$

Hence the lines $v_2=0$ break off from the octic and the generated curve is a sextic. Since the lines $v_2=0$ split off from the octic, A_3 becomes a double point and the intersections of $x_3=0$ with (5) become ordinary points of the curve. The equation (8) reduces to $v_2^2=0$. Therefore there are no double points outside of the triangle of reference.

THEOREM II. *The generated curve is a plane sextic with one quadruple point and three double points, if A_1 is on the first base conic. The equation of this sextic is*

$$u_2^2v_2 + u_1x_1x_3(u_1v_0x_1x_3 - u_2v_1) = 0. \tag{9}$$

Just as in the construction of conics by means of projective pencils the line corresponding to the line joining one vertex to the other is a tangent to the conic at the other vertex, so also here. The tangents to the sextic at the quadruple point A_1 are the lines corresponding to A_3A_1 and the tangents at the double point A_3 are the lines corresponding to A_1A_3 in the projectivity set up. The equation of the tangents to the sextic at A_1 is

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[2] Ibid. pp. 77–78.

$$c'u_1^2x_3^2 - 2g'u_1u_2x_3 + a'u_2^2 = 0.$$

Let the line A_3A_1 cut the second base conic in B_1 and B_2. Then there can be distinguished seven kinds of quadruple points as follows:

1. *Quadruple point of the* 1*st kind*: All four tangents are real and distinct. The lines A_2B_1 and A_2B_2 cut the first base conic in real and distinct points.

2. *Quadruple point of the* 2*nd kind*: Two of the tangents coincide and the others are real and distinct. One of the lines A_2B_1 or A_2B_2 is tangent to the first base conic and the other one cuts it in real and distinct points.

3. *Quadruple point of the* 3*rd kind*: Two pairs of tangents coincide. The two lines A_2B_1 and A_2B_2 are tangent to the first base conic.

4. *Quadruple point of the* 4*th kind*: All four tangents coincide. The lines A_2B_1 and A_2B_2 coincide and are tangent to the first base conic.

5. *Quadruple point of the* 5*th kind*: Two of the tangents coincide and the other two are imaginary. One of the lines A_2B_1 or A_2B_2 is tangent to the first base conic and the other one cuts it in imaginary points.

6. *Quadruple point of the* 6*th kind*: Two of the tangents are real and distinct and the other two are imaginary. One of the lines A_2B_1 or A_2B_2 cuts the first base conic in real and distinct points and the other one cuts it in imaginary points.

7. *Quadruple point of the* 7*th kind*: All four tangents are imaginary. A_2B_1 and A_2B_2 cut the first base conic in imaginary points.

Let A_1A_3 cut the first base conic in B_3 and let A_2B_3 cut the second base conic in B_4 and B_5. Then A_3B_4 and A_3B_5 are the tangents to the sextic at A_3. Their equation is

$$4g^2v_0x_1^2 - 2cgv_1x_1 + c^2v_2 = 0.$$

The tangents are real and distinct, coincident, or imaginary or A_3 is a node, cusp, or conjugate point according as

$$c^2(h'^2 - a'b') + 4g^2(f'^2 - b'c') - 4cg(f'h' - b'g') > 0, = 0, < 0,$$

or as A_2B_3 cuts the second base conic in real and distinct, coincident, or imaginary points.

The equations of the tangents to the sextic at the double points P_1, P_2, the intersections of $x_1=0$ with the first base conic, show that P_1 and P_2 are nodes, cusps, or conjugate points according as

$$f'^2 - b'c' > 0, = 0, < 0,$$

or as $x_1=0$ cuts the second base conic in real and distinct, coincident, or imaginary points. P_1 and P_2 are consecutive, if $x_1=0$ is tangent to the first base conic. It follows from the expansion and the equation of the tangents at this point that this consecutive double point is a tacnode, node-cusp, or a tacnode isolated according as $x_1=0$ cuts the second base conic in real and distinct, coincident, or in imaginary points.

Likewise A_3 is a tacnode, node-cusp, or a tacnode isolated, if A_3 is on the first base conic and $x_1=0$ cuts the second base conic in real and distinct, coincident, or in imaginary points.

All three double points coincide at A_3, if A_3 is on the first base conic and $x_1=0$ is tangent to this conic at A_3. A_3 is then an oscnode, tacnode-cusp, or an oscnode isolated according as $x_1=0$ cuts the second conic in real and distinct, coincident, or in imaginary points.

P_1 and P_2 are imaginary, if $x_1=0$ cuts the first base conic in imaginary points.

The combinations of all these conditions for the different kinds of multiple points result in the following different types of sextics:

63 sextics with one quadruple point and three real and distinct double points;

57 sextics with one quadruple point, two consecutive double points and one real and distinct double point;

18 sextics with one quadruple point and three consecutive double points;

21 sextics with one quadruple point, one real and two imaginary double points.

Sextics with one quadruple point and four double points. Let A_1 be again on the first base conic (4) and in addition let $x_1-x_3=0$ on A_2 be a common tangent of the base conics. Then

$$b = (f+h)^2/(c+2g), \qquad b' = (f'+h')^2/(a'+c'+2g'),$$

and two ordinary points coincide forming a double point P_3.[8] Thus we have the following theorem.

THEOREM III. *The generated curve is a plane sextic with one quad-*

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[8] Ibid. p. 78.

ruple point and four double points, if A_1 is on the first base conic and A_2 is on a common tangent of the base conics.

The only double point to be investigated here is P_3. The equation of the tangents at this point and the construction show that P_3 is a node or a conjugate point according as A_2 is between the points of contact of the common tangent with the conics or outside of this segment.

If we give the triangle of reference all possible positions it may take, there result the following different types of sextics:

63 types with one quadruple point and four real and distinct double points;

41 types with one quadruple point, two consecutive and two real and distinct double points;

12 types with one quadruple point, three consecutive double points and one real and distinct double point;

27 types with one quadruple point, two real and distinct double points and two imaginary double points.

The sextic generated when A_2 is on one base conic is essentially the same as the sextic discussed in this section.

Sextics with nine double points. Let A_1A_3 pass through two intersections of the base conics and let the polar of A_1 with respect to the first base conic and the polar of A_3 with respect to the second base conic pass through A_2. Then $a'=a, c'=c, g'=g, f'=h=0$. The double line $x_2^2=0$ breaks off from the octic, for it is a factor of the equation (6) if the above values are substituted in (6). The generated curve is therefore a sextic. It has double points at A_1 and A_3, at the intersections P_1, P_2 of $x_1=0$ with the first base conic (4) and at the intersections P_3, P_4 of $x_3=0$ with the second base conic (5). The equation (8) reduces to

$$x_2(b'x_2 + 2h'x_1)^2 = 0.$$

Hence the curve has three double points outside of the triangle of reference, one of which is consecutive with the double point A_1. Two of them are on the line $b'x_2+2h'x_1=0$. This line cuts the sextic in the points given by

$$x_2^4(bx_2 + 2fx_3)^2 = 0,$$

that is, in two consecutive double points at A_3 and in one double point P_5 on the line $bx_2+2fx_3=0$.

THEOREM IV. *The generated curve is a plane sextic with two pairs of two consecutive and five real and distinct double points, if A_1A_3 passes* *through two intersections of the base conics and if the polar lines of* A_1 *with respect to the first base conic and of* A_3 *with respect to the second base conic pass through* A_2.

The equations of the tangents at A_1 and A_3 and the expansion at these points show that A_1 and A_3 are tacnodes or tacnodes isolated, according as the line A_1A_3 passes through two real and distinct or imaginary intersections of the base conics. Since the polar lines of A_1 with respect to the first base conic and of A_3 with respect to the second base conic pass through A_2, A_1A_3 may not pass through a point of contact of the base conics. For this would cause A_2 to be on the conics and consequently a degeneration of the sextic would follow. Hence A_1 and A_3 cannot be node-cusps.

The tangents to the sextic at P_1 and P_2 are real and distinct, coincident, or imaginary, that is, P_1 and P_2 are nodes, cusps, or conjugate points according as $-b'c>0$, $=0$, <0. But b' and c may not be zero. Therefore P_1 and P_2 are nodes or conjugate points according as $x_1=0$ cuts the second base conic in real and distinct or in imaginary points.

Similarly P_3 and P_4 are nodes or conjugate points according as $-ab>0$, $=0$, <0, or as $x_3=0$ cuts the first base conic in real and distinct or in imaginary points.

P_1 and P_2 are consecutive and form a tacnode or a tacnode isolated, if $x_1=0$ is tangent to the first and cuts the second base conic in real and distinct or in imaginary points. Likewise P_3 and P_4 coincide forming a tacnode or a tacnode isolated, if $x_3=0$ is tangent to the second and cuts the first base conic in real and distinct or in imaginary points. P_1 and P_2, P_3 and P_4 are imaginary double points, if $x_1=0$ cuts the first and $x_3=0$ cuts the second base conic in imaginary points.

The double point P_5 has the coordinates $b'f$, $-2fh'$, bh'. It follows from the equation of the tangents to the sextic at P_5 that P_5 is a node or a conjugate point according as $x_2=0$ passes through two real intersections of the base conics or through two imaginary ones. P_5 coincides with A_1, if $h'=0$, that is, if the polar of A_1 with respect to the second base conic passes through A_2. A_1 is then an oscnode or an oscnode isolated according as $x_2=0$ passes through two real or through two imaginary intersections of the base conics.

Taking into consideration all the possible combinations of the different kinds of double points discussed, there can be generated the following types of sextics:

6 types with two pairs of two consecutive and five real and distinct double points;

6 types with three pairs of two consecutive and three real and distinct double points;

2 types with four pairs of two consecutive and one real and distinct double point;

4 types with three consecutive, two consecutive and four real and distinct double points;

2 types with three consecutive, two pairs of two consecutive and two real and distinct double points;

3 types with two imaginary, two pairs of two consecutive and three real and distinct double points;

2 types with two imaginary, three pairs of two consecutive double points and one real and distinct double point;

3 types with two imaginary, three consecutive, two consecutive and two real and distinct double points;

1 type with two imaginary, three consecutive and two pairs of two consecutive double points;

2 types with four imaginary, two pairs of two consecutive double points and one real and distinct double point;

1 type with four imaginary, three consecutive and two consecutive double points.

Sextics with ten double points. Let the position of the triangle of reference with regard to the base conics be chosen as in the preceding section and let in addition A_2 be on a common tangent of the conics. The generated sextic will have one more double point P_6 caused by the coincidence of two ordinary points.

THEOREM V. *The generated curve is a plane sextic having two pairs of two consecutive and six real and distinct double points, if A_1A_3 passes through two intersections of the base conics, if the polar lines of A_1 with respect to the first and of A_3 with respect to the second base conic pass through A_2, and if A_2 is on a common tangent of the base conics.*

The double point P_6 has the coordinates h', $-(a+c+2g)$, f, if $x_1-x_3=0$ is the common tangent. It follows from the equation of the tangents to the sextic at this point that P_6 is a node or a conjugate point according as A_2 is between the points of contact of the common tangent with the conics or outside of this segment.

The conditions for the different kinds of double points of the other singularities of the curve are the same as those discussed in the last section. Thus if we give the triangle of reference all possible positions with regard to the base conics, there result the following types of sextics:

7 types with two pairs of two consecutive and six real and distinct double points;

2 types with three pairs of two consecutive and four real and distinct double points;

1 type with four pairs of two consecutive and two real and distinct double points;

4 types with two imaginary, two pairs of two consecutive and four real and distinct double points;

3 types with two imaginary, three pairs of two consecutive and two real and distinct double points;

3 types with four imaginary, two pairs of two consecutive and two real and distinct double points.

A far greater number of sextics having nine or ten double points can be derived by this method, if the polar lines of A_1 with respect to the first and of A_3 with respect to the second base conic do not pass through A_2. But then it is very difficult to find the geometrical conditions for the different kinds of double points, which is important for the construction of these curves.

Holy Ghost Academy

THE ARITHMETICAL INVARIANTS OF QUADRATIC FORMS

GORDON PALL

1. Introduction. Anyone who has tried to read the developments by H. J. S. Smith [9][1] or H. Minkowski [6] of the systems of invariants (for a general positive integer n) associated with a genus of integral n-ary quadratic forms must be discouraged by their complexity. Further, from the standpoint of practical computation for a given form, these invariants are quite tedious to obtain, requiring as a first step the determination of a very special "characteristic form."

These complications are perhaps due essentially to the insistence on integral coefficients at all stages of the theory of integral forms. It is remarkable that, although Gauss began the systematic development of the theory of integral quadratic forms about the year 1800, it was not until over a hundred years later (in Hensel's work on p-adic numbers) that it was realized that there might be a considerable advantage in studying quadratic forms first in the field of rationals, and applying results thus obtained to the more difficult problems in the ring of integers. It turns out in fact that the complicated "simultaneous characters" of Smith and Minkowski are closely related to certain easily computed invariants under rational, linear transformations, and this fact enormously simplifies their use.

Our main purpose in this lecture will be to set forth a complete and easily computable system of invariants for a genus of integral n-ary quadratic forms. By our methods, anyone used to working with Legendre-Jacobi symbols should be able to obtain in less than fifteen minutes a complete system of invariants of a given integral form in (say) six variables, provided the coefficients are not unreasonably large. One has little more to do than "complete squares" by the method of elementary algebra, and then to read off the values of the invariants.

In an article scheduled to appear in the Duke Mathematical Journal, B. W. Jones shows how to reduce any integral quadratic form to a unique canonical form modulo 2^r (r large), and so embodies a complete set of invariants in a form which is in some respects equivalent but not quite so handy as ours.

Before describing these invariants, we shall recall some of the general properties of a genus of integral quadratic forms.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

2. **Properties of a genus.** A quadratic form $f=x'Ax=\sum a_{ij}x_ix_j$ is called *integral* if its *coefficients* a_{ii} and $2a_{ij}$ $(i, j=1, \cdots, n)$ are integers. We shall assume that the determinant, $|f|$ or $|A|$, of f is not zero.

The linear transformation $x=Ty$ replaces f by the form g of matrix $T'AT$. We shall single out the cases where the elements of the matrix T are (a) real, (b) rational, (c) rational but with denominators prime to a given integer k, (d) integral with $|T|$ prime to k, (e) integral with $|T|=\pm 1$, (f) integral with $|T|=1$. In each case we assume $|T|\neq 0$. In cases (e) and (f) we shall call T respectively *unit-modular* and *unimodular*.

If f is replaced by $\sum b_iy_i^2$, by a real transformation, the number of negative coefficients b_i is invariant (under real transformations), will be denoted here by ι, and called the *index* of f. We learn in algebra that n and ι comprise a complete invariant system under real transformations.

Gauss called f and g *equivalent* if T is unimodular. All forms equivalent to a given one constitute a *class*. Some recent writers have defined class by means of what we have named unit-modular transformations. There is no essential difference, and any statement about one kind of class can be immediately interpreted in terms of the other kind.

A genus will consist of a finite number of classes, and can be defined in several ways. Since we shall have a later use for the result, we shall recall Gauss's original definition of genus for binary quadratic forms, or rather a slight modification of his definition. Any integral b.q.f. can be written as $f=d_1(ax_1^2+bx_1x_2+cx_2^2)=d_1f_1$, where d_1 is a positive integer, and a, b, c are coprime integers. The quantities d_1, $D=b^2-4ac$, and index ι, are invariants of f under unit-modular transformations. These quantities satisfy the conditions $(-1)^\iota D<0$; $D\equiv 0$ or 1 mod 4; $\iota=0$, 1, or 2. Besides these, the following quantities, called *generic characters*, are invariant:

$$\begin{array}{lll} & (f_1\,|\,p) & \text{for each odd prime } p \text{ in } D,\\ (1) & (-1\,|\,f_1) & \text{if } \omega \geqq 4, \text{ or if } \omega = 2 \text{ and } \rho = 1,\\ & (-2\rho\,|\,f_1) & \text{if } \omega \geqq 5, \text{ or if } \omega = 3.\end{array}$$

Here $-D=2^\omega e$, where e is odd, and ρ stands for $(-1\,|\,e)$. (What we mean is that if f_1 is replaced in the character $(f_1|p)$ by any number n represented by f_1 and prime to p, the value of the Legendre symbol $(n|p)$ is always the same; similarly if $p=2$, n being odd and represented by f_1.) The preceding invariants are not (unless D is a square) independent, but are easily shown to satisfy the single relation

$$(2|f_1)^{\omega}(-1|f_1)^{(e+1)/2} = (-1)^{[\iota/2]}(f_1|e_1), \tag{2}$$

where $e_1 = |e|$. This complicated relation will be given a more elegant form in §7.

Gauss defined two binary forms f and g as in the same *genus* if they have the same values for their invariants d_1, D, ι, and (1). This definition has the merit that the invariants are easily computable, and so one can tell in a jiffy whether two given forms are, or are not, in the same genus. Then Gauss proved, for binaries, two simple properties which were later extended to three, and then to n, variables, by Smith, Minkowski, and Siegel:

THEOREM 2.1. *Call two n-ary forms f and g equivalent modulo k, if f can be transformed by an integral transformation of determinant prime to k into a form congruent (coefficient for coefficient) modulo k to g. Then f and g are in the same genus if and only if they have the same index ι, and are equivalent modulo k, for every positive integer k.*

THEOREM 2.2. *Two n-ary forms f and g are in the same genus if and only if, for every positive integer k, each can be transformed into the other by a rational transformation with denominators prime to k.*

Minkowski [6, p. 71] in effect *defined* a genus by means of the property in Theorem 2.1. He then used the obvious fact that the numbers of solutions of $f \equiv n \bmod k$ and $g \equiv n \bmod k$ must be the same for every n and k, to obtain a complete (but complicated) system of invariants, which he showed would imply the equivalence mod k of f and g for every k. Eisenstein [2] (for ternaries of odd determinant) and Smith (for n-aries) had previously carried on in the Gauss tradition, and defined a genus by means of directly computable invariants. Smith no doubt obtained these invariants by studying the number of solutions of the congruences $f \equiv n \bmod p^r$, for any prime p. He knew the truth of both Theorems 2.1 and 2.2, although he published complete proofs only for $n = 3$. Siegel [8] was the first to prove Theorem 2.2 for a general n.

Both the properties in the above theorems imply that f and g have the same determinant d. If we assume they have the same determinant d, both statements can be improved. Thus, two forms of integral matrix are in the same genus if they have the same determinant d, index ι, n, and are equivalent modulo $8d$; or again, if they have the same determinant d, and one can be transformed into the other by a rational transformation with denominators prime to $2d$.

Hel Braun [1] considered the complete system of generic invariants given by n, ι, a determinant d such that $(-1)^{\iota}d > 0$, $q_0 = 8d^3$, and a matrix-residue S mod q_0; and showed that such a system corresponds to an actually existing form, if and only if there exists an integer x_1 prime to q_0 such that $|S| \equiv x_1^2 d$ mod q, and

$$\sum_{x \bmod q_0} \exp 2\pi i x' S x / q_0 = \exp \frac{\pi i}{4}(n - 2\iota) \cdot (2q_0)^{n/2} \cdot d^{1/2}.$$

The latter condition was obtained by applying the reciprocity law for Gauss sums, and it is not surprising therefore that we can replace it by condition (7) of §5, which is closely related to the quadratic reciprocity law.

3. **The essentials of p-adic numbers and Hilbert norm-residue symbols obtained by an independent method.** There are at least three interesting points in which our treatment differs from that of Hensel [4, chap. 12] and Hasse [3]. First, we shall define a symbol, which, although essentially equal to Hilbert's norm-residue symbol, is defined more simply and symmetrically. Second we shall not use p-adic numbers as such, but shall replace them by rational congruences. Third, we can get all the results of Hasse (on rational quadratic forms) by strictly elementary methods (cf. §6).

Let p be a given prime. Two nonzero rational numbers a and b will be said to be in the same *p-adic class* if, for each positive integer r, there exists a rational number x such that

$$ax^2 \equiv b \bmod p^r. \tag{1}$$

By (1) we mean that $(ax^2 - b)/p^r$ is a rational number whose denominator is prime to p, that is, an integer modulo p. It is easily shown that if p is odd there are exactly four p-adic classes, containing the respective numbers (called p-adic kernels)

$$1, \ p, \ \nu, \ p\nu, \tag{2}$$

where ν denotes any given quadratic non-residue mod p; and if $p = 2$, there are exactly eight p-adic classes, with the p-adic kernels

$$1, \ 3, \ -1, \ -3, \ 2, \ 6, \ -2, \ -6. \tag{3}$$

To find the p-adic class of any nonzero rational number we need only write it in the form $s^2 p^\alpha k$, where s is a rational number, $\alpha = 0$ or 1, and k is an integer prime to p; and replace k by a number of the same quadratic character, 1 or ν if $p > 2$, ± 1 or ± 3 if $p = 2$.

It is useful to introduce a conventional "prime" p, called the prime ∞ or p_∞; and for this to understand that the solvability of (1) for every r means the solvability of

$$ax^2 = b \tag{4}$$

with x real. For the prime p_∞ it is evident that there are two p-adic classes, one consisting of all positive, the other of all negative rationals.

Example. 2 and $-7/2$ are in the same p-adic class for $p=2$, 11; in different p-adic classes if $p=\infty$, 3, 5, 7.

If a and b are any nonzero rational numbers, we define the symbol $(a, b)_p$ to have the value $+1$ or -1 according as the congruence

$$ax^2 + by^2 \equiv 1 \bmod p^r \tag{5}$$

has or has not, for each r, rational solutions x_r and y_r.

Two remarks should be made. First, our symbol $(a, b)_p$ has the same value as Hilbert's symbol $\left(\frac{a,b}{p}\right)$ in those cases in which Hilbert's symbol is defined. Second, in strict analogy to the Hensel p-adic background, which we are here replacing by rational congruences, we might have supposed that it would be necessary to restrict the solution numbers x_r and y_r in (5) to have the power of p in their denominators bounded independently of r. It is of interest to note in passing that this restriction is unnecessary, not only in the case of (5), but in the more general case of representation of forms by forms.

It follows at once from (5) that

$$(a, b)_p = (a', b')_p \tag{6}$$

if a' and b' are in the p-adic classes of a and b respectively. Hence $(a, b)_p$ can be evaluated by considering only a few cases, and we now give the final results:

$$(a, b)_\infty = -1 \text{ if and only if } a \text{ and } b \text{ are negative;} \tag{7}$$

$$(p^\alpha m, p^{\alpha'} m')_p = (-1 \mid p)^{\alpha\alpha'}(m \mid p)^{\alpha'}(m' \mid p)^{\alpha} \quad \text{if} \quad p > 2; \tag{8}$$

$$(2^\alpha m, 2^{\alpha'} m')_2 = (2 \mid m)^{\alpha'}(2 \mid m')^{\alpha}(-1)^{(m-1)(m'-1)/4} \quad \text{if} \quad p = 2. \tag{9}$$

Here m and m' denote integers prime to p; α and α' are 0 or 1.

Example. $(2, -5/2)_p$ has the value -1 if $p=5$; $(+1)(-1)(+1) = -1$ if $p=2$; $+1$ for all other primes p.

The following properties of the symbol $(a, b)_p$, valid for all primes p, are easily verified by use of (7)–(9), or can be proved independently:

$$(a, b)_p = (b, a)_p; \tag{10}$$

$$(a, b)_p = 1 \text{ if } a, b, \text{ or } a + b \text{ is in the } p\text{-adic class of } 1; \tag{11}$$

$$(a, -a)_p = 1, \qquad (a, a)_p = (a, -1)_p; \tag{12}$$

$$(13)\quad (a, b_1b_2)_p = (a, b_1)_p(a, b_2)_p, \qquad (a_1a_2, b)_p = (a_1, b)_p(a_2, b)_p.$$

For any odd prime p, $(a, b)_p$ is evidently $+1$ unless p actually appears in a or b. One easily sees that for given a and b, $(a, b)_p$ is -1 for only a finite even number (possibly zero) of primes p (counting in p_∞). This fact is usually written in the simple form

$$(14)\qquad \prod_p (a, b)_p = +1,$$

and will be found to be equivalent to various cases of the quadratic reciprocity law.

4. The invariants of a rational quadratic form under rational transformations. Minkowski [6, pp. 219–239] obtained a complete system of such invariants, in 1890, on the basis of his theory for integral forms. The system which we shall now describe is essentially that of Hasse (1923), the binary and ternary case having been due to Hensel. Hasse gave many beautiful applications, and to appreciate the following developments properly, one should read his work. An exposition containing a number of novel features will be given in a forthcoming book by the author; proofs will be found there of all results stated here without proof. The following remarks should be sufficient for the applications made in §6 to integral forms.

Besides applying rational linear transformations T to f, we are interested in the effect of multiplying f by a nonzero rational number. If $t = |T|$, evidently $|f|$ is multiplied by t^2. Also, $|\lambda f| = \lambda^n |f|$. This proves the following theorem.

THEOREM 4.1. *The squarefree integer part d of the determinant D_n of an n-ary form f is invariant under nonsingular linear transformations with rational coefficients. If n is even, d is also invariant under multiplication by rational constants.*

To obtain further invariants under rational transformations, Hasse wrote f in the form $\sum a_i x_i^2$, and showed that the product of the $n(n+1)/2$ Hilbert symbols $(a_i, a_j)_p$ $(i \leqq j;\ i, j = 1, \cdots, n)$ is an invariant of f under rational transformations. The restricted form of f made his proof [3, pp. 209–216] unnecessarily complicated. Instead, it is easy to prove, for any form $f = \sum a_{ij}x_ix_j$, that if D_i $(i = 1, \cdots, n)$ denotes the leading principal minor determinant of order i in the matrix (a_{ij}) of f, then if none of the D_i vanishes, the quantity

$$(1)\qquad c_p = c_p(f) = (-1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p$$

is invariant under rational transformations. Indeed this remains true, when properly interpreted, even if some of the D_i other than D_n vanish, provided no two consecutive D_i vanish; for then, if we interpret any product such as $(D_{j-1}, -0)_p(0, -D_{j+1})_p$ as though it were $(D_{j-1}, -h)_p(h, -D_{j+1})_p$ with any value h not zero, we obtain the correct value of $c_p(f)$.

THEOREM 4.2. *Two forms f and g are rationally equivalent if and only if they have the same values for their invariants*

$$(2) \qquad n,\ i,\ d,\ c_p \text{ for every } p.$$

These invariants are not independent of one another, and it is easily seen that they satisfy the relations

$$(3) \quad 0 \leqq \iota \leqq n, \quad (-1)^\iota d > 0, \quad c_\infty = (-1)^{[(\iota-1)/2]}, \quad \prod_p c_p = +1;$$

$$(4) \qquad \text{if } n = 1, \quad c_p = (-1, -d)_p \text{ for every } p;$$

$$(5) \qquad \text{if } n = 2, \quad c_p = 1 \text{ if } p \text{ satisfies } (-d \mid p) = 1.$$

Conversely, Hasse obtains many applications of the existence theorem that if n, ι, and d are integers, d squarefree, and (c_p) is a complex of signs satisfying (3)–(5), then there exists a form f with these assigned values for its invariants.

Besides (3_4) it should be noted that if f is integral and p is an odd prime not dividing $|f|$, then $c_p(f) = +1$.

The following formulae are easily proved, and show how easily the invariants c_p can be handled in applications. If λ is any nonzero rational number, then

$$(6) \qquad c_p(\lambda f) = \begin{cases} (\lambda, (-1)^{(n+1)/2})_p c_p(f) & \text{if } n \text{ is odd,} \\ (\lambda, (-1)^{n/2} D_n)_p c_p(f) & \text{if } n \text{ is even.} \end{cases}$$

If $f = f_1(x_1, \cdots, x_r) + f_2(x_{r+1}, \cdots, x_n)$, where the variables do not overlap, and $d_1 = |f_1|$, $d_2 = |f_2|$, then

$$(7) \qquad c_p(f) = c_p(f_1)c_p(f_2) \cdot (-1, -1)_p(d_1, d_2)_p.$$

In particular, if $f = a_1x_1^2 + \phi(x_2, \cdots, x_n)$, and $d = |f|$,

$$(8) \qquad c_p(f) = (a_1, d)_p c_p(\phi).$$

5. The invariants of a genus of integral quadratic forms.

LEMMA 5.1. *Let p be any finite prime. Every integral n-ary quadratic form f can be expressed (by means of a rational transformation which is integral modulo p and has determinant* 1) *in the form*

$$(1) \qquad p^{e_1}\phi_1 + p^{e_2}\phi_2 + \cdots + p^{e_s}\phi_s, \qquad e_1 < e_2 < \cdots < e_s,$$

where each ϕ_i *denotes a form with integral coefficients modulo* p, *the variables in different* ϕ_i *do not overlap,* $|\phi_i|$ *is prime to* p $(i=1, \cdots, s)$, *and the* e_i *are integers. Also,* $e_1 \geqq 0$, *except that* $e_1 = -1$ *if* $p=2$ *and* f *has any odd cross-product coefficient. If* n_i *denotes the number of variables in* ϕ_i, $\sum n_i = n$, *and* $\sum e_i n_i$ *is the exponent to which* p *appears in* $|f|$.

We give the proof (cf. [7, p. 38; 6, pp. 22 seq.]), since it is part of the technique of computing the generic invariants. It amounts to little more than judiciously completing squares. We can write $f = p^{e_1} f_1$, where the matrix (a_{ij}) of f_1 is integral, but that of f_1/p is not. If $p>2$ and any a_{ii} is prime to p, we can suppose it to be a_{11}; if every a_{ii} is divisible by p, then some $2a_{ij}$ is prime to p, and we can suppose this true of $2a_{12}$; we replace x_2 by x_1+x_2, and secure a_{11} to be prime to p. If $p=2$, and any a_{ii} is odd, we take it to be a_{11}; but if every a_{ii} is even we can suppose a_{12} odd, and that $a_{11} \neq 0$.

Thus the matrix of f_1 is of the form

$$\begin{bmatrix} A & B \\ B' & C \end{bmatrix}, \quad \text{where } A = [a_{11}] \text{ or } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad |A| \text{ prime to } p,$$

the latter case occurring only if $p=2$. Then the transformation

$$\begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \text{ replaces } f_1 \text{ by the form of matrix } \begin{bmatrix} A & 0 \\ 0 & C - B'A^{-1}B \end{bmatrix},$$

and this transformation is integral modulo p.

In the first case, where $A = (a_{11})$, this process is tantamount to completing squares by the identity of elementary algebra:

$$\sum a_{ij}x_ix_j = a_{11}\left(x_1 + \sum_{2}^{n} a_{11}^{-1} a_{1j}x_j\right)^2 + a_{11}^{-1} \sum_{j,k=2}^{n} (a_{11}a_{jk} - a_{1k}a_{j1})x_jx_k,$$

absorbing into a square the terms involving x_1. In the second case, it is not difficult to show that one arrives at the same form (of matrix $C - B'A^{-1}B$) by merely completing squares twice, once for the variable x_1, then for the variable x_2; the denominators 2 which appear in the partial linear transformations disappear in their product. For example, we have

$$\begin{aligned} f &= 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \\ &= 2(x_1 + x_2/2 + x_3/2)^2 + \phi/2, \end{aligned}$$

where

$$\phi = 3x_2^2 + 2x_2x_3 + 3x_3^2 = 3(x_2 + x_3/3)^2 + (8/3)x_3^2,$$

and can assert that f is transformed into $2y_1^2+2y_1y_2+2y_2^2+(4/3)y_3^2$ by a linear transformation which is *integral* modulo 2 and has determinant 1.

By repeated applications of the preceding process, we evidently arrive at a form (1), where each ϕ_i is either a sum of binary quadratic forms

$$2(ax^2+bxy+cy^2),\ b \text{ odd, the determinant } 4ac-b^2 \text{ odd,}$$

or contains at least one term ax^2, a odd. In these respective cases we shall term ϕ_i of *type B or A*. We can now read off from f the complete system of its generic invariants:

THEOREM 5.2. *Let $p>2$. In the form* (1), *the quantities*

$$s,\ e_1, \cdots, e_s,\ n_1, \cdots, n_s,\ \left(\frac{|\phi_i|}{p}\right) \qquad (i=1, \cdots, s) \tag{2}$$

constitute a complete system of invariants of f, under integral transformations of determinant prime to p.

That is, two forms f and g with the same values for their invariants (2) are equivalent modulo p^r, r arbitrarily large.

THEOREM 5.3. *If $p=2$, the following quantities are invariant, in the form $2^{e_1}\phi_1+\cdots+2^{e_s}\phi_s$, under integral transformations of odd determinant:*

(3) $s,\ e_1, \cdots, e_s,\ n_1, \cdots, n_s$, *the type A or B of each ϕ_i;*

(4) *the residue* mod 8 *of Δ_{1k} if ϕ_k is followed by a rise of* 8;

(5) *the residue* mod 4 *of Δ_{1k} if ϕ_k is followed by a rise of* 4;

(6) *the quantity* $\left(\dfrac{2}{\Delta_{1k}}\right)^{l-1}\left(\dfrac{2}{\Delta}\right)^{\nu} c_2(\psi)$

for every block

$$\psi=\phi_k+2\phi_{k+1}+2^2\phi_{k+2}+\cdots+2^{l-1}\phi_{k+l-1}$$

or

$$\phi_{k-1}/2+\phi_k+2\phi_{k+1}+\cdots+2^{l-1}\phi_{k+l-1},$$

where $\phi_k, \phi_{k+1}, \cdots, \phi_{k+l-1}$ are all of type A and $l \geqq 1$, and in the second case ϕ_{k-1} is of type B; and the block cannot be enlarged into another block of the same type. Here Δ_{1k} denotes the product of the odd determinants $|\phi_1| \cdots |\phi_k|$; a "rise of 8" means that either $k=s$, or ϕ_k is of type A and $8 \mid 2^{e_{k+1}-e_k}\phi_{k+1}$, or ϕ_k is of type B (so that $\phi_k/2$ is integral) and $4 \mid 2^{e_{k+1}-e_k}\phi_{k+1}$; a "rise of 4" means that ϕ_k is of type B or that ϕ_k is *of type A and* $4 \mid 2^{e_{k+1}-e_k}\phi_{k+1}$. *In* (6), Δ *and* v *are defined by* $|\psi| = 2^v\Delta$, *where* Δ *is odd; and clearly* $v \equiv \sum n_{k+1}$ (*i odd*, $1 \leqq i \leqq l-1$) mod 2. *Two forms with the same invariants listed here are equivalent* mod 2^r, *r arbitrarily large.*

THEOREM 5.4. *Two integral n-ary forms with the same index and determinant are in the same genus if and only if they have the same values for the invariants listed in the preceding two theorems for the prime* 2 *and for each odd prime in their determinant.*

THEOREM 5.5. *Let* n, ι, d *be given integers*, $0 \leqq \iota \leqq n$, $(-1)^\iota d > 0$. *For each prime* p *in* d, *assign a form-residue* $p^{e_1}\phi_1 + \cdots + p^{e_s}\phi_s$, *where* $\sum e_i n_i$ *is the exponent of* p *in* d, *and assign values to the invariants listed in Theorems* 5.2 *and* 5.3, *subject to the restriction that* $|\phi_1|\,|\phi_2| \cdots |\phi_s|$ *has the same quadratic character as* $d/p^{e_1 n_1 + \cdots + e_s n_s}$. *Then there exists an integral form with the invariants so assigned, if and only if*

$$\prod_p c_p(f) = +1. \tag{7}$$

This should be compared with Hel Braun's result mentioned in §2.

As an example of the computation of the invariants of a genus, consider the form $f = 5x_0^2 + 4x_1^2 + 4x_2^2 + 4x_3^2 + 2x_0x_1 - 4x_0x_2 + 4x_0x_3 + 4x_1x_3$ $= 5(x_0 + x_1/5 - 2x_2/5 + 2x_3/5)^2 + (19x_1^2 + 16x_2^2 + 16x_3^2 + 4x_1x_2 + 16x_1x_3$ $+ 8x_2x_3)/5 = 5y_0^2 + \phi/5$, where $\phi = 19(x_1 + 2x_2/19 + 8x_3/19)^2 + 60\psi/19$, where $\psi = 4x_3^2 + 2x_3x_2 + 5x_2^2 = 5(x_2 + x_3/5)^2 + (19/5)x_3^2$. The determinant is therefore $5(1/5)\cdot(19)(60/19)^2(5)(19/5) = 144$. Hence the only primes that need be considered are 2 and 3, and since the fractional coefficients above happen to be integral modulo 2 and 3, we can read off without further work the forms $5y_0^2 + (19/5)y_1^2 + (60/19)y_2^2 + (12/5)y_3^2$, or what is essentially equivalent,

$$5z_0^2 + 7z_1^2 + 4(5z_2^2 + 7z_3^2) \bmod 2^r,$$

and

$$2z_0^2 + 2z_1^2 + 3(2z_2^2 + 2z_3^2) \bmod 3^r.$$

The invariants (2) relating to the prime 3 are therefore $s = 2$, $e_1 = 0$, $e_2 = 1$, $n_1 = n_2 = 2$, $(4|3) = 1$, $(4|3) = 1$. The invariants (3)–(6) relating to the prime 2 are $s = 2$, $e_1 = 0$, $e_2 = 2$, $n_1 = n_2 = 2$, both ϕ_1 and ϕ_2 of type A, the residue mod 8 of $|\phi_1|\,|\phi_2|$ is 1, the residue mod 4 of $|\phi_1|$ is 3, $c_2(\phi_1) = (-5, -7)_2 = +1$, $c_2(\phi_2) = +1$. The form f is thus seen to be in the same genus as $x_0^2 + 3x_1^2 + 4x_2^2 + 12x_3^2$. It may be remarked that f occurred, along with many more complicated forms, in an investigation of all systems of generalized quaternions permitting unrestricted factorization, and that there are exactly 39 such systems in which the norm-form (a quaternary quadratic form) is positive definite and in a genus of one class.

6. **Some remarks on the methods of proof.** Certain quantitative results concerning genera of quadratic forms, such as the weight formulae of Smith, Minkowski, and Siegel, have never been proved by strictly elementary methods. It may however be conjectured that any qualitative properties can be so proved. For example, the author has proved all the results of this article by elementary means, without using (for example) Dirichlet's theorem on the existence of primes in an arithmetical progression. This theorem seems to have evaded all efforts to find an elementary proof. Many writers have used Dirichlet's theorem in work on quadratic forms, when they might instead have used the following theorem of Gauss, on the existence of a genus of binary quadratic forms:

THEOREM 6.1. *If d, D, ι, and the characters of* §2 (1) *are assigned, subject to the necessary conditions* $(-1)^{\iota}D<0$, $D\equiv 0$ *or* 1 mod 4, $0\leqq\iota\leqq 2$, *and* §2 (2), *then there exists an integral binary quadratic form f with the assigned values for its invariants.*

Now if a is any number represented by f_1, it will be found by use of (7)–(9) of §3 that

$$(1)\qquad \begin{aligned} &(a, D)_\infty = (-1)^{[\iota/2]}, \qquad (a, D)_2 = (2\,|\,f_1)^{\omega}(-1\,|\,f_1)^{(e+1)/2},\\ &(a, D)_p = (f_1\,|\,p) \text{ if } p > 2 \text{ and } p \text{ divides } D \text{ to an odd power.}\end{aligned}$$

The existence condition §2 (2) therefore takes the simpler form (cf. §5 (7))

$$(2)\qquad \prod_p (a, D)_p = 1.$$

An analysis of Gauss's existence theorem will be found to yield the following theorem.

THEOREM 6.2. *Let d be a nonzero rational number, and let j_p be assigned equal to* $+1$ *or* -1 *for every prime p, including the prime* ∞. *However, let j_p be* -1 *for only a finite even number of primes p, and let j_p be* $+1$ *whenever d is in the p-adic class of* 1. *Then there exists a nonzero rational number h such that*

$$(3)\qquad (h, d)_p = j_p \text{ for every } p.$$

In many problems, it will be found that either the construction of a binary quadratic form of a suitably determined genus, or the choice of a number represented by such a form, or a direct use of Theorem 6.2, will serve the same end as the choice of a prime by means of Dirichlet's theorem.

We shall illustrate this by giving a proof, noteworthy for its simplicity, of Legendre's theorem that every positive integer not of the form $4^h(8n+7)$ is a sum of three squares.

Let a denote a positive squarefree integer not of the form $8n+7$. It will suffice to prove that a is represented by a positive, properly primitive form of determinant 1, since every such form is equivalent to $x_1^2+x_2^2+x_3^2$ (cf. [5, p. 121]).

I. *Case* $a\equiv 3 \bmod 8$. We obtain by Gauss's theorem a positive, primitive form $\psi=C_1x^2+Rxy+B_1y^2$ of discriminant $R^2-4B_1C_1=-a$, with characters such that

$$(\psi \mid p) = (-2 \mid p) \quad \text{if} \quad p \mid a. \tag{4}$$

This is consistent with the condition of possibility §2 (2), since

$$\prod_{p \mid a} (\psi \mid p) = \prod_{p \mid a} (-2 \mid p) = 1,$$

the number of prime factors in a of the form $8k+5$ or 7 being even. We can take C_1 prime to $2a$, and then by a translation $x\rightarrow x+hy$ secure $R\equiv a \bmod 2a$, and hence (comparing discriminants) $B_1\equiv 0 \bmod a$. Set $C=2C_1$, $B=2B_1$. In view of (4) we can solve

$$-C \equiv t^2 \pmod{a} \tag{5}$$

for an integer t. We set $s=0$, and write $b=(C+t^2)/a$, $c=(B+s^2)/a$, $r=(st-R)/a$. Then $ax^2+by^2+cz^2+2ryz+2szx+2txy$ has determinant 1 and solves our problem.

II. *Case* $a\equiv 1$ or $2 \bmod 4$. We can choose a positive primitive form $\psi=Cx^2+2Rxy+By^2$ of discriminant $-4a=4R^2-4BC$, with characters such that

$$\begin{aligned} &(\psi \mid p) = (-1 \mid p) \text{ if } p \mid a,\ p > 2; \quad &&(-1 \mid \psi) = 1 \text{ if } a \equiv 1 \bmod 4,\\ &(2 \mid \psi) = 1 \text{ if } a \equiv 2 \bmod 8, &&(-2 \mid \psi) = -1 \text{ if } a \equiv 6 \bmod 8. \end{aligned} \tag{6}$$

This is consistent with §2 (2), since the number of prime factors $4k+3$ in a is correctly adjusted. As before, we can secure $R\equiv 0$, $B\equiv 0 \bmod a$, C being prime to $2a$. We have (5), and so on, exactly as in case I.

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ON THE COMPOSITION OF ALGEBRAIC FORMS OF HIGHER DEGREE

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1. **Introduction.** No discussion of composition can properly be introduced except by mentioning the complex numbers. From the relation

$$x_1 + ix_2 = (y_1 + iy_2)(z_1 + iz_2),$$

$$x_1 = y_1z_1 - y_2z_2, \qquad x_2 = y_1z_2 + y_2z_1,$$

we have, upon taking norms,

$$x_1^2 + x_2^2 = (y_1^2 + y_2^2)(z_1^2 + z_2^2).$$

Thus the quadratic form $y_1^2+y_2^2$ is composed with itself.

Similarly by using quaternions we achieve the composition of a sum of four squares with itself,[1] and by using Cayley's algebra of order 8, we obtain the composition of a sum of eight squares with itself. No further progress in this direction is possible.[2]

In general let us assume that

$$x_k = \sum_{i,j} c_{ijk}y_iz_j \qquad (i, j, k = 1, 2, \cdots, n) \tag{1}$$

where the c_{ijk} are numbers of a commutative ring $\mathfrak{R}$ with unit element, and the y's and z's are indeterminates. If there exist three homogeneous forms f, g, h of degree k with coefficients in $\mathfrak{R}$ such that

$$f(x) = g(y)\cdot h(z) \tag{2}$$

is an identity by virtue of the bilinear transformation (1), we say that g and h are *composable*, that f is *composite*, and that the transformation (1) is *admissible.*

The theory of composition appears in the mathematical literature under two rather distinct special cases, first where $\mathfrak{R}$ is a field, and second where $\mathfrak{R}$ is the ring of rational integers. The field case is the simpler and progress has been carried much further, but this is not the case in which we are here primarily interested.

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[1] It seems certain that a knowledge of the fact that a sum of four squares is composable with itself, which had been proved by Euler in 1748, was one of the main clues which led Hamilton to the discovery of quaternions.

[2] See L. E. Dickson, Ann. of Math. (2) vol. 20 (1919) pp. 155–171.

The case where $\mathfrak{R}$ is the ring of rational integers is intimately connected with some of the basic concepts of number theory, particularly with the class number in an algebraic field, and in spite of the large number of special results which have been obtained, composition theory here still appears to be in a primitive condition.

Fortunately it is not necessary to attempt a summary of the literature, for it is exhaustively given in Dickson's *History of the theory of numbers*.[3]

The aim of the present paper is very modest. While the problem of composition may be stated for an abstract ring, it seems as if only in the case of a principal ideal ring are there enough properties at hand to make the problem amenable to our methods. The comprehensive Dedekind-Weber composition, involving ideal classes in algebraic fields of degree n, is approached with the calculus of matrices with the result that certain aspects of the theory are thrown into higher relief. Finally the theory of ideals in more general algebras is shown to yield compositions, sometimes of a somewhat unorthodox type. But the existence of these compositions is sufficient to show that the complete determination of all compositions over a ring is likely to prove to be a complicated problem.

2. **The Dedekind-Weber theorem.** Let $\mathfrak{F}$ be an ordinary algebraic field of order n over the rational field $\mathfrak{R}$, and let $[\mathfrak{F}]$ be its maximal integral ring, with $[\mathfrak{R}]$-basis

$$\epsilon_1, \epsilon_2, \cdots, \epsilon_n.$$

We shall call

$$t(y) = y_1\epsilon_1 + y_2\epsilon_2 + \cdots + y_n\epsilon_n,$$

where the y's are indeterminates, the *general number* of $[\mathfrak{F}]$. By means of the multiplication table

$$\epsilon_i\epsilon_j = \sum_k c_{ijk}\epsilon_k$$

of $[\mathfrak{F}]$, it is possible to write

$$t(x) = t(y)\cdot t(z) \tag{3}$$

where

$$x_k = \sum_{i,j} c_{ijk}y_iz_j.$$

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[3] L. E. Dickson, *History of the theory of numbers*, vol. 3, Washington, 1923, chaps. 3 and 14.

The norm $n(x)$ of $t(x)$ is an algebraic form of degree n in n indeterminates with coefficients in $[\mathfrak{R}]$, and from (3) it follows that

$$n(x) = n(y) \cdot n(z).$$

Thus the norm form $n(y)$ is composable with itself, and is the most obvious example of composition which we have.

This process is readily extensible to linear algebras which have the norm property, whether they are associative or not, and leads to the composition of forms in n indeterminates with themselves. If the algebra is associative but not commutative, these forms are of degree less than n.

But not all composition is of a form with itself, and the most general result in this direction is due to Dedekind and Weber.[4]

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of $[\mathfrak{F}]$ with respective $[\mathfrak{R}]$-bases

$$\alpha_1, \alpha_2, \cdots, \alpha_n, \qquad \beta_1, \beta_2, \cdots, \beta_n$$

and let the product ideal $\mathfrak{c} = \mathfrak{a}\mathfrak{b}$ have the basis $\gamma_1, \gamma_2, \cdots, \gamma_n$. Then rational integers c_{ijk} exist such that

$$\alpha_i\beta_j = \sum_k c_{ijk}\gamma_k.$$

Define

$$t(\mathfrak{a}) = y_1\alpha_1 + y_2\alpha_2 + \cdots + y_n\alpha_n, \qquad t(\mathfrak{b}) = z_1\beta_1 + z_2\beta_2 + \cdots + z_n\beta_n$$

where the y's and z's are indeterminates. Then

$$t(\mathfrak{a}) \cdot t(\mathfrak{b}) = t(\mathfrak{c})$$

where

$$t(\mathfrak{c}) = x_1\gamma_1 + x_2\gamma_2 + \cdots + x_n\gamma_n, \qquad x_k = \sum_{i,j} c_{ijk}y_iz_j.$$

Let $nt(\mathfrak{a})$ denote the norm of the number $t(\mathfrak{a})$ in $[\mathfrak{F}]$. It can be shown that

$$nt(\mathfrak{a}) = n(\mathfrak{a}) \cdot g(y)$$

where $n(\mathfrak{a})$ is the norm of the ideal $\mathfrak{a}$ and $g(y)$ is an n-ary n-ic form with coefficients in $[\mathfrak{R}]$. Similarly

$$nt(\mathfrak{b}) = n(\mathfrak{b}) \cdot h(z), \qquad nt(\mathfrak{c}) = n(\mathfrak{c}) \cdot f(x).$$

Since $nt(\mathfrak{c}) = nt(\mathfrak{a}) \cdot nt(\mathfrak{b})$, it follows that

$$f(x) = g(y) \cdot h(z), \qquad x_k = \sum c_{ijk}y_iz_j.$$

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[4] H. Weber, *Lehrbuch der Algebra*, 2d ed., Braunschweig, 1908, p. 335.

We are thus led to the composition of the forms $g(y)$ and $h(z)$, which are in general different.

It is shown, further, that if $\mathfrak{a}$ and $\mathfrak{a}'$ are ideals of the same class, the corresponding forms $g(y)$ and $g'(y)$ are equivalent, and that a change of basis of $\mathfrak{a}$ leads to an equivalent $g(y)$. But if $\mathfrak{a}$ and $\mathfrak{a}'$ belong to different classes, $g(y)$ and $g'(y)$ are not equivalent. Thus as many non-equivalent forms are obtained in this way as there are classes of ideals in $[\mathfrak{F}]$.

3. The general problem. In the type of composition which is accomplished through ideal multiplication in an algebraic field, the admissible bilinear transformations (1) satisfy two conditions[5] which are sometimes but not always demanded in composition theory—first that the $x_1, x_2, \cdots, x_n$ shall be linearly independent when the y's and z's are independent indeterminates, and second that there shall exist elements a_{ijk} in the principal ideal ring $\mathfrak{R}$ such that

$$y_i z_j = \sum_k a_{ijk} x_k. \tag{4}$$

We might say that the composition is *non-trivial* if these conditions are satisfied. We shall show that one simple hypothesis will insure both of these conditions.

Let us stream-line the notation by introducing the vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix}, \qquad z = \begin{bmatrix} z_1 \\ z_2 \\ \cdots \\ z_n \end{bmatrix},$$

and the matrices

$$\begin{aligned} R(y) &= y_1R_1 + y_2R_2 + \cdots + y_nR_n, \\ S(z) &= z_1S_1 + z_2S_2 + \cdots + z_nS_n \end{aligned}$$

where

$$R_i = (c_{isr}), \qquad S_i = (c_{ris}).$$

In this notation the bilinear transformation (1) can be written

$$x = R(y)\cdot z = S^{\mathrm{T}}(z)\cdot y. \tag{5}$$

The hypothesis which we shall introduce is that *the matrices* $S_1, S_2, \cdots, S_n$ *are relatively prime on the right.*

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[5] Weber, loc. cit.

From this hypothesis it follows[6] that n matrices A_i with elements in $\mathfrak{R}$ exist such that

$$A_1S_1 + A_2S_2 + \cdots + A_nS_n = I, \qquad A_i = (a_{sir}).$$

That is,

$$\sum_{i,j} a_{ijr}c_{ijs} = \delta_{rs}$$

where δ_{rs} is Kronecker's delta. Now a relation $\sum b_k x_k = 0$ would imply

$$\sum_{i,j,k} b_k c_{ijk} y_i z_j = 0$$

and, since the y's and z's are independent indeterminates,

$$\sum_k b_k c_{ijk} = 0 \qquad (i, j = 1, 2, \cdots, n).$$

Then

$$\sum_{i,j,k} a_{ijr} b_k c_{ijk} = \sum_k b_k \delta_{rk} = b_r = 0$$

for every r, so that $x_1, x_2, \cdots, x_n$ are linearly independent.

We may now verify that equations (4) with the a_{ijk} as defined above actually constitute a solution of (1). For upon substituting we have

$$\sum_{i,j,l} c_{ijk}a_{ijl}x_l = \sum_l \delta_{kl}x_l = x_k.$$

Thus our hypothesis is sufficient that (1) be non-trivial.

That it is necessary is quite evident, for if (4) exists,

$$x_k = \sum_{i,j} c_{ijk} \sum_l a_{ijl}x_l,$$

and if the x's are linearly independent,

$$\sum_{i,j} c_{ijk}a_{ijl} = \delta_{kl},$$

which in matric form is

$$A_1S_1 + A_2S_2 + \cdots + A_nS_n = I.$$

An equivalent condition is that the matrices $R_1, R_2, \cdots, R_n$ be relatively prime on the left.

An implication of this condition is that we may assume f, g and h to be primitive. For if

$$f(x) = a \cdot f'(x), \qquad g(y) = b \cdot g'(y), \qquad h(z) = c \cdot h'(z)$$

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[6] Bull. Amer. Math. Soc. vol. 39 (1933) p. 573.

where a, b and c are the contents and f', g' and h' primitive, then by (1)

$$f'(x) = p(y, z)$$

where p has coefficients in $\mathfrak{R}$. Since

$$a \cdot p(y, z) = bc \cdot g'(y) \cdot h'(z),$$

it follows that $a \mid bc$. By (4)

$$g'(y) \cdot h'(z) = q(x)$$

has coefficients in $\mathfrak{R}$ so that

$$bc \cdot q(x) = a \cdot f'(x),$$

whence $bc \mid a$. Then $bc = ua$ where u is a unit of $\mathfrak{R}$, and

$$u \cdot f'(x) = g'(y) \cdot h'(z).$$

A linear homogeneous transformation on the variables of a form f may be written

$$x = Ax', \qquad |A| \neq 0, \tag{6}$$

where A is a matrix with elements in $\mathfrak{R}$. If $|A|$ is a unit of $\mathfrak{R}$, then

$$x' = A^{-1}x$$

is again a transformation of the same type. If by virtue of (6)

$$f(x) = f'(x'), \qquad A \text{ unimodular,}$$

then the form $f'(x)$ is called *equivalent* to $f(x)$. Clearly the totality of numbers of $\mathfrak{R}$ which f and f' represent is the same.

If $M_1, M_2, \cdots, M_n$ are $n \times n$ matrices with elements in $\mathfrak{R}$, then

$$| x_1 M_1 + x_2 M_2 + \cdots + x_n M_n |$$

is a homogeneous form of degree n in the n indeterminates $x_1, x_2, \cdots, x_n$. A form which can be written as the determinant of such a matrix will be called *amenable.*

Let us consider the effect of a transformation of the variables on the bilinear transformation (1). If $y = Ay'$, then from (5)

$$x = S^{\mathrm{T}}(z) \cdot Ay',$$

so that

$$S'(z) = A^{\mathrm{T}} \cdot S(z). \tag{7}$$

Similarly if we take $z = Bz'$, we have

$$R'(y) = R(y) \cdot B. \tag{8}$$

The following theorem was obtained by Gauss[7] for quadratic forms and generalized by Dedekind.

If $f(x)=g(y)\cdot h(z)$ *by virtue of* (1), *every irreducible factor of* $h(z)$ (*or of* $g(y)$) *of degree greater than* 0 *is a factor of* $|S^{\mathrm{T}}(z)|$ (*or of* $|R(y)|$). *If* $h(z)$ (*or* $g(y)$) *is irreducible of degree* n, *then it is amenable.*

From (5) we have

$$S^{\mathrm{T}}(z)\cdot y = x.$$

Upon multiplying through by adj $S^{\mathrm{T}}(z)$, we have

$$|S^{\mathrm{T}}(z)|\cdot y = \operatorname{adj} S^{\mathrm{T}}(z)\cdot x.$$

Since $g(y)$ is homogeneous of degree k,

$$|S^{\mathrm{T}}(z)|^{k}\cdot g(y) = P(z, x)$$

where $P(z, x)$ is a homogeneous polynomial in x and z with coefficients in $\mathfrak{R}$. Upon multiplying by $h(z)$ and making use of (2), we have

$$|S^{\mathrm{T}}(z)|^{k}\cdot f(x) = P(z, x)\cdot h(z).$$

If $\mathfrak{R}$ is a ring in which factorization into primes is unique, the same is true in the polynomial ring $\mathfrak{R}[x, z]$. Then

$$h(z) \,\Big|\, |S^{\mathrm{T}}(z)|^{k},$$

and the distinct irreducible factors of $h(z)$ occur among the distinct irreducible factors of $|S^{\mathrm{T}}(z)|$. If in particular $h(z)$ is irreducible of degree n,

$$h(z) = c\cdot |S^{\mathrm{T}}(z)|, \qquad c \in \mathfrak{R}.$$

Thus $h(z)$ is amenable. Similarly we find that, if $g(y)$ is irreducible of degree n,

$$g(y) = c\cdot |R(y)|, \qquad c \in \mathfrak{R}.$$

Another basic theorem in the theory of composition is the Theorem of Dickson.[8]

Let $f(x)=g(y)\cdot h(z)$ *by virtue of* (1). *If* $|S^{\mathrm{T}}(z)|$ *is not identically zero,* $g(y)$ *is equivalent to* $c\cdot f(x)$ *in the quotient field of* $\mathfrak{R}$. *If* $|R(y)|$ *is not identically zero,* $h(z)$ *is equivalent to* $c\cdot f(x)$ *in the quotient field of* $\mathfrak{R}$. *If furthermore* $h(z)$ *represents a unit* u *of* $\mathfrak{R}$ *and is irreducible of degree* n *in* n *variables,* $g(y)$ *is equivalent to* $u\cdot f(x)$ *in* $\mathfrak{R}$. *Similarly if* $g(y)$

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[7] See, for instance, the exposition of A. Speiser, *Festschrift Heinrich Weber*, Teubner, Leipzig, 1912, pp. 375–395.

[8] L. E. Dickson, C. R. Acad. Sci. Paris vol. 172 (1921) pp. 636–640.

represents a unit u of $\mathfrak{R}$ and is irreducible of degree n in n variables, $h(z)$ is equivalent to $u\cdot f(x)$ in $\mathfrak{R}$.

Let $k_1, k_2, \cdots, k_n$ be n numbers of $\mathfrak{R}$ such that $|S^{\mathrm{T}}(k)|\neq 0$. In the relations

$$x = S^{\mathrm{T}}(z)\cdot y, \qquad f(x) = g(y)\cdot h(z)$$

set $z\rightarrow k$. Then

$$y = S^{-\mathrm{T}}(k)\cdot x, \qquad f(x) = g(y)\cdot h(k).$$

The first equation represents a transformation of the type $y=Ay'$ where $y'=x$, so that the second equation becomes

$$f(y') = h(k)\cdot g(y) = c\cdot g(y).$$

If $h(z)$ represents a unit of $\mathfrak{R}$, let $k_1, k_2, \cdots, k_n$ be so chosen that $h(k)=u$. Then $f(x)=u\cdot g(y)$. If $h(z)$ is irreducible of degree n, we have seen that $h(z)=c\cdot|S^{\mathrm{T}}(z)|$, so that $|S^{\mathrm{T}}(k)|$ is a unit of $\mathfrak{R}$ and the transformation $y=S^{-\mathrm{T}}(k)\cdot x$ is unimodular.

4. **The theory of ideals.** The writer has approached the theory of ideals in linear associative algebras over a principal ideal ring through the matric calculus.[9] This leads to a theory of composition which extends the known theory.

First, when applied in algebraic fields it enriches the Dedekind-Weber theory by bringing to light the matrices whose determinants were the essential items in the older theory. The algebraic form is defined in terms of the basis of the ideal class rather than in terms of the single ideal, which comes more directly to the essential point of composition of classes.

Second, in the general case where the class group does not exist, some quite unorthodox examples of composition are obtained.

We assume that $\mathfrak{F}$ is the quotient field of the principal ideal ring $\mathfrak{R}$, and that $\mathfrak{A}$ is a finite algebra over $\mathfrak{F}$ with the $\mathfrak{F}$-basis $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$, and the constants of multiplication c_{ijk} which belong to $\mathfrak{R}$. These n^3 numbers can be arranged into sets of n matrices by defining

$$Q_i = (c_{rsi}), \qquad R_i = (c_{isr}), \qquad S_i = (c_{ris})$$

where r denotes the row and s the column in which an element lies.

The numbers

$$\alpha = a_1\epsilon_1 + a_2\epsilon_2 + \cdots + a_n\epsilon_n$$

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[9] Monatshefte für Mathematik und Physik vol. 48 (1939) pp. 293–313. Amer. J. Math. vol. 64 (1942) pp. 646–652.

of $\mathfrak{A}$ whose coördinates a_i lie in $\mathfrak{R}$ constitute an integral domain of $\mathfrak{A}$ which we shall denote by $[\mathfrak{A}]$. Corresponding to each number α there are three matrices

$$\begin{aligned} Q(\alpha) &= a_1Q_1 + a_2Q_2 + \cdots + a_nQ_n, \\ R(\alpha) &= a_1R_1 + a_2R_2 + \cdots + a_nR_n, \\ S(\alpha) &= a_1S_1 + a_2S_2 + \cdots + a_nS_n, \end{aligned}$$

all of whose elements belong to $\mathfrak{R}$.

We shall assume that $\mathfrak{A}$ is a Frobenius algebra, that is, that there is in $\mathfrak{A}$ at least one number α such that $Q(\alpha)$ is nonsingular. In this case each of the correspondences

$$\alpha \rightleftarrows R(\alpha), \qquad \alpha \rightleftarrows S(\alpha)$$

is an isomorphism, the so-called first and second regular representations of $[\mathfrak{A}]$.

An additive group $\mathfrak{M}$ of numbers of $[\mathfrak{A}]$ which is closed under multiplication by the numbers of $\mathfrak{R}$ is called an *$\mathfrak{R}$-module.* An $\mathfrak{R}$-module which is closed under multiplication on the left by the numbers of $[\mathfrak{A}]$ is a *left ideal.* A *right ideal* is similarly defined.

Every left or right ideal $\mathfrak{a}$ of $[\mathfrak{A}]$ has an $\mathfrak{R}$-basis $\alpha_1, \alpha_2, \cdots, \alpha_n$ where

$$\alpha_i = a_{i1}\epsilon_1 + a_{i2}\epsilon_2 + \cdots + a_{in}\epsilon_n, \qquad a_{ij} \in \mathfrak{R}.$$

We define the matrix $A = (a_{rs})$, and say that A corresponds to the ideal $\mathfrak{a}$ by a *Poincaré correspondence.* If A is nonsingular, it is unique up to a left factor U which is a unimodular matrix with elements in $\mathfrak{R}$. The matrix $S(\alpha)$ corresponds to the principal left ideal $(\alpha]$, and $R^{\mathsf{T}}(\alpha)$ corresponds to the principal right ideal $[\alpha)$.

Let $\mathfrak{a}$ be a left ideal with $[\mathfrak{A}]$-basis $\alpha_1, \alpha_2, \cdots, \alpha_n$, and let A be the greatest common right divisor of the matrices

$$S(\alpha_1), S(\alpha_2), \cdots, S(\alpha_k),$$

which is unique up to a unimodular left factor U. Then A corresponds to $\mathfrak{a}$ by the Poincaré correspondence, and the rows of A determine an $\mathfrak{R}$-basis of $\mathfrak{a}$.

A necessary and sufficient condition in order that a matrix A with elements in $\mathfrak{R}$ shall correspond to a left ideal is that there shall exist n matrices D_i with elements in $\mathfrak{R}$ such that

$$AR_i^{\mathsf{T}} = D_i^{\mathsf{T}}A \qquad (i = 1, 2, \cdots, n).$$

Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *equivalent*, or to belong to the same *class*, provided two numbers τ_1 and τ_2 of $[\mathfrak{A}]$ exist such that

$$\mathfrak{a}\tau_1 = \mathfrak{b}\tau_2.$$

At least in the case where $\mathfrak{F}$ is the rational field, $\mathfrak{R}$ the ring of rational integers, and $\mathfrak{A}$ is semi-simple, the number of classes of left (or right) ideals is finite.[10]

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two left ideals with $\mathfrak{R}$-bases $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\beta_1, \beta_2, \cdots, \beta_n$, and let $\mathfrak{a}\rightleftarrows A$ so that matrices H_i exist such that $S(\alpha_i) = H_i A$. If and only if $\mathfrak{b}$ is equivalent to $\mathfrak{a}$, there exists a matrix B corresponding to $\mathfrak{b}$ such that

$$S(\beta_i) = H_i B \qquad (i = 1, 2, \cdots, n).$$

Then also

$$A\overset{\mathsf{T}}{R}_i = \overset{\mathsf{T}}{D}_i A, \qquad B\overset{\mathsf{T}}{R}_i = \overset{\mathsf{T}}{D}_i B.$$

If $\mathfrak{A}$ is a Frobenius algebra, every ideal class has a basis of order n in the following sense. There exist n matrices $B_1, B_2, \cdots, B_n$ such that every matrix

$$b_1 B_1 + b_2 B_2 + \cdots + b_n B_n, \qquad b_i \in \mathfrak{R},$$

corresponds to an ideal of the class, and conversely every ideal of the class corresponds (not uniquely) to a matrix of this linear system.

It is not usually true if $\mathfrak{a}\rightleftarrows A$ and $\mathfrak{b}\rightleftarrows B$ that $\mathfrak{a}\times\mathfrak{b}\rightleftarrows AB$. But if $[\mathfrak{A}]$ is of class number h, the Poincaré correspondence can be amplified so that to every ideal $\mathfrak{a}$ there correspond h matrices $A^1, A^2, \cdots, A^h$, one for each class, with the following property. If $\mathfrak{b}$ is in class b, and if $\mathfrak{a}\times\mathfrak{b} = \mathfrak{c}$, and if $\mathfrak{c}\rightleftarrows C$ by the Poincaré correspondence, then

$$A^b B = C.$$

Thus every ideal $\mathfrak{a}$ has, besides its principal norm

$$n(\mathfrak{a}) = \|A\|$$

where the double bars indicate the absolute value of the determinant, h other norms

$$n^i(\mathfrak{a}) = \|A^i\|.$$

If $\mathfrak{b}$ is in class b and $\mathfrak{a}\times\mathfrak{b} = \mathfrak{c}$, then

$$n^b(\mathfrak{a})\cdot n(\mathfrak{b}) = n(\mathfrak{c}).$$

5. **The algebraic field.** Let us now restrict $\mathfrak{F}$ to be the rational

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[10] M. Deuring, *Algebren*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 4, no. 1, Springer, Berlin, 1935, p. 90.

field, $\mathfrak{R}$ to be the ring of rational integers, and $[\mathfrak{A}]$ to be the maximal integral domain of a (commutative) algebraic field of order n over $\mathfrak{F}$. We can connect our results with the Dedekind-Weber theorem.

It is now true that every ideal is two-sided and the norm of the product of two ideals is equal to the product of their norms. Consequently all the norms of an ideal $\mathfrak{a}$ are equal,

$$n(\mathfrak{a}) = n'(\mathfrak{a}) = n^2(\mathfrak{a}) = \cdots = n^h(\mathfrak{a}).$$

Select two ideal classes, the ath and the bth, whose product class is the cth class. Select bases

$$A_1, A_2, \cdots, A_n; B_1, B_2, \cdots, B_n; C_1, C_2, \cdots, C_n$$

for these three classes, and define the n-ary n-ic forms

$$g(y) = | y_1A_1 + y_2A_2 + \cdots + y_nA_n |,$$
$$h(z) = | z_1B_1 + z_2B_2 + \cdots + z_nB_n |$$

where the y's and z's are indeterminates. The matrix A_i corresponds to an ideal $\mathfrak{a}_i$ by the Poincaré correspondence, and $\mathfrak{a}_i$ corresponds to A_i^b under the extended correspondence. It can, moreover, be shown that

$$g(y) = | y_1\overset{b}{A}_1 + y_2\overset{b}{A}_2 + \cdots + y_n\overset{b}{A}_n |.$$

Since each product $A_i^bB_j$ corresponds to an ideal in the cth class, there exist rational integers c_{ijk} such that

$$\overset{b}{A}_iB_j = \sum_k c_{ijk}C_k.$$

Thus

$$g(y)\cdot h(z) = \left| \sum_{i,j} y_iz_j\overset{b}{A}_iB_j \right| = \left| \sum_k x_kC_k \right| = f(x)$$

where

$$x_k = \sum_{i,j} c_{ijk}y_iz_j.$$

Thus we have the composition of n-ary n-ic forms in a manner which is directly related to the composition of ideal classes.

This may be clearer to the reader if an example is given. In the quadratic field $\mathfrak{F}((-31)^{1/2})$, the maximal integral domain has the basis

$$\epsilon_1 = 1, \qquad \epsilon_2 = (1 + (-31)^{1/2})/2.$$

The class number is 3, and each of the ideals[11] (1), $(2, 2^{-1}(1+(-31)^{1/2})$

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[11] L. W. Reid, *The elements of the theory of algebraic numbers*, Macmillan, 1910, p. 444.

$(4, (1+(-31)^{1/2})/2)$ is in a different class, which we shall denote as the first, second and third classes, respectively. The matrices

$$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 0 & 1 \\ -8 & 1 \end{bmatrix}$$

form a basis for the first or principal class, while

$$A_1 = \begin{bmatrix} -1 & 1 \\ -4 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the second class, and

$$B_1 = \begin{bmatrix} -1 & 1 \\ -2 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

for the third class. Let

$$\begin{aligned} f(x) &= | x_1S_1 + x_2S_2 | = x_1^2 + x_1x_2 + 8x_2^2, \\ g(y) &= | y_1A_1 + y_2A_2 | = 4y_1^2 - y_1y_2 + 2y_2^2, \\ h(z) &= | z_1B_1 + z_2B_2 | = 2z_1^2 - z_1z_2 + 4z_2^2. \end{aligned}$$

These represent the three positive reduced forms of discriminant -31.[12] Now

$$\overset{3}{A}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}, \qquad \overset{3}{A}_2 = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix},$$

and

$$| y_1\overset{3}{A}_1 + y_2\overset{3}{A}_2 | = g(y).$$

We have

$$(y_1\overset{3}{A}_1 + y_2\overset{3}{A}_2)(z_1B_1 + z_2B_2) = x_1S_1 + x_2S_2$$

where

$$\begin{aligned} x_1 &= y_1z_1 - 4y_1z_2 - 2y_2z_1, \\ x_2 &= - y_1z_1 + y_2z_2. \end{aligned}$$

6. **A noncommutative ring.** A simple instance of an integral domain of a noncommutative algebra of class number greater than 1 was given by E. J. Finan.[13] The complete matric algebra of order 4 over the rational field has a non-maximal integral domain with basis

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[12] L. E. Dickson, *Introduction to the theory of numbers*, University of Chicago Press, 1929, p. 140.

[13] Duke Math. J. vol. 1 (1935) pp. 484–490.

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\quad \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\quad \begin{bmatrix}0 & 0\\ 3 & 0\end{bmatrix}.$$

The class number of this domain is 3, and the three left ideal classes may be represented by the matrices

$$A=\begin{bmatrix}a & b & 0 & 0\\ 3d & c & 0 & 0\\ 0 & 0 & c & d\\ 0 & 0 & 3b & a\end{bmatrix},\quad B=\begin{bmatrix}3a & b & 0 & 0\\ 3d & c & 0 & 0\\ 0 & 0 & c & d\\ 0 & 0 & b & a\end{bmatrix},\quad C=\begin{bmatrix}a & b & 0 & 0\\ d & c & 0 & 0\\ 0 & 0 & 3c & d\\ 0 & 0 & 3b & a\end{bmatrix}$$

where A represents the principal class.

Denote by $\mathfrak{a}_1$ the ideal whose basis is A with $a=1$, $b=c=d=0$; by $\mathfrak{a}_2$ the ideal whose basis is A with $a=0$, $b=1$, $c=d=0$, and so on. These basic ideals are singular, and in fact $\mathfrak{a}_1=\mathfrak{c}_1$, $\mathfrak{a}_2=\mathfrak{c}_2$, $\mathfrak{a}_3=\mathfrak{b}_3$, $\mathfrak{a}_4=\mathfrak{b}_4$. A nonsingular ideal can belong to but one class.

	A_1	A_2	A_3	A_4	B_1	B_2	B_3	B_4	C_1	C_2	C_3	C_4
A_1	C_1	C_2	C_3	$3C_4$	$3C_1$	C_2	C_3	$3C_4$	C_1	C_2	C_3	C_4
A_2	$3C_1$	$3C_2$	C_3	$3C_4$	$3C_1$	C_2	C_3	$3C_4$	$3C_1$	$3C_2$	$3C_3$	$3C_4$
A_3	B_1	$3B_2$	B_3	B_4	B_1	B_2	B_3	B_4	B_1	$3B_2$	$3B_3$	B_4
A_4	B_1.	$3B_2$	$3B_3$	$3B_4$	$3B_1$	$3B_2$	$3B_3$	$3B_4$	B_1	$3B_2$	$3B_3$	B_4
B_1	B_1	$3B_2$	$3B_3$	$3B_4$	$3B_1$	$3B_2$	$3B_3$	$3B_4$	B_1	$3B_2$	$3B_3$	B_4
B_2	B_1	$3B_2$	B_3	B_4	B_1	B_2	B_3	B_4	B_1	$3B_2$	$3B_3$	B_4
B_3	B_1	$3B_2$	B_3	B_4	B_1	B_2	B_3	B_4	B_1	$3B_2$	$3B_3$	B_4
B_4	B_1	$3B_2$	$3B_3$	$3B_4$	$3B_1$	$3B_2$	$3B_3$	$3B_4$	B_1	$3B_2$	$3B_3$	B_4
C_1	C_1	C_2	C_3	$3C_4$	$3C_1$	C_2	C_3	$3C_4$	C_1	C_2	C_3	C_4
C_2	$3C_1$	$3C_2$	C_3	$3C_4$	$3C_1$	C_2	C_3	$3C_4$	$3C_1$	$3C_2$	$3C_3$	$3C_4$
C_3	$3C_1$	$3C_2$	C_3	$3C_4$	$3C_1$	C_2	C_3	$3C_4$	$3C_1$	$3C_2$	$3C_3$	$3C_4$
C_4	C_1	C_2	C_3	$3C_4$	$3C_1$	C_2	C_3	$3C_4$	C_1	C_2	C_3	C_4

From the accompanying multiplication table it may be noted that multiplication is associative[14] but that the class group fails to exist. The examples of composition furnished by this ring are in a sense trivial, but they serve to show that the qualifying conditions attached to our earlier theorems are essential, and that the problem of determining all forms which admit composition in a ring is a complicated one.

By composing the third ideal class with the first, we obtain

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[14] This follows from the definition of the product of two modules. Monatshefte für Mathematik und Physik, loc. cit. p. 304.

$$3(y_1+y_2+y_3+y_4)^2(y_1+3y_2+3y_3+y^4)^2(z_1z_3-3z_2z_4)^2=3(x_1x_3-x_2x_4)^2.$$

That each side would be a perfect square was predictable, since the algebra is of rank 2. This composition is essentially

$$x_1x_3 - x_2x_4 = (y_1 + y_2 + y_3 + y_4)(y_1 + 3y_2 + 3y_3 + y_4)(z_1z_3 - 3z_2z_4).$$

Let us see if this composition is in accord with the Theorem of Gauss. Here $|S^T(z)|$ vanishes identically and

$$|R(y)| = 3(y_1 + 3y_2 + 3y_3 + y_4)^2(y_1 + y_2 + y_3 + y_4)^2$$

so that $g(y)$ divides $|R(y)|$, while $h(z)$ divides $|S^T(z)|$ trivially.

The example is also of interest in connection with the Theorem of Dickson. Clearly $g(y)$ is not equivalent to $f(x)$, and since $|S^T(z)|$ vanishes identically, it is not expected to be. Since $|R(y)|$ does not represent ±1, $h(z)$ is not equivalent to $f(x)$ in the ring of rational integers, but is equivalent to $f(x)$ in the field of rational numbers.

UNIVERSITY OF WISCONSIN

BOOK REVIEWS

Fourier series. By G. H. Hardy and W. W. Rogosinski. (Cambridge Tracts in Mathematics and Mathematical Physics, no. 38.) New York, Macmillan; Cambridge University Press, 1944. 8+100 pp. \$1.75.

The object of the authors, as stated in their preface, was to write a book on Fourier series "in a modern spirit, concise enough to be included in the (Cambridge) series" of tracts, yet complete enough to serve as an introduction to Zygmund's *Trigonometrical series.* How well their object has been achieved is readily seen on examining the wealth of material covered in some ninety pages. At the outset they define what is expected of the reader and what aspects of the subject they propose to omit. The reader is expected to have a knowledge of Lebesgue integral theory as treated in Chapters X–XII of Titchmarsh's *Theory of functions*; and while it is recognized "unofficially" that the reader will undoubtedly have some knowledge of the theory of trigonometrical series, officially the tract is self-contained.

Certain topics are frankly omitted: for example, the inequalities of Young and Hausdorff, M. Riesz's theorem on conjugate series, theorems on Cesàro summability of general order, and uniqueness theorems for summable series. Nor is Denjoy's theory of general trigonometrical series considered.

Chapter I, entitled *Generalities* lays the groundwork for all subsequent developments. At the outset trigonometrical series and trigonometrical Fourier series are defined, and at the same time their relation to harmonic and analytic functions is brought out. The emphasis on this relationship is reinforced at various points throughout the tract by the application of Fourier series theorems to analytic function theory. The essential results of Lebesgue integral theory and the theory of the spaces L^p which are needed for the sequel are stated succinctly.

Chapter II is directed toward the classical L^2 theory of Fourier series. First the situation for general normal orthogonal systems is considered. Here the Riesz-Fischer theorem, the Parseval theorem, and the relation between closure and completeness find their natural setting. From this point on, attention is focussed on trigonometrical series. The completeness of the trigonometrical system (T): $1/2$, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cdots$ in $L(0, 2\pi)$ is demonstrated; and the completeness of (x^n) for $n=N, N+1, \cdots$ in $L(a, b)$ and the Weierstrass approximation theorems appear as corollaries.

The theorems of Chapter III fall into two main categories. The first category is concerned with general properties of Fourier series: the theorems which relate the Fourier constants of a given function of $L(-\pi, \pi)$ and the Fourier constants of functions associated with it in certain simple ways; the Riemann-Lebesgue Theorem; the theorem on the term-by-term integration of a Fourier series; the Gibbs phenomenon. The second category of theorems consists of two classes, the first dealing with properties of the Fourier constants which appear as consequences of special hypotheses on the function; the second dealing with trigonometrical series whose coefficients are positive and decreasing to zero. Properties of Fourier constants are given for functions f which satisfy one of the following conditions: f is non-negative in $(0, 2\pi)$, is odd and non-negative in $(0, \pi)$ (the results in these two cases yield Carathéodory's inequalities for the coefficients of a function analytic and with non-negative real part in the interior of the unit circle and the theorem of Dieudonné and Rogosinski on schlicht functions), is monotone in $(0, 2\pi)$, is convex in $(0, 2\pi)$, satisfies a uniform Hölder condition in $(0, 2\pi)$, is of bounded variation, and so on. In the case of functions of bounded variation the example of Hille and Tamarkin is given of a function which is continuous and of bounded variation and yet has Fourier coefficients which are not $o[|n|^{-1}]$ where n is the index.

Chapter IV is concerned with the convergence of a Fourier series and its associated conjugate series. Various known sufficient conditions for the pointwise convergence of a Fourier series are given, such as Dini's test, Jordan's test, Lebesgue's test. Related problems for the conjugate series are considered. An example is given for a function whose Fourier series diverges at a point of continuity.

The summability of a Fourier series is treated in Chapter V, which is introduced by a brief discussion of linear regular methods of summation. General methods of summation of Fourier series are discussed and the $(C, 1)$ and A methods are developed as special cases of the general theory.

Chapter VI deals with applications of the results of Chapter V. Here the theorem is proved that there are Fourier series which diverge almost everywhere. Other topics treated are Fourier series with positive coefficients, gap theorems, the Gibbs set, the existence of the conjugate function.

The final chapter is devoted to the theory of general trigonometrical series. The classical uniqueness theorem for trigonometrical series associated with the names of Riemann, Cantor, and du Bois-Reymond is established, and the remainder of this chapter is devoted to proving that if a trigonometrical series converges except in an enumerable set to a finite and integrable function, then it is the Fourier series of this function.

M. H. HEINS

The advanced theory of statistics. Vol. 1. By Maurice G. Kendall. Philadelphia, Lippincott, 1944. 12+457 pp. $16.00.

Modern statistics is built around sampling theory. It is not well presented by books in the tradition of a quarter-century ago which exalted an extreme and sterile empiricism and ignored or deprecated probability and mathematics in general, even when such books are revised to mention modern developments. Neither is it adequately presented in the books, now becoming available to research workers in an increasing number of fields, which give sound practical advice and formulae, but without the derivations. For the serious student of statistics nothing is wholly satisfactory short of a treatment starting from first principles and proceeding by fully stated definitions and derivations to the methods needed for the entire array of situations with which statistics deals. To get on as far as possible with this program has been the object of only a few of the many books on statistics. Of these, Mr. Kendall's is the largest, and covers in fullest detail the subjects treated.

In spite of the title, this book is not "advanced" in the sense of requiring of the reader a prior knowledge of statistics. It does call for more of a mathematical background than is possessed by most students of statistics. Anyone who has mastered advanced calculus should get along fairly well with it, though there are occasional uses of such relatively advanced mathematics as complex integration and evaluation of integrals by residues, the Euler-Maclaurin sum formula, Stirling's formula (which is used without proof or explicit statement), the integral form of the remainder in Taylor's theorem, and definite quadratic forms. For a reader who is not troubled by these purely mathematical matters the style is unusually clear and explicit. There are many illustrations drawn from actual observations. Each chapter ends with a collection of problems; these are of a superior quality, suitable for testing mathematical skill and mastery of the material, not merely numerical data to be substituted in formulae. The work therefore possesses qualifications as an excellent textbook for suitably prepared students in the hands of a suitable teacher. The teacher, however, should not only be prepared to help the students over the mathematical hurdles noted and others, but should also be enough of a specialist in statistics to apportion due emphasis to the various topics treated in relation to the wider theory and to applications. The book will also need supplementation from other sources, at least until the second volume appears, in order to cover some of the most essential parts of statistics.

Chapter 1, *Frequency-distributions*, provides a good introduction, and Chapter 2, *Measures of location and dispersion*, is rather exhaustive, though these two chapters seem to be influenced too much by tradition in contrast with the modern tendency to distinguish early and sharply between "sample" and "population," that is, between observations and probability distribution. Chapter 3, *Moments and cumulants*, includes a good account of Sheppard's corrections. Chapter 4, *Characteristic functions*, presents a careful and adequate treatment following in part that of Paul Lévy and including some of the limit theorems. Chapters 5 and 6 describe the binomial, negative binomial, Poisson, hypergeometric, normal, Pearson and Gram-Charlier distributions. Chapter 7 is entitled *Probability and likelihood* and makes some attempt to deal with fundamentals, though the main attack on them is postponed to the second volume. The chapter ends with a curiously misplaced proof by Cramér, under Lindeberg's condition, of approach to normality; this would seem logically to belong three chapters earlier. Chapter 8, *Random sampling*, deals chiefly with practical problems involved in making a sample random. Chapter 9 develops a considerable number of formulae for standard errors, exact and approximate. Chapter 10, *Exact sampling distributions*, illustrates various methods of obtaining such distributions, including the use of geometrical methods, of characteristic functions, and of mathematical induction. Chapter 11, *Approximations to sampling distributions*, contains a most intelligible and comprehensive account of Fisher's cumulants and the theory connected with them. There is an extensive table of cumulants of cumulants. Chapter 12 deals with the χ^2 distribution, with applications and approximations. Chapter 13 is on *Association and contingency*, Chapter 14 on *Product moment correlation*, Chapter 15 on *Partial and multiple correlation*, and Chapter 16 on *Rank correlation*. These chapters are comprehensive treatments and include nearly all published results of importance in their fields.

It is stated in the preface that the second volume will deal with the theory of estimation, regression, analysis of variance, tests of significance, multivariate analysis, theories of statistical inference, and time series. If these subjects are treated as comprehensively as those in the first volume, a substantial treatise covering the main body of the theory will be available.

The treatment is occasionally lacking in exactitude and rigor, though it compares favorably in this respect with many books on mathematics and mathematical physics, and is enormously above the general level of books on statistics. The volume is printed in England, with a rather luxuriously pleasing use of display formulae and wide margins on large pages. However there is no table of contents in this first volume, and typographical errors seem unusually numerous. A particularly troublesome one is the omission of the subscript n from v in the second integral in line 7 of p. 112; this should be remedied by each owner of a copy, since the meaning of the Helly-Shohat-Fréchet theorem is involved. The continued fraction for the normal probability integral is derived on p. 129 by Laplace's method instead of by the more satisfactory argument of Jacobi, and one of the useful series for this integral is not included at all. The advice on p. 336 not to use the standard error of the correlation coefficient unless n exceeds 500 is excessively conservative when ρ is small.

The author follows the practice of defining the variance of a sample as the simple mean of the squares of the deviations from the mean. This saves trouble for the author and the reader, in their capacities as such, but makes no end of trouble for them or anyone else who undertakes to substitute observations in the formula thus simply laid down. Use of a denominator less by unity than the sample number has distinct advantages.

Previous relevant writings are occasionally ignored—as often happens in a science built up from contributions scattered through multitudinous journals devoted largely to applications. The statement on p. 250 that no previous systematic account had been given of methods for deriving sampling distributions neglects a paper by B. H. Camp in the Annals of Mathematical Statistics for 1937.

With these qualifications, which can easily be taken care of by a competent teacher, or in later editions by the author and publishers, the book is an outstanding landmark and promises to be of the utmost value in advancing knowledge of the theory of statistics.

HAROLD HOTELLING

A handbook of perspective drawing. By James C. Morehead and James C. Morehead, Jr. Pittsburgh, J. C. Morehead, 1941. 3+166 pp. $4.50.

Perspective being a central projection, the *Handbook* denotes by *picture plane*, *station point*, and *station-point distance* the plane of projection, the center of projection, and the distance between these, respectively. The perspective of a point P may be obtained by solving the two problems of geometry of position: (1) find the line l determined by P and the station point S; (2) find the point P' in which l pierces the picture plane π. The aggregate of the projections P' of the points P of an object constitutes the perspective of this object. An important point of a line is its *vanishing point* where a parallel to it through S pierces π, and an important line of a plane is its *vanishing line* where a plane through S parallel to it cuts π. In the *Handbook* this line is called a *horizon.* All lines of a two-parameter family of parallel lines have the same vanishing point V.

Let us now think of an architectural object like a house which contains lines (edges) belonging to one or another of three mutually perpendicular two-parameter families of parallel straight lines, those of one family being vertical. For a station point S of general position and a picture plane π of general orientation, there will, in general, be three finite vanishing points in π. Such a picture is called a three-point perspective. If π is vertical there will be only two finite vanishing points and these will be on the vanishing line (horizon) of the horizontal planes. The picture is then called a two-point perspective. Finally, if π is still vertical and contains a line of one of the other families, there will be only one finite vanishing point, and the resulting picture is called a one-point perspective.

Early in the book the notion of *measure line* is introduced. It is a line which lies in or is parallel to the picture plane and thus shows distances in their true size or magnified or diminished by a constant factor. The first method given for drawing perspectives is that called *direct projection.* This consists essentially in solving problems (1) and (2) stated above by the Mongean method of descriptive geometry with the vertical plane of that method taken as the picture plane. The *mixed method* is the next one described. Here the perspectives of the vertical edges are obtained as in the direct method. However, the perspectives of the corners on these edges are found with the aid of a measure line.

Chapter II concerns itself with what is called the *45°-line method.* The essential feature of this method is to find the vanishing points of the diagonals of the faces of a cube which are parallel to the floor and walls of a rectangular building.

In Chapter III the book describes various constructions and mechanical devices for drawing lines to distant vanishing points. In Chapter IV use is made in two-point perspective of the 45°-line method in floor planes. This method reduces to the use of a single vanishing point the lay-out procedure for finding the perspectives of the horizontal dimensions of a rectangular building. Chapter V is devoted to three-point perspective. The picture plane π is here assumed to be *not* vertical. The orientation of this plane with respect to the rectangular building to be represented is determined by two (parametric) angles A and B.

Chapter VI concerns itself with the perspectives of curves such as circles, conics and helices. Application is made to such objects as loggias with circular arches, viaducts with parabolic arches, groined vaults, cylindrical bays with conical roofs and niches with domed ceilings, water wheels and helical stairways.

In Chapter VII some of the methods already described are applied to the determination of the perspectives of the shadows and reflections of the objects represented.

The reviewer has seldom seen a book on drawing in which the figures are executed more carefully and beautifully than they are in this book. For this reason as well as for the precise descriptions of the method involved, a copy of this book should be in the possession of everyone who is interested in perspective drawing.

WILLIAM H. ROEVER

NOTES

The issue of Science for December 15, 1944, contains the address delivered by Professor M. H. Stone of Harvard University on August 13, 1944, at the dinner held in connection with the Summer Meeting of the Society in Wellesley, Massachusetts. The published article bears the title *American mathematics in the present war.*

Professor Emeritus W. D. Cairns of Oberlin College, for twenty-seven years secretary-treasurer and for the past two years president of the Mathematical Association of America, was elected honorary president for life on November 26, 1944, at the Chicago meeting.

Professor Marston Morse of the Institute for Advanced Study has received a citation from the War Department for outstanding performance as a civilian in government work in furtherance of the nation's war effort.

The School of Mathematics of the Institute for Advanced Study will allocate a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year 1945–1946. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, N. J., and should be returned at the earliest possible date.

Associate Professor L. E. Babcock of Newberry College, Newberry, South Carolina, has been appointed to an assistant professorship at the University of Richmond.

Dr. C. L. Clark of the University of Virginia has been appointed to an assistant professorship at Oregon State College.

Assistant Professor W. M. Davis of Cornell College, Mount Vernon, Iowa, has been promoted to a professorship.

Assistant Professor H. A. Doeringsfeld of the University of Minnesota has been promoted to an associate professorship.

Associate Professor Mary Goins of Western College, Oxford, Ohio, has been appointed to an associate professorship at DePauw University.

Associate Professor F. W. John of New York University has been promoted to a professorship.

Adjunct Professor W. S. Krogdahl of the University of South Caroline has been promoted to an associate professorship.

Assistant Professor F. E. Miller of the University of Minnesota has been promoted to an associate professorship.

Miss Helen K. Milleson of Carleton College, Northfield, Minnesota, has been appointed statistician in the Bureau of Census, Washington, D. C.

Dr. C. D. Olds of Purdue University has been appointed to an assistant professorship at San Jose State College, San Jose, California.

Mr. H. E. Wahlert of New York University has been promoted to an assistant professorship.

The following appointments to instructorships are announced: Coe College, Cedar Rapids, Iowa: Miss Aletha C. Gaddis; University of Kentucky: Mr. F. W. Donaldson; University of Nebraska: Dr. J. F. Heyda; North Dakota Agricultural College: Mr. Neil Lockwood; University of Oregon: Mrs. Olga A. Freeman, Miss Marie Ridings; Southeastern Louisiana College, Hammond, Louisiana: Dr. Margaret R. Davis.

Professor H. A. DoBell of New York State College for Teachers died December 8, 1944, at the age of forty-eight years. He had been a member of the Society since 1928.

Professor W. P. Ott of the University of Alabama died December 25, 1944, at the age of sixty-eight years. He had been a member of the Society since 1924.

Professor Emeritus G. T. Sellew of Knox College died December 27, 1944, at the age of seventy-eight years. He had been a member of the Society since 1899.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

51. E. W. Barankin: *Bounds for the characteristic roots of a matrix.*

With the application of Schwarz's inequality (a principal tool in the more recent investigations of bounds for matrix roots) to the n linear equations $\lambda x_r = \sum_s a_{rs} x_s$, where λ is a root of the arbitrary matrix $A = \|a_{rs}\|$, and $\{x_r\}$ is an eigen-vector of A belonging to λ, it is proved that $|\lambda| \leqq \max (R_r T_r)$, where $R_r \equiv \sum_s |a_{rs}|$, $T_r \equiv \sum_s |a_{sr}|$. Generalizations of this theorem, to give stronger upper bounds, are indicated; and the independent result, that $|\lambda| \leqq \max R_r$, $|\lambda| \leqq \max T_r$, is established. The first theorem was recently proved for matrices of order 2 by A. B. Farnell (Bull. Amer. Math. Soc. vol. 50 (1944) pp. 789–794). (Received December 23, 1944.)

52. Garrett Birkhoff: *Lattice-ordered Lie groups.*

A Lie l-group is defined as a lattice-ordered group which is a Lie group; a Lie tl-group, as a Lie l-group whose lattice operations are continuous in the topology; a Lie l-algebra, as a lattice-ordered Lie algebra whose set of positive elements is invariant under all inner automorphisms. Conditions that the Lie algebra of a Lie group be a Lie l-algebra are found. A Lie algebra of dimension r can be made into a Lie l-algebra if and only if it has a chain of invariant subalgebras of length r; hence only if it is solvable. A simply ordered topological l-group is one-dimensional. The only Lie tl-groups are the powers R^n of the additive group of real numbers; hence they are all commutative. (Received January 3, 1945.)

53. A. L. Foster: *Boolean-like rings, a generalization of Boolean rings. The logical algebra of general commutative rings.*

This paper is mainly concerned with the study of a generalization (first touched on in: A. L. Foster, *The idempotent elements of a commutative ring form a Boolean algebra* · · · . *Ring duality and transformation theory*, to appear in Duke Math. J., March 1945) of the concept Boolean ring, a generalization in which many of the formal properties, both ring and "logical," of the latter are preserved, and one which arises naturally from the basic duality theory of rings previously introduced. The class of Boolean-like rings is discussed within the framework of a certain logical algebra of (general) rings. (Received January 8, 1945.)

54. G. N. Garrison: *Note on invariant complexes of a quasigroup.*

In an earlier paper (*Quasi-groups*, Ann. of Math. vol. 41 (1940) pp. 474–487) the writer discussed (§4.4) relations between two invariant complexes, H and K, of a finite quasigroup S. The purpose of the present note is to discuss the intersection $H\wedge K$ under much less restrictive hypotheses than were imposed earlier. If $H\wedge K=D$, D not vacuous, D is shown to be invariant. Also the order of D divides the order of H (and K) and non-overlapping sets D, Da, Db, $\cdots$ exist such that $H=D+Da+Db+\cdots$. When K is invariant, so is Ka for any a (previous result); therefore if $H\wedge K=0$, choose $a<S$ so that $H\wedge Ka=D\neq 0$, D invariant. In either case it is shown that, whenever $Hb\wedge Kc=D\neq 0$, D is invariant and any other intersection $Hp\wedge Kq$ is either vacuous or takes the form Dr for some r. (Received January 25, 1945.)

55. R. E. Johnson: *Systems of finite linear equations.*

In this paper the author studies the system of linear equations (1) $\sum_{r=1}^{n_s} a_{rs}x_r=y_s$, $s=1, 2, \cdots$ for a_{rs} and y_s in a commutative field P. It is shown that the matrix $A=(a_{rs})$ has associated with it a "quasi-inverse" A^*. A necessary and sufficient condition that (1) have a solution is that the idempotent matrix A^*A be a right unit of the matrix $Y=(y_{rs})$; $y_{1s}=y_s$, $y_{rs}=0$, $r>1$. If (1) has a solution, the general solution is obtained in terms of A and A^*. (Received January 23, 1945.)

56. J. N. B. Livingood: *A partition function with the prime modulus $p>3$.*

In this paper the author, generalizing a problem recently solved by J. Lehner, obtains a convergent series for $p_a(n)$, the number of partitions of n into summands which are congruent to $\pm a$ modulo p, p being a prime number greater than 3. The method is that of Hardy and Ramanujan in its recent refinement. Let $p_a(n)$ be the coefficient of a generating function $F_a(x)$ which is regular within the unit circle. All depends upon the asymptotic behavior of $F_a(x)$ near roots of unity. In order to determine this a modular transformation is applied to $F_a(x)$. The infinite product $F_a(x)$ goes over into another one whose asymptotic behavior is evident. The coefficients $p_a(n)$ are then computed by means of the Cauchy integral, subject to the Farey dissection. Since the modular forms in question are of dimension zero, the method, due to Rademacher, necessitates the introduction of the so-called Kloosterman sums, in order to lower the estimate of the sums of certain roots of unity. An interesting by-product of the investigation is the determination of the asymptotic ratios of the $p_a(n)$ for different a and for n tending to infinity. (Received January 5, 1945.)

57. M. F. Smiley: *A remark on metric boolean rings.*

Just as a metric lattice is modular (V. Glivenko, *Géométrie des systèmes de choses normées*, Amer. J. Math. vol. 58 (1936) pp. 799–828), so is a ring with unity element on which is defined a real valued function $\mu(a)$ satisfying (1) $\mu(a)>0$ *for every* $a\neq 0$ and (2) $\mu(a+b)+2\mu(ab)=\mu(a)+\mu(b)$ a metric boolean ring. An example shows that the assumption of a unity element is essential. (Received January 16, 1945.)

58. Ernst Snapper: *Partial differentiation and elementary divisors.*

Let $P=K[x_1, \cdots, x_n]$ be a polynomial ring, where K is a field of characteristic p ($=0$ or $\neq 0$). The operator ring $D=K[\partial/\partial x_1, \cdots, \partial/\partial x_n]$ and the "algebra of linear exponentials M" are defined. Elements of D are operators $\phi=\sum_{i_1\cdots i_n} a_{i_1\cdots i_n}\partial^{i_1+\cdots+i_n}/\partial x_1^{i_1}\cdots\partial x_n^{i_n}$, operating on the elements of M. A set of linear partial differential equations with constant coefficients is a set S of operators and its solutions are elements $\mu \in M$, where $\phi(\mu)=0$ for all $\phi \in S$. An "auxiliary" ideal $\mathfrak{a} \subseteq P$ corresponds to S. The general point ξ of an $(n-i)$-dimensional associated prime $\mathfrak{p}$ of a primary component $\mathfrak{q}$ of $\mathfrak{a}$ determines a "general". $(n-i)$-dimensional exponential solution $\exp(\xi) \in M$ of S. The differential exponent of $\mathfrak{p}$ is δ if a polynomial f with i variables and with coefficients in the field of rational functions of $\mathfrak{p}$, where $f \exp(\xi)$ is a solution of S, has at most degree $\delta-1$. The relationships are studied between δ, the ordinary exponent ρ of $\mathfrak{q}$, the Hentzelt exponent ν of $\mathfrak{q}$, and the multiplicity σ of the root which corresponds to $\mathfrak{p}$ of the $(n-i)$-dimensional elementary divisor of $\mathfrak{a}$. When $\mathfrak{p}$ is isolated and either $p=0$ or $\delta<p$, $\rho=\nu=\sigma=\delta$. When $p\neq 0$, and $\delta \geqq p$, $\rho \leqq \nu$, $p \leqq \nu$, $p \leqq \sigma$. (Received January 17, 1945.)

59. W. J. Sternberg: *On a special set of algebraic nonlinear equations.*

Some physical problems lead to algebraic nonlinear equations, such as the following set $\sum_{i=1}^{3} 1/(x_i/s_k+jy_i)=1/(a_k/s_k+jb_k)$, where $k=1, 2, 3$, $j=(-1)^{1/2}$, $s_1^2 \neq s_2^2$, $s_2^2 \neq s_3^2$, $s_3^2 \neq s_1^2$. An analogous set of two or four or more equations could also be treated. The unknowns are x_i, y_i, while a_k, b_k, s_k are given. The above complex equations are equivalent to six real equations. The transformation $u_i=y_i/x_i$, using the abbreviation $1/(1+s_k^2u_i^2)=f_k(u_i)$ $[i, k=1, 2, 3]$, leads to $\sum_{i=1}^{3} f_k(u_i)/x_i=A_k$, $\sum_{i=1}^{3} u_i f_k(u_i)/x_i=B_k$ where A_k, B_k can be computed from a_k, b_k, s_k. Since the above equations are linear with respect to $1/x_i$, eliminate these unknowns and obtain for the u_i three nonlinear equations, whose left sides are determinants. These determinants are rational alternating functions of the u_i. Simplify the said equations and introduce the elementary symmetric functions w_1, w_2, w_3 of the u_i. The point is that finally three linear equations are obtained for w_1, w_2, w_3. They are uniquely determined and the equation with the roots u_1, u_2, u_3 can be found. The problem is therefore reduced to one equation of third degree. (Received January 20, 1945.)

60. H. S. Wall: *Polynomials with real coefficients whose zeros have negative real parts.*

Let $P(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+ \cdots +a_n$ be a polynomial with real coefficients, and let $Q(z)$ be the polynomial obtained from $P(z)$ by dropping out the first, third, fifth, $\cdots$ terms of $P(z)$. Then, the zeros of $P(z)$ all have negative real parts if and only if the successive quotients obtained in applying to $P(z)$ and $Q(z)$ the euclidean algorithm for finding the greatest common divisor of two polynomials have the form c_1z+1, c_2z, c_3z, $\cdots$, c_nz, where c_1, c_2, $\cdots$, c_n are positive. On the basis of this result, a new proof is obtained of a theorem of A. Hurwitz (*Werke*, vol. II, p. 533 ff.). (Received January 24, 1945.)

ANALYSIS

61. Felix Bernstein: *The integral equations of the theta function.*

In 1920 the author showed that the integral equation $\vartheta * \vartheta - 2t\vartheta + \vartheta * 1 - 1 = 0$ $(\vartheta * \vartheta = \int_0^t \vartheta(\tau)\vartheta(t-\tau)dt)$ for the variable $h=e^{-\pi^2 t}$ has the theta zero function $\sum_{-\infty}^{+\infty} h^{n^2}$ as the only solution analytical and regular in the unit circle at the origin. In subsequent papers it has been brought out that this equation defines relationships of the theta zero function to the theory of heat and a number of new theorems. The transcendent theorems of addition have been derived. It has been shown that the Volterra theory of integral equations of this type can be essentially generalized to include functions as solutions whose domain of existence is limited. No relation could be established between the above equation and the central analytical property of the functions, namely the product representation of Jacobi. A new integral equation has been discovered of a different type which exhibits the product representation of one of the four fundamental theta functions by aid of the "Faltung" of two others. These new integral equations are related to Riemann's zeta function. (Received January 23, 1945.)

62. H. L. Garabedian: *Stieltjes integral transformations*. Preliminary report.

This paper involves a study of the integral transformation $v(x)=\int_0^\infty u(y)d\phi(y/x)$, which, with appropriate restrictions on the mass function $\phi(x)$, defines a regular method of summation $(H, \phi(x))$. Various formulations of inclusion relationships between H-methods of summation together with numerous examples are given. These formulations involve a Stieltjes integral equation of the type $\phi_a(u)=\int_0^\infty \phi_c(u/v)d\phi_b(v)$, sequences of the type $\int_0^\infty u^n d\phi(u)$ $(n=0, 1, 2, \cdots)$, and moment functions of the type $\int_0^\infty u^z d\phi(u), \mathfrak{R}(z)\geqq 0$. (Received January 2, 1945.)

63. Dunham Jackson: *The boundedness of certain sets of orthonormal polynomials in one, two, and three variables.*

A recent paper (Duke Math. J. vol. 11 (1944) pp. 351–365) has discussed the boundedness of orthonormal polynomials for which the domain of integration is (among other possibilities) an arc of a plane curve of the second degree. The corresponding problem is studied now for a curve of the form $y=x^m$ in which m is an arbitrary positive integer or has one or another of a few illustrative fractional values. When the exponent is a fraction there is an intimate connection with the theory of orthogonal polynomials in a single variable with auxiliary conditions (Trans. Amer. Math. Soc. vol. 48 (1940) pp. 72–81), calling for new developments of the latter theory. The length of the analysis which appears to be required in the case of even a very simple fractional exponent suggests that the problem as formulated for algebraic curves of any degree of generality is far from elementary. The methods used in connection with plane curves are found to be effective also for skew curves of corresponding simplicity in space of three dimensions. (Received December 11, 1944.)

64. E. R. Lorch: *The Cauchy-Schwarz inequality and self-adjoint spaces.*

The inequality of Cauchy-Schwarz is expanded to the following form. Let s_{ij} denote a skew-symmetric complex matrix and let $\phi(x)$ denote a real function of the complex vector $x=(x_1, \cdots, x_n)$. Suppose that $\max_y \left| \sum x_i\bar{y}_i + \sum s_{ij}x_i\bar{y}_j \right| /\phi(y) =\sum x_i\bar{x}_i/\phi(x)$. Then $s_{ij}=0$ and $\phi(x)=k(\sum x_i\bar{x}_i)^{1/2}$ where k is a positive constant. This result is pivotal in the proof of the fact that if for a linear vector space $\mathfrak{B}$ the equality $\mathfrak{B}=\mathfrak{B}^*$ holds, where $\mathfrak{B}^*$ is the space adjoint to $\mathfrak{B}$, then $\mathfrak{B}$ is either a unitary space or a Hilbert space. The equation $\mathfrak{B}=\mathfrak{B}^*$ is defined to have the following significance: There exists an algebraic and topological isomorphism T between $\mathfrak{B}$ and $\mathfrak{B}^*$, $Tf=f^*$, $f\in\mathfrak{B}$, $f^*\in\mathfrak{B}^*$ such that $(f, f^*)=\|f\|\cdot\|f^*\|$. Here (f, f^*) denotes the value of the functional f^* on the vector f. (Received January 2, 1945.)

65. Harry Pollard: *A theory of the Laplace integral.* Preliminary report.

It is easily verified that under simple conditions on $f(x)$ the formula $f(x)=\int_0^{k/t}(1-xt/k)^{k-1}L_{k,t}(f)dt$ is an identity, where $L_{k,t}(f)=(-1)^k k^{k+1}[k!t^{k+1}]^{-1}\cdot f^{(k)}(k/t)$ is the Post-Widder operator. If $k\rightarrow\infty$, $f(x)=\int_0^\infty e^{-xt}g(t)dt$, where $g(t)=\lim_{k\rightarrow\infty} L_{k,t}(f)$. Obtaining conditions for which this passage to the limit is valid is tantamount to establishing a representation theory for the Laplace integral. The final results coincide with Widder's theory, but new light is shed on the central role played by the operator $L_{k,t}(f)$. (Received January 20, 1945.)

66. P. R. Rider: *A new use for tables of the incomplete beta function.*

The solution of the problem of minimizing a definite integral having an integrand of the form $(1+y'^2)^m/y''$ has been given. This paper points out how tables of the incomplete beta function can be used to advantage in carrying out the solution in a numerical case. (Received January 22, 1945.)

APPLIED MATHEMATICS

67. C. A. Truesdell: *The membrane theory of shells of revolution.*

The differential equations of the bending theory, and hence of the membrane theory of shells of revolution, are derived as consequences of the equations of three-dimensional elasticity and certain additional assumptions. Less restrictive than customary boundary conditions at the apex of closed domes are proposed. By means of stress functions satisfying a simple ordinary differential equation, solutions of the stress equations are surveyed, and it is shown that the new boundary conditions can be satisfied for a large class of surfaces for which the old could not. Displacement functions are introduced which reduce the integration of the displacement equations to the integration of a fairly simple ordinary differential equation. Numerous exact solutions of the differential equations are given explicitly, with the aid of the stress and displacement functions, and some numerical examples are given. (Received January 10, 1945.)

GEOMETRY

68. S. B. Jackson: *The four-vertex theorem for surfaces of constant curvature.*

In an earlier paper by the writer (S. B. Jackson, *The four-vertex theorem for spherical curves*, Amer. J. Math. vol. 62 (1940) pp. 795–812) it was shown that on any simple closed spherical curve of class C''', not a circle, there are at least four geodesic vertices, that is, extrema of the geodesic curvature. The aim of the present paper is to extend this result to any surface of constant curvature. Specifically, it is shown that every simple closed curve of class C'', not a geodesic circle, in a simply connected region of a surface of constant curvature has at least four geodesic vertices. The technique of the paper is to map the given region of the surface onto the plane in such a way that the geodesic vertices of the given curve go into the vertices of the corresponding plane curve. The theorem then follows from the known facts about the vertices of plane curves. It is shown by example that the restriction that the curve be in a simply connected region of the surface is essential. Finally, it is proved that the theorem cannot be extended to any surfaces of variable curvature. (Received January 30, 1945.)

69. Edward Kasner and John DeCicco: *Some properties of minimal surfaces.*

It is known that for any minimal surface and also for a sphere, any parallel system of plane sections is an isothermal family on the surface. The authors prove that this property is characteristic for the minimal surfaces and the spherical surfaces. Moreover the following fundamental theorem is obtained. If a surface possesses more than four distinct isothermal systems of parallel plane sections, all of which are parallel to a given rectilinear direction L, then every parallel system of plane sections is isothermal, and the surface is either minimal or spherical. Some further results are the following. Any surface which has more than two simply-infinite families of plane geodesics of nonzero curvature is either a sphere or a plane. If a surface has a single isothermal system of parallel plane suections such that the mean curvature is the same at all points of any one of these sections, then it is a surface of revolution. (See for related theorems paper in Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) (Received January 24, 1945.)

STATISTICS AND PROBABILITY

70. E. J. Gumbel: *Expected values of a statistical variate.*

Let x be a continuous unlimited variate; let $F(x)$ be the probability of a value equal to, or less than, x. What values may be expected in a sample of size n? The n expected values $\bar{x}_m$ $(m=1, 2, \cdots, n)$ and the n most probable values $\tilde{x}_m$ are, respectively, the means and the modes of the distributions of the mth values. As the calculation of these means is in general extremely tiresome, and the calculation of the n modes laborious, it is proposed to use as approximations the values $\hat{x}_m$ which correspond to the probabilities F_m obtained from linear interpolation between F_1 and F_n. Here, F_1 and F_n are the probabilities of the most probable largest and of the most probable smallest value in a sample of size n. The probabilities F_m are subject to the inequalities $(m-1)/n<F_m<m/n$. The distribution of the values x_m converges, with increasing sample size, toward the theoretical distribution. For a symmetrical distribution the mean of the values $\hat{x}_m$ coincides with the mean of the population. (Received January 22, 1945.)

TOPOLOGY

71. R. H. Sorgenfrey: *A condition that a continuum be irreducible between two points.*

It is proved that in order that a (compact) continuum M be irreducible between two of its points, it is necessary and sufficient that of every three continua whose sum is M, either two fail to intersect or the sum of two contains the third. (Received December 4, 1944.)

72. P. A. White: *On new types of regular convergence.*

New types of convergence are introduced making use of other local properties in the same way that local connectedness was used in the definition of r-regular convergence by G. T. Whyburn (*On sequences and limiting sets*, Fund. Math. vol. 25 (1935)). In particular the sequence of compact sets $(M_n) \to M$ is said to converge r-coregularly (r-completely avoidably regular—r-c.a. regular) if for every $\epsilon > 0$ there exists a $\delta > 0$ ($\eta > \delta > 0$) and N such that for any $i > N$, point $p \in M_i$, and r-dimensional Vietoris cycle $x^r \in M_n - M_n \cdot S[p, \epsilon](M_n \cdot (F[p, \eta]))$, $x^r \sim 0$ on $M_n - M_n S(p, \delta) \cdot (M_n \cdot (S[p, \epsilon] - S[p, \delta]))$, where S and F represent the spherical neighborhood of a point and its boundary, respectively. If the convergence is i-regular ($i \leqq r-1$) and r-coregular, M contains no r-cut points. If the convergence is i-regular ($i \leqq r$) and r-c.a. regular, M is completely r-avoidable at each point. If each M_i is an r-dimensional closed Cantorian manifold and the convergence is i-regular ($i \leqq r$), i-c.a. regular ($i \leqq r-2$), and $(r-1)$-coregular, and dim $M \leqq n$, then M is a generalized closed manifold in the sense of R. L. Wilder. If the convergence is 0-c.a. regular, it is also 0-regular and 0-coregular, and if each M_i is a 2-dimensional compact manifold, then so is M. However M may not be homeomorphic with any of the M_i. (Received December 12, 1944.)

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SPANS IN LEBESGUE AND UNIFORM SPACES OF TRANSLATIONS OF STEP FUNCTIONS

RALPH PALMER AGNEW

1. **Introduction.** For each $p \geqq 1$, let L_p be the Lebesgue space whose elements are real or complex valued measurable functions $f(x)$, defined over $-\infty < x < \infty$, for which the integral

$$\int_{-\infty}^{\infty} |f(x)|^p dx \tag{1.1}$$

is finite. The distance $\|f_2 - f_1\|$ between two elements f_1 and f_2 of the space is defined by

$$\|f_2 - f_1\| = \left\{\int_{-\infty}^{\infty} |f_2(x) - f_1(x)|^p dx\right\}^{1/p}. \tag{1.2}$$

For each $p \geqq 1$, L_p is a linear metric complete separable space.

Let E be a set in L_p. The *linear manifold* $M(E)$ determined by E is the set of all linear combinations (finite) of elements of E. The *span* $S_p(E)$ of E in L_p is the closure in L_p of $M(E)$; an element ϕ of L_p belongs to $S_p(E)$ if and only if to each $\epsilon > 0$ corresponds an element f_ϵ of $M(E)$ such that $\|\phi - f_\epsilon\| < \epsilon$.

Let $f \in L_p$. For each real h, the *translation* $f(x+h)$ of $f(x)$ is also in L_p. Let T_f denote the set of translations of f. Wiener [2, pp. [illegible]][1] showed that if $f \in L_2$, then $S_2(T_f)$ is the whole space L_2 if and [illegible] if the real zeros of the Fourier transform of f form a set of measure 0. He [2, pp. 9–19] showed also (and this was much more difficult) that if $f \in L_1$, then $S_1(T_f)$ is the whole space L_1 if and only if the Fourier transform of f has no real zeros. He [2, p. 93] raised the question whether similar propositions hold for other values of p and expressed a "suspicion" that they do, at least when $1 \leqq p \leqq 2$.

In view of the similar suspicions held by Wiener and others, a result recently announced by Segal [1] is surprising. Segal has shown that if $1 < p < 2$, then there is an element f of L_p such that (i) the zeros of the Fourier transform of f form a set of measure 0 and (ii) the span $S_p(T_f)$ of the translations of f does *not* include the whole space L_p. This development will doubtless create interest in the search for criteria for $S_p(T_f) = L_p$. With the hope that both the result and the

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

special direct proof may throw light on the problem, the author presents the following result which he obtained ten years ago and has not previously announced.

THEOREM 1.3. *If* $p>1$ *and* F *is a simple step function, then* $S_p(T_F)=L_p$.

2. **Simple step functions.** A *simple step function* is a function $F(x)$ which has a nonzero constant value over a finite interval $a\leqq x<a+2b$ and is zero elsewhere. The positive constant $2b$ is the *width* of the step function. With each simple step function F of width $2b$, we may associate the *normal step function* G defined by

$$G(x) = \begin{cases} 1, & -b\leqq x<b, \\ 0, & \text{elsewhere.} \end{cases} \tag{2.1}$$

Clearly, $M(T_F)=M(T_G)$ and $S_p(T_F)=S_p(T_G)$.

The Fourier transform $g(u)$ of the function $G(x)$ in (2.1) is

$$g(u) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} G(x)e^{iux}dx = (2/\pi)^{1/2}\frac{\sin bu}{u}, \qquad u\neq 0, \tag{2.2}$$

and $g(0)=(2/\pi)^{1/2}b$. This transform is bounded, continuous, and in L_p for each $p>1$. Since $g(u)$ has real zeros, the second theorem of Wiener cited above implies that $S_1(T_G)$ is not the whole space L_1. Since the set of zeros has measure 0, Wiener's first theorem implies that $S_2(T_G)=L_2$. Hence, when $p=2$, the conclusion of Theorem 1.3 follows from Wiener's theory.

3. **Two lemmas.** We shall use the two following lemmas.

LEMMA 3.1. *Let* $p\geqq 1$ *and let* f *be an element of* L_p. *If* $S_p(T_f)$ *contains* g, *then* $S_p(T_f)$ *contains* $S_p(T_g)$.

This is a consequence of the fact that linear combinations of translations of linear combinations of translations of f are linear combinations of translations of f.

LEMMA 3.2. *Let* $p\geqq 1$ *and let* F *be an element of* L_p. *If* $S_p(T_F)$ *contains, for each* $h>0$, *a simple step function of width* h, *then* $S_p(T_F)=L_p$.

Let ϕ be an element of L_p. Let $\epsilon>0$. A fundamental theorem on approximation gives a step function $g(x)$, a linear combination of simple step functions, such that $\|\phi-g\|<\epsilon$. Since g is a linear combination of elements of $S_p(T_F)$, g itself is an element of $S_p(T_F)$. This establishes Lemma 3.2.

4. **Proof of Theorem 1.3.** Using Lemma 3.2, we see that Theorem 1.3 is a consequence of the following lemma.

LEMMA 4.1. *Let* $p>1$. *If* F *is a simple step function, then* $S_p(T_F)$ *contains, for each* $h>0$, *a simple step function of width* h.

Let c, a, and h_1 be the constants such that $F(x)=c$ when $a\leqq x<a+h_1$ and $F(x)=0$ otherwise. Let $h>0$, and let $g(x)$ be the simple step function for which $g(x)=1$ when $0\leqq x<h$ and $g(x)=0$ otherwise. Let $\epsilon>0$. We shall prove Lemma 4.1 by giving an explicit formula for a linear combination $g_\epsilon(x)$ of translations of $F(x)$ such that $\|g-g_\epsilon\|<\epsilon$.

Let m be the least integer for which $mh_1>h$ and let $H=mh_1$. Let

$$G(x) = \sum_{j=0}^{m-1} c^{-1}F(x - jh_1 - a). \tag{4.2}$$

Then

$$G(x) = 1, \qquad 0 \leqq x < H, \tag{4.3}$$

and $G(x)=0$ otherwise. Let n be the least positive integer such that $(nH+h)^{1/p}(2n+1)^{-1}<\epsilon$. Let

$$\begin{aligned} g_\epsilon(x) = \sum_{k=0}^{n-1}\Bigg[&\left(1 - \frac{2k+1}{2n+1}\right)G(x - kH) \\ &- \left(1 - \frac{2k+2}{2n+1}\right)G\left(x - kH - h\right)\Bigg]. \end{aligned} \tag{4.4}$$

An understanding of the manner in which $g_\epsilon(x)$ approximates $g(x)$ is enhanced by drawing graphs of the terms of the sum in (4.4) and of the function $g_\epsilon(x)$. The function $g_\epsilon(x)$ is so constructed that

$$|g(x) - g_\epsilon(x)| = (2n+1)^{-1}, \qquad 0 \leqq x < nH + h, \tag{4.5}$$

and $|g(x)-g_\epsilon(x)|=0$ elsewhere. Hence

$$\begin{aligned} \int_{-\infty}^{\infty} |g(x) - g_\epsilon(x)|^p dx &= \int_0^{nH+h} (2n+1)^{-p} dx \\ &= (nH + h)(2n+1)^{-p} \end{aligned} \tag{4.6}$$

and

$$\|g - g_\epsilon\| = (nH + h)^{1/p}(2n+1)^{-1} < \epsilon. \tag{4.7}$$

This completes the proof of Lemma 4.1 and hence also that of Theorem 1.3.

It is apparent that the above method for approximating $g(x)$ in L_p,

$p>1$, fails when $p=1$; when $p=1$ and ϵ is small, one is unable to choose a positive integer n such that the last inequality in (4.7) holds. As a matter of fact, no method can serve. For if $g(x)$ could be approximated in L_1 by linear combinations of translations of $G(x)$, then we could prove that $S_1(T_G)=L_1$; and this would contradict the second Wiener theorem cited in §1.

When h, H, and ϵ are fixed, ϵ being small, the number n of terms in (4.4) required to define the function $g_\epsilon(x)$ is a function of p which increases as p decreases. This indicates that the approximation in L_p of $g(x)$ by linear combinations of translations of $G(x)$ becomes more difficult as p decreases toward 1. This is in accord with the fact that the approximation is impossible when $p=1$.

5. **An indicator.** Let $p\geqq 1$ and let f be an element of L_p. The number $I^{(p)}(f)$ defined by

$$I^{(p)}(f) = \underset{g}{\text{l.u.b.}}\ \underset{F}{\text{g.l.b.}}\ \|g - F\|, \tag{5.1}$$

where the g.l.b. is taken for all F in $M(T_f)$ and the l.u.b. is then taken for all g in L_p for which $\|g\|=1$, is an *indicator* of the extent to which linear combinations of translations of $f(x)$ span L_p. If $S_p(T_f)=L_p$, then $I^{(p)}(f)=0$, and the converse holds. In any case, $0\leqq I^{(p)}(f)\leqq 1$. It would be interesting to have information about $I^{(p)}(f)$. The author has very little. It seems to be true that if f is a simple step function, then $I^{(1)}(f)=1$.

6. **A corollary of Theorem 1.3.** When one is seeking to determine whether $S_p(T_f)=L_p$, the following theorem may be helpful.

THEOREM 6.1. *Let $p>1$, and let f be an element of L_p. In order that $S_p(T_f)=L_p$, it is necessary and sufficient that $S_p(T_f)$ contain at least one simple step function.*

Necessity is trival. Sufficiency is a consequence of Theorem 1.3 and Lemma 3.1.

7. **Uniform approximation.** The theorems of this section deal with uniform approximation, over the infinite interval $-\infty<x<\infty$, of functions $g(x)$ by linear combinations of translations of a given simple step function.

THEOREM 7.1. *If $F(x)$ is a simple step function, then to each simple step function $g(x)$ and positive number ϵ corresponds a linear combination $g_\epsilon(x)$ of translations of $F(x)$ such that*

$$| g(x) - g_\epsilon(x) | < \epsilon, \qquad -\infty < x < \infty. \tag{7.2}$$

Let $\epsilon > 0$. In case $g(x) = 1$ over $0 \leqq x < h$ and $g(x) = 0$ otherwise, (4.5) shows that the function $g_\epsilon(x)$ defined explicitly in (4.4) has the required property provided n is chosen so great that $n^{-1} < \epsilon$. In case $g(x) = c$ over $a \leqq x < a + h$, the function $cg_\epsilon(x - a)$ has the required property. This proves Theorem 7.1.

THEOREM 7.3. *Let $F(x)$ be a simple step function. Then to each continuous function $f(x)$, for which $f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$, and positive number ϵ corresponds a linear combination $f_\epsilon(x)$ of translations of $F(x)$ such that*

$$|f(x) - f_\epsilon(x)| < \epsilon, \qquad -\infty < x < \infty. \tag{7.4}$$

To prove this theorem, let $\epsilon > 0$ and choose M such that

$$|f(x)| < \epsilon/2, \qquad |x| \geqq M. \tag{7.5}$$

Because $f(x)$ is uniformly continuous over $-M \leqq x \leqq M$, we can construct a step function $\phi(x)$, vanishing outside the interval $-M \leqq x \leqq M$, such that

$$|f(x) - \phi(x)| < \epsilon/2, \qquad -\infty < x < \infty. \tag{7.6}$$

Since $\phi(x)$ is a linear combination of simple step functions, it follows from Theorem 7.1 that there is a linear combination $f_\epsilon(x)$ of translations of $F(x)$ such that

$$|\phi(x) - f_\epsilon(x)| < \epsilon/2, \qquad -\infty < x < \infty. \tag{7.7}$$

The required representation (7.2) follows.

There are, of course, bounded functions which cannot be uniformly approximated by step functions, that is, by finite linear combinations of simple step functions. For example, $\sin x$ and $\sin x^{-1}$ (the latter function being defined to be 0 when $x = 0$) cannot be. However, the conclusion of Theorem 7.3 holds for all functions $f(x)$ so approximable.

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ON THE BLOCH-LANDAU CONSTANT FOR SCHLICHT FUNCTIONS

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The definition of the Bloch-Landau constant $\mathfrak{A}$ is based upon the following theorem due to Bloch:[1]

THEOREM 1. *There is an absolute positive constant P with the following property: Let $f(x)$ be regular for $|x| < 1$, $|f'(0)| = 1$. Then the map of $|x| < 1$ under $f(x)$ contains in a single sheet an open circle of radius P.*

In addition to the original Bloch constant $\mathfrak{B}$, the least upper bound of the P satisfying the above theorem, Landau[2] has defined two other constants, $\mathfrak{L}$ and $\mathfrak{A}$, in connection with this theorem. $\mathfrak{L}$ is the least upper bound of the P of Theorem 1 if the requirement that there be a circle of radius P contained in a single sheet is replaced by the requirement that there be a circle of radius P each point of which is an interior point of some sheet of the map. $\mathfrak{A}$ is the least upper bound of the P of Theorem 1 if it is required that the function $f(x)$ be schlicht for $|x| < 1$. It is immediately apparent that $\mathfrak{B} \leqq \mathfrak{L} \leqq \mathfrak{A}$.

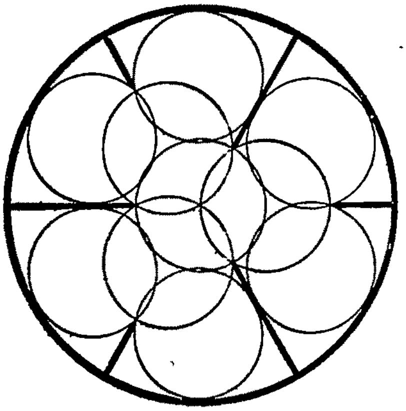

FIG. 1

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[1] Bloch, *Les théorèmes de M. Valiron sur les fonctions entières, et la théorie de l'uniformisation*, C. R. Acad. Sci. Paris vol. 178 (1924) pp. 2051–2052.

[2] E. Landau, *Über die Blochsche Konstante und zwei verwandte Weltkonstanten*, Math. Zeit. vol. 30 (1929) pp. 608–634. This paper includes a proof of Theorem 1.

The exact value of none of these three constants is known, although upper and lower bounds have been established for all three constants. Landau[2] proved that $\mathfrak{L}<\mathfrak{A}$; that $\mathfrak{B}<\mathfrak{L}$ was proved by Ahlfors.[3]

The best upper bound given so far for the constant $\mathfrak{A}$ is due to Robinson:[4] $\mathfrak{A}<0.658$. This result was obtained by mapping $|x|<1$ on to a circle with six radial slits, three each of two different lengths, symmetrically placed. (See Fig. 1.) The lengths of the slits are determined so as to make the ten circles shown in the region all maximal.

Robinson's procedure is capable of extension to an infinite sequence of mappings, which can be successfully exploited. The nth region of this sequence (see Fig. 2) is that portion of the slit plane enclosed by the circle $|y|=R_n$. The radial slits of the nth region fall along the lines through the origin and the $3\cdot 2^{n-1}$st roots of unity. The R_n are determined so that the maximal circles in every region shall be of radius unity.

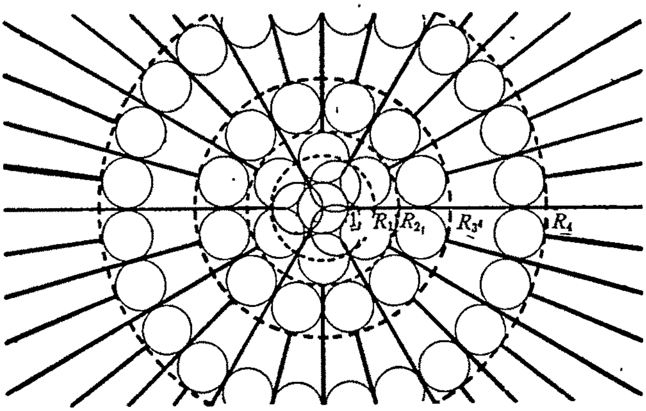

FIG. 2

A similar sequence of regions may be defined for every integer $s\geqq 3$, by beginning with s slits falling along the s lines through the origin and the sth roots of unity, and ending on the circle $|y|=1$. Call the regions of this sequence $C_{1,s}$, $C_{2,s}$, $C_{3,s}$, $\cdots$.

(Note that the region $C_{2,3}$, although it contains two sets of three slits each, is not exactly the region D of Fig. 1. The region $C_{2,3}$ contains only four maximal circles, since it is bounded by the circle $|y|=R_2=2.9557$, which passes through the ends of the third set of slits. The circle $|y|=2.9561$ which bounds D is tangent to the six outer maximal circles in D.)

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[3] L. V. Ahlfors, *An extension of Schwarz's lemma*, Trans. Amer. Math. Soc. vol. 43 (1938) pp. 359–364.

[4] R. M. Robinson, *The Bloch constant* $\mathfrak{A}$ *for a schlicht function*, Bull. Amer. Math. Soc. vol. 41 (1935) pp. 535–540.

Let $y=f_{n,s}(x), f_{n,s}(0)=0$, be the function which maps $|x|<1$ on to $C_{n,s}$ ($s\geqq 3$; $n=1, 2, 3, \cdots$).

It is clear that the upper bound $B_{n,s}$ for $\mathfrak{A}$ furnished by mapping $|x|<1$ on to $C_{n,s}$ is

$$\mathfrak{A} \leqq B_{n,s} = 1/|f'_{n,s}(0)|.$$

THEOREM 2. *Let $f(x)$ and $g(x)$ be regular for $|x|<1, f(0)=g(0)=0$. Let $f(x)$ map $|x|<1$ on to F and $g(x)$ map $|x|<1$ on to G, where $F\subsetneq G$. Then $|f'(0)|<|g'(0)|$.*

This result is readily proved by noticing that the function $g^{-1}(f(x))$ satisfies the conditions of Schwarz's lemma.[5]

Since $C_{1,s}\subset C_{2,s}\subset C_{3,s}\subset \cdots$, it is clear, by Theorem 2, that the sequence $\{B_{1,s}, B_{2,s}, B_{3,s}, \cdots\}$ is monotonically decreasing to a positive limit $B_{\infty,s}$.

THEOREM 3. *If $s>3$, then $B_{\infty,s}>B_{\infty,3}$.*

PROOF. Let $P_{1,s}$ be the entire plane with the s slits of the region $\sum_{j=1}^{\infty} C_{j,s}$ which end on the unit circle. Let $y=g(x)$, $g(0)=0$, map $|x|<1$ on to $P_{1,s}$. Now $P_{1,s}\supset C_{n,s}$ for fixed s and every n. By Theorem 2 it follows that

$$B_{\infty,s} \geqq 1/|g'(0)|.$$

Since the function

$$k(x) = x/(1-x)^2$$

transforms $|x|<1$ into the entire plane slit along the negative real axis from $-\infty$ to $-1/4$, therefore

$$g(x) = (-4k(x^s))^{1/s},$$

whence

$$|g'(0)| = 4^{1/s}.$$

Thus

$$1/|g'(0)| = 1/4^{1/s} > 0.707 \qquad (s \geqq 4).$$

By Theorem 2, $B_{\infty,3}<B_{2,3}$, and the value $B_{2,3}<0.658$ is found by computation. (For the method of computation, see reference 4.) Combination of the above inequalities gives Theorem 3.

Since the case $s=3$ is the only one hereafter of interest, we shall from now on drop the subscript "3" from the notations $f_{n,3}(x)$, $C_{n,3}$, $B_{n,3}$. Thus the mapping regions C_n are now exactly those of Fig. 2.

THEOREM 4. $\mathfrak{A}\leqq B_\infty<0.65647$.

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[5] See L. R. Ford, *Automorphic functions*, p. 166.

PROOF. The function of the sequence $\{f_n(x)\}$ which maps $|x|<1$ on to C_4 is

$$f_4(x) = R_4K^{1/12}(p_4k(K^2(p_3k(K^2(p_2k(-K(p_1k(x^3)))))))),$$

where

$$k(x) = x/(1-x)^2, \qquad K(x) = \text{the inverse of } k(x).$$

The constants p_j appearing in $f_4(x)$ are determined, in terms of the radii R_j, so as to adjust properly the lengths of the slits in C_4:

$$p_4 = -4k(-(R_3/R_4)^{12}),$$

$$p_3 = 4M\left(\frac{1}{p_4}k((R_2/R_4)^{12})\right),$$

$$p_2 = 4M\left(\frac{1}{p_3}N\left(\frac{1}{p_4}k((R_1/R_4)^{12})\right)\right),$$

$$p_1 = E/(E+p_2), \qquad E = 4N\left(\frac{1}{p_3}N\left(\frac{1}{p_4}k((1/R_4)^{12})\right)\right),$$

where

$$M(b) = (4b^2+b)^{1/2} - 2b, \qquad N(b) = (4b^2+b)^{1/2} + 2b,$$

and where

$$R_0 = 1, \quad R_1 \doteq 2, \quad R_2 = 2.9557, \quad R_n = 1 + \csc(\pi/3\cdot 2^{n-1}) \text{ for } n \geqq 3.$$

Since

$$1/B_4 = |f_4'(0)| = R_4p_4^{1/12}p_3^{1/6}(p_2p_1)^{1/3},$$

we find

$$\mathfrak{A} \leqq B_\infty < B_4 < 0.65647.$$

It is possible to show that the upper bound of B_∞ given in Theorem 4 is accurate to four significant figures, by considering the functions $g_n(x)$, $g_n(0)=0$, which map $|x|<1$ on to the regions P_n, defined as follows: Let P_n be the entire plane with all those slits of the region $C_\infty = \sum_{j=1}^\infty C_j$ which end on the circles $|y| = R_k$ $(k=0, 1, 2, \cdots, n-1;$ $R_0=1)$. Let $D_n = 1/|g_n'(0)|$. Then by Theorem 2 the sequence $\{D_n\}$ increases monotonically to a limit $D_\infty \leqq B_\infty$.

By observing[6] that the sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ are both normal families on the unit circle, we can conclude that they converge regularly on $|x|<1$ to a common limit function $\phi(x)$. Indeed, $\phi(x)$ maps $|x|<1$ on to C_∞, the common kernel of the sequences of regions $\{C_n\}$ and $\{P_n\}$. Thus

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[6] See Carathéodory, *Conformal representation*, chap. 5.

$$D_\infty = 1/|\phi'(0)| = B_\infty.$$

It is interesting, however, to note that we can prove this last statement directly, without resort to the more powerful theorems on the behaviour of normal families:

THEOREM 5. $D_\infty = B_\infty$.

PROOF. If Q_n is the region into which C_n is transformed by the function

$$u_n(x) = 4^{1/\mu}R_n(-k(-(x/R_n)^\mu))^{1/\mu} \qquad (\mu = 3\cdot 2^{n-1}),$$

then $Q_n \supset P_n$. For Q_n is easily seen to be the whole plane with μ slits along the same lines as the μ slits of P_n. Let y_0 be the end point of one of the slits in Q_n. The end point of the slit in P_n along this same line is some point $R_h e^{2m\pi i/\mu}$ $(0 \leqq h < n;\ 0 \leqq m < \mu)$. Now

$$|y_0| = |u_n(R_h e^{2m\pi i/\mu})| = 4^{1/\mu} R_h R_n^2/(R_n^\mu + R_h^\mu)^{2/\mu} > R_h,$$

whence indeed $Q_n \supset P_n$.

Now the function $w_n(x) = u_n(f_n(x))$ transforms $|x| < 1$ into Q_n. Furthermore,

$$|w_n'(0)| = |u_n'(0)| \cdot |f_n'(0)|.$$

Now $|u_n'(0)| = 4^{1/\mu}$. Thus, by applying Theorem 2 to the relation $Q_n \supset P_n \supset C_n$, we have

$$|w_n'(0)| = 4^{1/\mu}|f_n'(0)| > |g_n'(0)| > |f_n'(0)| \qquad (\mu = 3\cdot 2^{n-1}).$$

But $4^{1/\mu} \to 1$ as $n \to \infty$, whence Theorem 5 is true.

THEOREM 6. $\qquad 0.65646 < B_\infty < 0.65647.$

PROOF. The function of the sequence $\{g_n(x)\}$ which maps $|x| < 1$ on to P_3 is

$$g_3(x) = (q_3 k(K^2(q_2 k(-K(-q_1 k(x^3))))))^{1/6},$$

where

$$q_3 = 4R_2^6, \qquad q_2 = 4M(R_1^6/q_3), \qquad q_1 = 4N(1/q_3)/(4N(1/q_3) + q_2),$$

where $M(b)$, $N(b)$, R_1, R_2 are as before. Since

$$1/D_3 = |g_3'(0)| = q_3^{1/6}(q_2 q_1)^{1/3},$$

we have the result stated in the theorem:

$$0.65646 < D_3 < D_\infty = B_\infty.$$

Robinson has proved[7] that if $f(x)$, $f'(0)=1$, $f(0)=0$, maps $|x|<1$ on to R, where the largest open circle in R has exactly the radius $\mathfrak{A}$, then R has the following property: If the analytic arc D is part of the boundary of R, then there are points of R on each side of D in the neighborhood of any point of D. It is interesting to note that our best upper bound B_∞ for $\mathfrak{A}$ is furnished by a region with this property.

Although the numerical improvement over Robinson's result is not very marked, the mapping of the unit circle $|x|<1$ on the open slit plane C_∞ has some features of interest. Each ray through the origin which does not fall on a slit (that is, a ray whose angle with the positive real axis is different from $2\pi m/3\cdot 2^n$) determines a "prime end."[8] Such a ray is mapped on to a curve which has a well defined end point on the unit circle. Through the introduction of prime ends the mapping of the interior of domains can be extended to the boundary points.[9] The points on the boundary of the unit circle are mapped either into the slits in C_∞ or into prime ends. The set S of all points on the unit circle which go over into prime ends is a nowhere dense perfect Cantor set, while the points which are mapped on the slits form denumerably many intervals which lie everywhere dense.

Now the set S is of measure 0. Indeed, let $y=\phi(x)$ map $|x|<1$ on C_∞. Then $\phi(x)\to\infty$ when x approaches a point of S. Thus the function

$$\Phi(x) = x/\phi(x)$$

is regular and bounded in $|x|<1$, since on $|x|<1$, $|\Phi(x)|\leqq 1/a$, where a is the shortest distance from the origin to a point on a slit. Now by a theorem of Fatou, a regular and bounded function approaches a limit almost everywhere on the unit circle. We can therefore extend $\Phi(x)$ to the boundary (as we have already done indirectly for $\phi(x)$ by the introduction of prime ends). The extended $\Phi(x)$ is 0 on S. By a theorem of F. and M. Riesz,[10] we know that a bounded analytic function, which is not identically 0, can attain the value 0 on the unit circle in at most a set of measure 0. Thus $mS=0$.

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[7] R. M. Robinson, *Bloch functions*, Duke Math. J. vol. 2 (1936) pp. 458–459.

[8] See C. Carathéodory, *Über die Begrenzung einfach zusammenhangender Gebiete*, Math. Ann. vol. 73 (1912) pp. 323–370.

[9] See Hurwitz-Courant, *Funktionentheorie*, pp. 400–405.

[10] F. and M. Riesz, *Über die Randwerte einer analytischen Funktion*, Compte Rendu du Quatrième Congrès des mathématiciens scandinaves, Stockholm, 1916, pp. 27–44. See also G. Szegö, *Über die Randwerte einer analytischen Funktion*, Math. Ann. vol. 84 (1921) pp. 232–244.

CONCERNING THE DEFINITION OF HARMONIC FUNCTIONS

E. F. BECKENBACH

1. **Introduction.** A real function $u(x, y)$, defined in a domain (non-null connected open set) D, is said to be harmonic in D provided $u(x, y)$ and its partial derivatives of the first and second orders are continuous and the Laplace equation,

$$\Delta u \equiv \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0, \tag{1}$$

is satisfied throughout D. A function is said to be harmonic at a point provided it is harmonic in a domain containing the point.

It has been shown [1][1] that if $u(x, y)$ is continuous in D and if the second order partial derivatives $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ exist and satisfy the Laplace equation (1) throughout D, then $u(x, y)$ is harmonic in D.

We shall show that if $u(x, y)$ and its partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$ are continuous in D, if $\partial u/\partial x$ and $\partial u/\partial y$ are differentiable, or even have finite Dini derivates, with respect to x and y at all points of D except at most at the points of a denumerable set of points in D, and if the Laplace equation (1) is satisfied at almost all points of D at which $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ exist, then $u(x, y)$ is harmonic in D.

Our result is comparable with the Looman-Menchoff theorem [3, pp. 9–16; 5, pp. 198–201] concerning the Cauchy-Riemann first order partial differential equations and analytic functions of a complex variable. Ridder [4] has stated that harmonic functions can be given a Looman-Menchoff characterization; but a generalization of the Looman-Menchoff theorem on which his proof is based is invalid, for there are functions having isolated singularities which satisfy the hypotheses of the generalization without satisfying the conclusion. For a generalization of the Looman-Menchoff theorem, see Maker [2].

2. **Notation and lemmas.** By $C(Q)$ we shall denote a square, by $C(R)$ a rectangle, having sides parallel to the coordinate axes. The set consisting of the points of $C(Q)$, or of $C(R)$, plus its interior, will be denoted by Q, or R, respectively.

Let F be a non-null set closed with respect to the domain D, and $C(Q)$ any square with Q lying in D, with sides of positive length and parallel to the coordinate axes, and with center at a point of F. Then the points common to F and Q will be called a *portion* of F.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

We shall use as lemmas the following known results.

LEMMA 1. *If $u(x, y)$ is continuous at (x_0, y_0) and harmonic in a deleted neighborhood of (x_0, y_0), then $u(x, y)$ is harmonic at (x_0, y_0).*

PROOF. This follows from the fact that the function can be expanded in a two-way power series in a deleted neighborhood of (x_0, y_0).

LEMMA 2. *If $u(x, y)$ is harmonic in the finite domain D, then for each rectangle $C(R)$ such that R lies in D we have*

$$\int_{C(R)} \frac{du}{d\nu} ds = 0, \tag{2}$$

where $d/d\nu$ denotes differentiation in the direction of the outward normal.

PROOF. The result follows directly from Green's theorem,

$$\int_{C(R)} \frac{du}{d\nu} ds = \int\int_R \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dx dy.$$

LEMMA 3. *If $u(x, y)$ and its partial derivatives of the first order are continuous in D and if for every square $C(Q)$ having sides parallel to the coordinate axes and such that Q lies in D we have*

$$\int_{C(Q)} \frac{du}{d\nu} ds = 0, \tag{3}$$

then $u(x, y)$ is harmonic in D.

PROOF. The mean-value function,

$$u^{(r)}(x, y) \equiv \frac{1}{4r^2} \int_{-r}^{r} \int_{-r}^{r} u(x + \xi, y + \eta) d\xi d\eta,$$

has continuous partial derivatives of the second order in the part $D^{(r)}$ of D in which $u^{(r)}(x, y)$ is defined. We have

$$\frac{\partial^2 u^{(r)}(x, y)}{\partial x^2} = \frac{1}{4r^2} \int_{-r}^{r} \left[\frac{\partial u(x + r, y + \eta)}{\partial x} - \frac{\partial u(x - r, y + \eta)}{\partial x}\right] d\eta$$

and

$$\frac{\partial^2 u^{(r)}(x, y)}{\partial y^2} = \frac{1}{4r^2} \int_{-r}^{r} \left[\frac{\partial u(x + \xi, y + r)}{\partial y} - \frac{\partial u(x + \xi, y - r)}{\partial y}\right] d\xi,$$

whence

$$\Delta u^{(r)}(x, y) = \frac{1}{4r^2} \int_{C(Q_r)} \frac{du}{dv}\, ds, \tag{4}$$

where $C(Q_r)$ is the square with sides of length $2r$ and parallel to the coordinate axes, and with center at (x, y).

From (3) and (4) it follows that $u^{(r)}(x, y)$ is harmonic in $D^{(r)}$. Hence $u(x, y)$, the uniform limit of $u^{(r)}(x, y)$ on any closed and bounded subset of D as $r \to 0$, is harmonic in D.

LEMMA 4 (BAIRE'S THEOREM). *Let F be a non-null plane set lying in a domain D and closed with respect to D, and let $\{F_n\}$, $n = 1, 2, \cdots$, be a sequence of sets lying in D and closed with respect to D such that $\{F_n\}$ covers F; that is, each point of F is a point of at least one F_n. Then there is a member F_N of $\{F_n\}$ which contains a portion of F.*

PROOF. If the result were not valid, there would be a descending sequence $\{P_n\}$, $P_1 \supset P_2 \supset \cdots$, of portions of F such that P_n and F_n have no point in common, $n = 1, 2, \cdots$. Then $\prod_{n=1}^{\infty} P_n$ and $\sum_{n=1}^{\infty} F_n$ would have no point in common. But this is impossible since $\prod_{n=1}^{\infty} P_n$ contains a point of F and $\sum_{n=1}^{\infty} F_n$ covers F.

LEMMA 5. *Let $C(Q)$ be a square having sides parallel to the coordinate axes, let F be a closed non-null set in Q, let $C(R)$ be the smallest rectangle (which may be degenerate) having sides parallel to the coordinate axes and satisfying the condition that F is contained in R; and let the vertices of $C(R)$ have coordinates*

$$(x_1, y_1),\ (x_2, y_1),\ (x_2, y_2),\ (x_1, y_2), \qquad x_1 \leqq x_2,\ y_1 \leqq y_2.$$

If the real function $w(x, y)$ is defined on the set Q, if the first order partial derivatives of $w(x, y)$ exist, or even if the Dini derivates are finite, at every point of Q except at most at the points of a denumerable set of points in Q, and if for the finite constant N we have

$$| w(x_0 + h, y_0) - w(x_0, y_0) | \leqq N | h |,$$
$$| w(x_0, y_0 + k) - w(x_0, y_0) | \leqq N | k |,$$

for all (x_0, y_0) in F and all (x_0+h, y_0), (x_0, y_0+k) in Q, then

$$\left| \int_{x_1}^{x_2} [w(x, y_2) - w(x, y_1)]dx - \iint_F \frac{\partial w}{\partial y}\, dxdy \right| \leqq 5N \operatorname{meas} (Q - F),$$

$$\left| \int_{y_1}^{y_2} [w(x_2, y) - w(x_1, y)]dy - \iint_F \frac{\partial w}{\partial x}\, dxdy \right| \leqq 5N \operatorname{meas} (Q - F).$$

PROOF. A proof of Lemma 5 may be found in Saks [**5**, pp. 198–199] or Menchoff [**3**, pp. 10–12].

We note, relative to Lemma 5, that since $w(x, y)$ is differentiable, or has finite Dini derivates, with respect to x and y at all points of Q except at most at the points of a denumerable set of points in Q, it follows [5, pp. 236, 272] that $\partial w/\partial x$ and $\partial w/\partial y$ exist almost everywhere in Q and are integrable.

3. **Theorem.** We shall establish the following result.

THEOREM. *If the real function $u(x, y)$ and its first order partial derivatives with respect to x and y are continuous in the finite domain D, and if $\partial u/\partial x$ and $\partial u/\partial y$ are differentiable, or even have finite Dini derivatives, with respect to x and y at all points of D except at most at the points of a denumerable set of points in D, and if the Laplace equation,*

$$\Delta u \equiv \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0,$$

is satisfied at almost all points at which $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ exist in D, then $u(x, y)$ is harmonic in D.

PROOF. Suppose that there is a point of D at which $u(x, y)$ is not harmonic; we shall obtain a contradiction.

Denote the set of points of D at which $u(x, y)$ is not harmonic by F. Since by definition the set of points at which $u(x, y)$ is harmonic is open, it follows that F is closed with respect to D. Further, by Lemma 1, F has no isolated points. Hence F is perfect with respect to D.

For each positive integer n, let F_n be the set of points (x, y) of F for which

$$\left| \frac{\partial u(x+h, y)}{\partial x} - \frac{\partial u(x, y)}{\partial x} \right| \leqq n |h|,$$

$$\left| \frac{\partial u(x, y+h)}{\partial x} - \frac{\partial u(x, y)}{\partial x} \right| \leqq n |h|,$$

$$\left| \frac{\partial u(x+h, y)}{\partial y} - \frac{\partial u(x, y)}{\partial y} \right| \leqq n |h|,$$

$$\left| \frac{\partial u(x, y+h)}{\partial y} - \frac{\partial u(x, y)}{\partial y} \right| \leqq n |h|,$$

for all h satisfying $|h| \leqq 1/n$. Since $\partial u/\partial x$ and $\partial u/\partial y$ are continuous in D, it follows that the sets F_n are closed with respect to D; and since $\partial u/\partial x$ and $\partial u/\partial y$ are differentiable, or have finite Dini derivates, with respect to x and y at all points of D except at most at the points of a denumerable set of points in D, it follows that the sets $\{F_n\}$,

$n=1, 2, \cdots$, cover all of F except the points of a set H which is at most denumerable.

Then we have

$$F = \sum_{n=1}^{\infty} F_n + H. \tag{5}$$

It follows from (5) and Lemma 4 that thee is a portion P of F either consisting of a single isolated point of H, or contained in an F_N of $\{F_n\}$. But since F is perfect with respect to D, the former alternative is impossible, so that the latter alternative holds.

The above portion P of F is contained in F_N, and is the common part of F and a set Q_0 in D, where $C(Q_0)$ is a square in D with center at a point of F and with sides parallel to the coordinate axes.

Let $C(Q)$ be any square lying in Q_0 and having its sides parallel to the coordinate axes, and let $F \cdot Q$ be the common part of F and Q. Let the sides of $C(Q)$ be divided into n equal parts, with n so large that the length of each part is less than or equal to $1/N$. Lines through the points of division parallel to the coordinate axes divide Q into n^2 squares. Let $Q_{p,n}$, $p=1, 2, \cdots, l$; $l \leqq n^2$, denote those of the n^2 squares having points in common with F.

For each $Q_{p,n}$ let $C(R_{p,n})$ be the smallest rectangle (which may be degenerate) having sides parallel to the coordinate axes and such that $R_{p,n}$ contains $F \cdot Q_{p,n}$. Let the vertices of $C(R_{p,n})$ have coordinates

$$(x_{1,p,n}, y_{1,p,n}), \quad (x_{2,p,n}, y_{1,p,n}), \quad (x_{2,p,n}, y_{2,p,n}), \quad (x_{1,p,n}, y_{2,p,n}),$$
$$x_{1,p,n} \leqq x_{2,p,n}, \; y_{1,p,n} \leqq y_{2,p,n}.$$

By Lemma 2 and the uniform continuity of $u(x, y)$ in Q, we have

$$\int_{C(Q)} \frac{du}{dv} ds = \sum_{p=1}^{l} \int_{C(R_{p,n})} \frac{du}{dv} ds. \tag{6}$$

Since

$$\int_{C(R_{p,n})} \frac{du}{dv} ds = \int_{x_{1,p,n}}^{x_{2,p,n}} \left[\frac{\partial u(x, y_{2,p,n})}{\partial y} - \frac{\partial u(x, y_{1,p,n})}{\partial y}\right] dx$$
$$+ \int_{y_{1,p,n}}^{y_{2,p,n}} \left[\frac{\partial u(x_{2,p,n}, y)}{\partial x} - \frac{\partial u(x_{1,p,n}, y)}{\partial x}\right] dy,$$

by Lemma 5 we have

$$\left| \int_{C(R_{p,n})} \frac{du}{dv} ds - \iint_{F \cdot Q_{p,n}} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dxdy \right| \tag{7}$$
$$\leqq 10N \operatorname{meas} (Q_{p,n} - F \cdot Q_{p,n}).$$

From (6) and (7) we obtain

$$\left|\int_{C(Q)} \frac{du}{d\nu}\,ds - \iint_{F\cdot Q}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)dxdy\right| \leqq 10N \operatorname{meas}\left(\sum_{p=1}^{l} Q_{p,n} - F\cdot Q\right). \tag{8}$$

Since the lengths of the sides of the squares $C(Q_{p,n})\rightarrow 0$ as $n\rightarrow\infty$, it follows that

$$\lim_{n\rightarrow\infty} \operatorname{meas}\left(\sum_{p=1}^{l} Q_{p,n} - F\cdot Q\right) = 0. \tag{9}$$

By hypothesis and the note at the end of §2, (1) is satisfied almost everywhere in D. Consequently from (8) and (9) we obtain

$$\int_{C(Q)} \frac{du}{d\nu}\,ds = 0.$$

Hence by Lemma 3, $u(x, y)$ is harmonic in Q_0. But the center of Q_0 is a point of F, so that $u(x, y)$ is not harmonic in Q_0. Thus the supposition that there is a point of D at which $u(x, y)$ is not harmonic has led to a contradiction.

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THE UNIVERSITY OF TEXAS

THE CONVERGENCE AND VALUES OF PERIODIC CONTINUED FRACTIONS

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1. Introduction. Much of the recent progress in the study of continued fractions is due to the consideration of the continued fraction as a product of linear fractional transformations, as opposed to the older approach in which attention is focused on the numerators and denominators, A_n and B_n, of the approximants.

The use of the transformation point of view, besides simplifying the proofs of theorems, often sheds additional light on the significance of the results. This advantage is particularly noticeable in the case of the periodic continued fraction, as the following theorem and its proof will show. The essential portions of the theorem were given by Stolz [1].[1] An improved but lengthy proof was later given by Pringsheim [2]. This was followed by Perron's shorter proof [3], in which ideas related to the transformation point of view were virtually suppressed. The proof given here makes full use of linear fractional transformations and is even shorter than Perron's proof.

THEOREM. *Consider the k-term periodic continued fraction*

$$\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_k}{b_k}+\frac{a_1}{b_1}+\cdots, \tag{1.1}$$

for which $a_1a_2\cdots a_k\neq 0$. *Let* F_ν *denote the* ν*th approximant, so that* $F_\nu=A_\nu/B_\nu$. *Let* S *denote the linear fractional transformation*

$$z'=\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots+\frac{a_k}{b_k}+\frac{z}{1}, \tag{1.2}$$

and let x_1 *and* x_2 *be the fixed points of* S. *The continued fraction converges if and only if* x_1 *and* x_2 *are finite numbers satisfying one of the following two conditions:*

$$x_1=x_2 \tag{1.3}$$

or

$$F_r\neq x_2,\ r=0,1,2,\cdots,k-1, \text{ and } |F_{k-1}-x_1|<|F_{k-1}-x_2|. \tag{1.4}$$

If the continued fraction converges, its value is x_1. *Furthermore,* $F_{nk+r}=x_1$ *for some* r, $0\leqq r\leqq k-1$, *and for* $n=1, 2, 3, \cdots$, *if and only if* $F_r=x_1$.

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

2. Two lemmas. The theorem will be proved by means of two lemmas on linear fractional transformations.

Let T denote the nonsingular linear fractional transformation

$$z' = \frac{az - b}{cz - d}, \qquad bc - ad \neq 0. \tag{2.1}$$

Let x_1 and x_2 be the fixed points of T. A necessary and sufficient condition that both x_1 and x_2 be finite is that $c \neq 0$.

Suppose that $c \neq 0$, and let $f = T(\infty)$. Obviously f is not a fixed point of T; hence f is finite and distinct from x_1 and x_2. Without loss in generality we can require that $|f - x_1| \leqq |f - x_2|$. Furthermore, if x is a fixed point of T, then

$$z' - x = \frac{az - b}{cz - d} - \frac{ax - b}{cx - d} = \frac{bc - ad}{cx - d} \cdot \frac{z - x}{cz - d} \cdot$$

But if x is a fixed point of T, it is also a fixed point of T^{-1}, so that

$$cx - d = c\left(\frac{dx - b}{cx - a}\right) - d = -\frac{bc - ad}{cx - a},$$

or

$$a - cx = \frac{bc - ad}{cx - d} \cdot$$

Hence

$$z' - x = (a - cx)\frac{z - x}{cz - d} \cdot \tag{2.2}$$

This equation will be used in the proof of each of the two lemmas.

Lemma 2.1. *If $x_1 = x_2$, where x_1 is finite, then T^n can be written*

$$\frac{1}{z' - x_1} = \frac{1}{z - x_1} + \frac{n}{f - x_1} \cdot$$

Proof. From (2.2) we have

$$\frac{1}{z' - x_1} = \frac{1}{a - cx_1} \cdot \frac{cz - d}{z - x_1} = \frac{1}{a - cx_1}\left(\frac{cx_1 - d}{z - x_1} + c\right).$$

But if T has only one fixed point, x_1, then $x_1 = (a + d)/2c$, so that $2cx_1 = a + d$, or $cx_1 - d = a - cx_1$. Hence

$$\frac{1}{z' - x_1} = \frac{1}{z - x_1} + \frac{c}{a - cx_1} \cdot$$

But $f=T(\infty)=a/c$. Hence T can be written

$$\frac{1}{z'-x_1}=\frac{1}{z-x_1}+\frac{1}{f-x_1}.$$

Upon repeated application of this formula, Lemma 2.1 follows at once.

LEMMA 2.2. *If x_1 and x_2 are finite and distinct, then T^n can be written*

$$\frac{z'-x_1}{z'-x_2}=\left(\frac{f-x_1}{f-x_2}\right)^n\frac{z-x_1}{z-x_2}.$$

PROOF. From (2.2) we obtain by division the equation

$$\frac{z'-x_1}{z'-x_2}=\frac{a-cx_1}{a-cx_2}\cdot\frac{z-x_1}{z-x_2}.$$

But $f=T(\infty)=a/c$. Hence T can be written

$$\frac{z'-x_1}{z'-x_2}=\frac{f-x_1}{f-x_2}\cdot\frac{z-x_1}{z-x_2}.$$

Upon repeated application of this formula, Lemma 2.2 follows at once.

3. **Proof of the theorem.** We observe that the right-hand member of (1.2) can be regarded as a terminating continued fraction for which $a_{k+1}=z$ and $b_{k+1}=1$, so that S can be written

$$(3.1)\qquad z'=\frac{A_k+zA_{k-1}}{B_k+zB_{k-1}}.$$

This observation immediately enables us to obtain the following conclusions:

(i) S is nonsingular; for the value of its determinant is $A_kB_{k-1}-A_{k-1}B_k=(-1)^{k-1}a_1a_2\cdots a_k$, which is nonvanishing by hypothesis.

(ii) $F_{k-1}=S(\infty)$; consequently if x_1 and x_2 are finite, it follows that F_{k-1}, the transform of the point at infinity, is finite and not a fixed point of S.

(iii) $F_{nk+r}=S^n(F_r)$, where (as in the remainder of this proof) $n=1, 2, \cdots$, and r is limited to the k values $0, 1, 2, \cdots, k-1$.

(iv) From (iii) it follows that $F_{nk+r}=x_1$ if and only if $F_r=x_1$; consequently no approximant of (1.1) actually takes on the value x_1 unless one of the first k approximants does so.

The rest of the theorem will be proved by considering separately four cases which are classified by the nature of the fixed points of S.

Case I. *The point at infinity a fixed point of* S. From (3.1) we see that this case occurs only if $B_{k-1}=0$. But if $B_{k-1}=0$, then $A_{k-1}\neq 0$, since S is nonsingular. Hence $F_{k-1}=\infty$. Hence $F_{nk+k-1}=\infty$ for all n, so that (1.1) diverges. We remark in passing, however, that if $B_{k-1}=0$, $A_k\neq 0$, and $|A_{k-1}|>|B_k|$, then the reciprocal of (1.1) converges to the value zero.

Case II. $x_1=x_2$, *where* x_1 *is finite.* In this case, Lemma 2.1 applies, so that observation (iii) above enables us to write

$$\frac{1}{F_{nk+r}-x_1}=\frac{1}{F_r-x_1}+\frac{n}{F_{k-1}-x_1}.$$

Since F_{k-1} and x_1 are finite, the right-hand member of this equation can be made arbitrarily large in absolute value by taking n sufficiently large. But if the reciprocal of $|F_{nk+r}-x_1|$ is arbitrarily large, then $|F_{nk+r}-x_1|$ itself is arbitrarily small. Hence $\lim_{n\to\infty} F_{nk+r}=x_1$. It has already been remarked that $F_{nk+r}=x_1$ if and only if $F_r=x_1$.

Case III. x_1 *and* x_2 *distinct finite numbers,* $|F_{k-1}-x_1|<|F_{k-1}-x_2|$. In this case, Lemma 2.2 applies, so that observation (iii) enables us to write

$$\frac{F_{nk+r}-x_1}{F_{nk+r}-x_2}=\left(\frac{F_{k-1}-x_1}{F_{k-1}-x_2}\right)^n\frac{F_r-x_1}{F_r-x_2}=K^n\frac{F_r-x_1}{F_r-x_2}, \tag{3.2}$$

where $|K|<1$ by hypothesis, so that $|K^n|$ can be made arbitrarily small by taking n sufficiently large. Hence if $F_r\neq x_2$, the entire right-hand member of (3.2) can be made arbitrarily small in absolute value. If it is made less than unity, then $|F_{nk+r}-x_1|=\epsilon_n|F_{nk+r}-x_2|$, or $|F_{nk+r}-(x_1-\epsilon_n^2x_2)/(1-\epsilon_n^2)|=\epsilon_n|x_1-x_2|/(1-\epsilon_n^2)$, where $0<\epsilon_n<1$, and $\lim_{n\to\infty}\epsilon_n=0$. It follows that $\lim_{n\to\infty}F_{nk+r}=x_1$. Hence if $F_r\neq x_2$ for each of the k values of r, then $\lim_{n\to\infty}F_{nk+r}=x_1$ for all r, so that (1.1) converges to the value x_1.

Suppose, however, that $F_r=x_2$ for one or more of the k values of r; then $F_{nk+r}=x_2$ for those values of r, since x_2 is a fixed point of S. Consequently, x_2 is a limit point of the sequence of approximants of (1.1). But F_{k-1} is not a fixed point of S, as has been pointed out in observation (ii). Consequently $F_{k-1}\neq x_2$, whence $\lim_{n\to\infty}F_{nk+k-1}=x_1$, so that x_1 is a limit point of the sequence of approximants of (1.1). Since $x_1\neq x_2$ by hypothesis, the sequence of approximants has two distinct limit points. Hence (1.1) diverges by oscillation.

Case IV. x_1 *and* x_2 *distinct and finite,* $|F_{k-1}-x_1|=|F_{k-1}-x_2|$. In this case (3.2) holds, where $K=e^{i\theta}$, $K\neq 1$. Since θ is not an integral multiple of 2π, it follows that the quantities K^n have at least two distinct values and do not approach a limit, finite or infinite.

But if the approximants F_{nk+k-1} approach a limit as n increases without limit, then the left-hand member of (3.2) must approach a limit, finite or infinite, as n increases, when $r=k-1$; and since the K^n do not approach a limit, this is possible only if $F_{k-1}=x_1$ or $F_{k-1}=x_2$, which is impossible since F_{k-1} is not a fixed point of S. Hence the sequence of approximants of (1.1) has at least two distinct limit points. The fraction (1.1) therefore diverges by oscillation.

This completes the proof of the theorem.

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EXTENSION OF A THEOREM OF BOCHNER ON EXPRESSING FUNCTIONALS AS RIEMANN INTEGRALS

BROCKWAY McMILLAN AND PACO LAGERSTROM

Introduction. S. Bochner[1] has shown that an additive homogeneous functional defined over a sufficiently large class C of functions can be realized as a Riemann integral with respect to a finitely additive measure V in the space X over which the functions are defined. His proof makes use of the fact that the constant function belongs to C, as a result, $V(X)$ is finite. It is the purpose of this note to show that a similar theorem holds even when $V(X)$ turns out to be infinite. A modification of Bochner's proof would suffice for this stronger theorem. We have chosen rather to treat it as a problem of extending the domain of definition of the given functional.

Throughout we have used the symbol → to be read as "implies." The equality $\equiv$ is used to denote an equality which holds by definition.

Notations. We consider a space X of points x, and real-valued point functions $f, g, \cdots$ over X. Given f, g, and real numbers a, b, we shall write

$$|f|,\ af+bg,\ fg,\ f\wedge g,\ f\vee g,\ f^+,\ f^-,$$

respectively, for those functions whose values for each x are given by

$$|f(x)|,\quad af(x)+bg(x),\quad f(x)g(x),\quad \inf\,[f(x),g(x)],$$
$$\sup\,[f(x),g(x)],\quad \sup\,[f(x),0],\quad \sup\,[-f(x),0].$$

We shall write a for the constant function $f(x)=a$, and write $f\geqq g$ if for each x, $f(x)\geqq g(x)$. The function which coincides with f on a set A and is equal to 0 in $X-A$ will be denoted by f_A. In particular we write 1_A for the characteristic function of the set A. The symbol $\emptyset$ will denote the empty set.

It is clear that $f=f^+-f^-$, and that

$$(f_A)^+=(f^+)_A,\qquad (f_A)^-=(f^-)_A.$$

1. R-measure.

1.1. By an R-measure in X we shall mean a set function $V(E)$ defined for sets E of a family A with the following properties:

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[1] S. Bochner, *Additive set functions on groups*, Ann. of Math. vol. 40 (1939) pp. 769–799. The theorem in question occurs in paragraph 4.

If E, E_1, $E_2 \in \mathbf{A}$, then

(1) $E_1 \cup E_2 \in \mathbf{A}$,
(2) $X - E \in \mathbf{A}$,
(3) $0 \leqq V(E) \leqq \infty$,
(4) $V(E) = 0$, and $B \subset E \rightarrow B \in \mathbf{A}$,
(5) $E_1 \cap E_2 = \emptyset \rightarrow V(E_1 \cup E_2) = V(E_1) + V(E_2)$.

Also

(6) there exists an $E \in \mathbf{A}$ with $0 < V(E) < \infty$.

1.2. *Remark.* (1), (2) imply $E_1 \cap E_2 \in \mathbf{A}$, $E_1 - E_2 \in \mathbf{A}$, $\emptyset \in \mathbf{A}$, $X \in \mathbf{A}$.

2. **The Riemann integral.** Let Δ be the class of all partitions δ of X into finitely many pairwise disjoint sets of $\mathbf{A}$. Given any $f \geqq 0$, bounded on $E \in \mathbf{A}$ with $V(E) < \infty$, we define

2.1
$$S_u(f, E, \delta) \equiv \sum_{D \in \delta} V(D \cap E)(\sup \{f_E(x) \mid x \in D\}),$$
$$S_l(f, E, \delta) \equiv \sum_{D \in \delta} V(D \cap E)(\inf \{f_E(x) \mid x \in D\}),$$

2.2
$$S_u(f, E) \equiv \inf \{S_u(f, E, \delta) \mid \delta \in \Delta\},$$
$$S_l(f, E) \equiv \sup \{S_l(f, E, \delta) \mid \delta \in \Delta\},$$

2.3
$$S_u(f) \equiv \sup \{S_u(f, E) \mid E \in \mathbf{A},\ V(E) < \infty\},$$
$$S_l(f) \equiv \sup \{S_l(f, E) \mid E \in \mathbf{A},\ V(E) < \infty\}.$$

We define the function classes

2.4
$$R_E \equiv \{f \mid S_u(f^+, E) = S_l(f^+, E) < \infty,\ S_u(f^-, E) = S_l(f^-, E) < \infty\},$$
$$R \equiv \{f \mid S_u(f^+), S_u(f^-) < \infty \quad \text{and} \quad (V(E) < \infty \rightarrow f \in R_E)\}.$$

Finally,

2.5
$$f \in R \rightarrow \int f \equiv S_u(f^+) - S_u(f^-).$$

It is easily shown that the supremum and infimum in 2.2 are in fact monotone limits over the directed set of partitions $\delta \in \Delta$, Δ being ordered by refinement. From this fact and from the definition it then follows that (when E, E_1, $E_2 \in \mathbf{A}$ and $V(E)$, $V(E_i) < \infty$)

2.6 $f \geqq 0$ and $E_1 \subset E_2 \rightarrow 0 \leqq S_u(f, E_1) \leqq S_u(f, E_2)$,

2.7 $f \geqq 0 \rightarrow S_u(f, E) = S_u(f_E)$, $S_l(f, E) = S_l(f_E)$,

2.8 $f \in R_E \rightarrow \left(f_E \in R \text{ and } \int f_E = S_l(f^+, E) - S_l(f^-, E)\right)$, 2.9 $f \in R$ and $f \geqq 0 \rightarrow \int f = \sup \left\{ \int f_E \mid E \in A, V(E) < \infty \right\}$,

2.91 $f \in R \rightleftarrows f^+, f^- \in R$,

2.92 $\int af + bg = a \int f + b \int g$.

3. Modules.

3.1. A class C of real-valued functions over X, together with a real-valued linear functional L defined over C, is called a module if it satisfies conditions 3.1 (1)–(11) below. (f, g denote elements of C; a, a real number.)

(1) Each f in C is bounded.
(2) $f+g \in C$.
(3) $af \in C$.
(4) $f \wedge 0 \in C$.
(5) $f \wedge 1 \in C$.
(6) $|L(f)| < \infty$.
(7) $L(f+g) = L(f) + L(g)$.
(8) $L(af) = aL(f)$.
(9) $f \geqq 0 \rightarrow L(f) \geqq 0$.
(10) There exists an $f \in C$ with $L(f) > 0$.
(11) $\operatorname{Inf}_{a>0} L(f \wedge a) \leqq 0$.

The main theorem of this paper is:

3.2. *If C is a module, there exists an R-measure $V(E)$ in X such that* (1) $C \subset R$, (2) $f \in C \rightarrow L(f) = \int f$, (3) *given* $e > 0$ *and* $g \in R$, *with* $g \geqq 0$, *there exists an* $f \in C$ *such that* $0 \leqq f \leqq g$ *and* $L(f) \leqq \int g < L(f) + e$.

Before constructing the R-measure we prove some elementary properties of a module C.

3.3 $f, g \in C \rightarrow f \vee g,\ f \wedge g \in C$. For example,
$f \vee g = g - (g - f) \wedge 0$.

3.4 $f \in C,\ a > 0 \rightarrow f \wedge a \in C$, for $f \wedge a = a \cdot (1/a) f \wedge 1$.

3.5 $f, g \in C, f \geqq g \rightarrow L(f) \geqq L(g)$, for $L(f) - L(g) = L(f - g) \geqq 0$.

3.6 $f, 1_A \in C \rightarrow f_A = f \cdot 1_A \in C$, for $0 \leqq f(x) \leqq b \rightarrow f_A = f \wedge b1_A$.

4. Completion of a module. In 4.1–4.5 below, f, h denote elements of a module C, while g may be any function.

4.1. $L_u(g) \equiv \inf \{L(h) \mid h \geqq g\}$ (if there exists an h, such that $h \geqq g$).

4.2. $L_l(g) \equiv \sup \{L(f) \mid f \leqq g\}$ (if there exists an f such that $f \leqq g$).

4.3. $C^* \equiv \{g \mid L_u(g) = L_l(g)\}$.

4.4. $L^*(g) \equiv L_u(g) = L_l(g)$ (for $g \in C^*$).

4.5. $C \subset C^*$ and $L^*(f) = L(f)$.

4.6. C^* is a module. We show (except for some obvious cases) that C^* has properties (1)–(11) of 3.1.

(3) and (8): Suppose $g \in C^*$ and, say, $a<0$. Then

$$\{f \mid f \leqq ag\} = \{ah \mid h \geqq g\}.$$

Hence

$$L_l(ag) = \sup \{L(ah) \mid h \geqq g\} = a \inf \{L(h) \mid h \geqq g\} = aL^*(g).$$

Similarly

$$L_u(ag) = aL^*(g).$$

(2) and (7): Suppose $g_1, g_2 \in C^*$. Then

$$\{f_1 + f_2 \mid f_i \leqq g_i\} \subset \{f \mid f \leqq g_1 + g_2\}.$$

Hence $L_l(g_1)+L_l(g_2) \leqq L_l(g_1+g_2)$ and, dually, $L_u(g_1+g_2) \leqq L_u(g_1) + L_u(g_2)$. (2) and (7) then follow from the fact that $L_l(g_1+g_2) \leqq L_u(g_1+g_2)$.

(4) and (5) follow from the inequality

$$h - f \geqq (h \wedge x) - (f \wedge x).$$

(11) follows from the fact that every $g \in C^*$ is covered by an $h \in C$, and that 3.5 does not depend on (11).

4.7. C^* is complete, in the sense that the process of extension described in 4.1–4.3 does not yield any new functions when applied to C^*.

Proof. It follows from 4.2 and 4.4 that

$$\sup \{L^*(f) \mid f \in C^*, f \leqq g\} = \sup \{L(f) \mid f \in C, f \leqq g\},$$

and similarly for the approximations from above.

4.8. Let C be any module. Given $f \in C$ and a number $a>0$, let 1_a be the characteristic function of the set $\{x \mid f(x) \geqq a\}$. For each $f \in C$ there exists an everywhere dense set S of real numbers $a>0$ such that $a \in S \rightarrow 1_a \in C^*$, where C^* is the completion of C. Since C^* is a module and is its own completion we have as a corollary the same theorem with the weaker assumption $f \in C^*$.

Proof. We shall prove the stronger result that there is at most a countable set $\{a_i\}$ of numbers $a_i>0$ such that 1_{a_i} is not in C^*. Given $e>0$, consider any $a \geqq e$ and numbers $b, c>0$ with $c \leqq e$. For any $d \geqq 0$ let $f^d = f \wedge d \in C$. Let $\phi(d) = L(f^d)$. We have

$$(1) \qquad c^{-1}\{f^a - f^{a-c}\} \geqq 1_a \geqq b^{-1}\{f^{a+b} - f^a\},$$

as may be seen by analyzing the three cases

$$f(x) \geqq a + b, \qquad a \leqq f(x) < a + b, \qquad f(x) < a.$$

Using 4.6 and (7), (8) of 3.1, we have from (1) that

$$(2) \quad c^{-1}[\phi(a) - \phi(a-c)] \geqq L_u(1_a) \geqq L_l(1_a) \geqq b^{-1}[\phi(a+b) - \phi(a)].$$

The outside inequalities imply that $\phi(a)$ is a convex function for $a \geqq e$. Taking limits in (2) as b, $c \to 0$, we have further that

$$D^-\phi(a) \geqq L_u(1_a) \geqq L_l(1_a) \geqq D^+\phi(a).$$

Since $\phi(a)$ is convex in the interval in question, $D^-\phi \neq D^+\phi$ at most at a countable number of points $\{a_i'\}$, $a_i' \geqq e$. Hence when $a \geqq e$ is not in $\{a_i'\}$, $D^-\phi = D^+\phi$ and $1_a \in C^*$ by 4.3. By taking successively $e = 1/n$, $n = 1, 2, \cdots$, we get at most a countable sum of countable sets—that is, at most a countable set $\{a_i\}$—in the interval $a > 0$ such that 1_{a_i} is not in C^*.

4.9. Let $\mathbf{A}^* = \{A \mid 1_A \in C^*\}$. Then

$$f \in C^* \to L^*(f) = \lim L^*(f_A),$$

where the limit is the limit taken on the directed system $\mathbf{A}^*$ ordered by $\supset$.

Proof. It is sufficient to prove 4.9 for $f \geqq 0$. By 4.8 there exists a sequence $a_n \downarrow 0$ such that the characteristic functions 1_n of the sets $\{x \mid f(x) \geqq a_n\}$ are all in C^*. Put $g_n = f - (f \wedge a_n)$. Then $g_n \in C^*$, $g_n \leqq f$, and $g_n \cdot 1_n = g_n$. Hence

$$L^*(g_n) \leqq L^*(f \cdot 1_n) \leqq L^*(f).$$

But $\inf L^*(f - g_n) = 0$ by 3.1, (11), 4.9 for $f \geqq 0$ now follows, since if $f \geqq 0$

$$L^*(f) \geqq \lim_A L^*(f_A) = \sup \{L^*(f_A) \mid A \in \mathbf{A}^*\}$$
$$\geqq \sup_n L^*(f \cdot 1_n) \geqq \sup_n L^*(g_n) = L^*(f).$$

5. Extension of L^* to "unbounded" functions.

5.1. $C^{**} \equiv \{f \mid A \in \mathbf{A}^* \to f_A \in C^*\}$.

5.2(a). $L^{**}(f) \equiv \lim_A L^*(f_A)$ (for $f \in C^{**}$, $f \geqq 0$). Here the limit is taken as in 4.9.

5.2. $L^{**}(f) \equiv L^{**}(f^+) - L^{**}(f^-) = \lim_A L^*(f_A^+) - \lim_A L^*(f_A^-) = \lim_A L^*(f_A)$ (for $f \in C^{**}$ and $L^{**}(f^+), L^{**}(f^-) < \infty$). Thus $|L^{**}(f)| < \infty$, except possibly if $f > 0$.

5.3. $C^* \subset C^{**}$ and $f \in C^* \rightarrow L^{**}(f) = L^*(f)$ (3.6, 4.9).

5.4. $f \geqq 0 \rightarrow L^{**}(f) \geqq 0$.

5.5. $f, g \in C^{**} \rightarrow af+bg \in C^{**}$ and (if $L^{**}(f), L^{**}(g) < \infty$) $L^{**}(af+bg) = aL^{**}(f)+bL^{**}(g)$, since $(af+bg)\cdot 1_A = a\cdot f1_A + b\cdot g1_A$, and L^{**} is defined as a limit on the directed set $\mathbf{A}^*$.

5.6. $f \leqq g \rightarrow L^{**}(f) \leqq L^{**}(g)$ (5.4, 5.5).

5.7. $0 \leqq f \leqq g$ and $g \in C^{**}$ and $L^{**}(g) = 0 \rightarrow f \in C^{**}$ and $L^{**}(f) = 0$. For $f_A \in C^*$ by 4.7. Hence by 5.4, 5.6

$$f \in C^{**}, \quad \text{and} \quad L^{**}(f) = 0$$

5.8. f, $1_E \in C^{**} \rightarrow f_E \in C^{**}$. For $1_A \in C^* \rightarrow (f_E)1_A = (f_A)(1_E 1_A) \in C^*$ by 3.6.

5.9. $f \in C^{**}$, $f \geqq 0 \rightarrow L^{**}(f) = \sup \{L^{**}(f_A) \mid 1_A \in C\}$ (5.6, 4.9, 5.3).

5.10. $1_X = 1 \in C^{**}$ (5.1).

Actually 5.8 is a special case of the following theorem, which however will not be needed for this paper:

5.11. $f, g \in C^{**} \rightarrow f\cdot g \in C^{**}$.

Proof. Assume $0 \leqq h$, $i \in C^*$ and $1_A \in C^*$. It follows from 4.8 that we may subdivide X into a finite number of sets E_ν such that $1_{E_\nu}\cdot 1_A \in C^*$ and that the oscillation of h and i on each set is less than e. Denoting by a_ν'', b_ν'', a_ν', b_ν' the maximum and minimum of h and i on E_ν we have

$$\sum a_\nu' \cdot b_\nu \cdot 1_{E_\nu} \cdot 1_A \leqq h \cdot i \cdot 1_A \leqq \sum a_\nu'' \cdot b_\nu'' \cdot 1_{E_\nu} \cdot 1_A.$$

Hence by the completeness of C^*: $h\cdot i\cdot 1_A \in C^*$. The theorem now follows since putting $f_A = h$ and $g_h = i$ we have that $f\cdot g\cdot 1_A = f_A \cdot g_A \in C^*$ for every $1_A \in C^*$.

6. The R-measure defined by L^{**}.

6.1(a). $\mathbf{A} \equiv \{E \mid 1_E \in C^{**}\}$.

6.1(b). $V(E) \equiv L^{**}(1_E)$ (for $E \in \mathbf{A}$).

6.1(c). $\mathbf{A}' \equiv \{E \mid E \in \mathbf{A}$ and $V(E) < \infty\}$.

From these definitions it follows:

6.2. V is an R-measure as defined in §1.

Proof. The properties (1)–(5) are obvious from §5. As for (6), we have from 3.1 (10) an $f \in C$ with $L(f) > 0$. We can assume $0 \leqq f(x) \leqq 1$. If $L^*(f1_A) = 0$ for all $1_A \in C^*$, then $L(f) = 0$ by 4.9, 4.5. Hence for one 1_A, $L^*(f_A) > 0$. But $1_A \geqq f_A$. Hence $L^*(1_A) = V(A) > 0$.

7. Comparison of $L^{**}(f)$ and $\int f(x)dV$.

7.1. $f \in C^*$, $f = f_E \geqq 0$, $E \in \mathbf{A}' \rightarrow f \in R_E$ and $\int f = L^*(f)$.

Proof. (a) If $V(E) = 0$, then $S_u(f_E) = S_l(f_E) = 0 = \int f$, since for some

$a\neq 0$, $0\leqq af^+\leqq 1_E$ and $0\leqq af^-\leqq 1_E$, we have $L^*(af)=0$ by 5.3, 5.7. Hence $L^*(f)=0=\int f$.

(b) If $0<V(E)<\infty$, given $e>0$, by 3.1 (1) and 4.8, there exists a partition δ of X into sets $A_0, \cdots, A_n\in\mathbf{A}$ such that

$$\sup\{f(x)-f(y)\mid x,y\in A_i\}<e[V(E)]^{-1},\qquad i=0,\cdots,n.$$

Let $E_i=EA_i$. Then $1_{E_i}=1_i\in C^{**}$ and hence $f_i=f1_i\in C^{**}$ by 5.8. Letting $b_i=\sup\{f(x)\mid x\in E_i\}$, we have

$$L^*(f)=L^{**}(f_E)=\sum_i L^{**}(f1_i)$$

and

$$S_u(f,E,\delta)=\sum_i V(E_i)b_i=\sum_i L^{**}(1_i\cdot b_i).$$

Hence $|L^*(f)-S_u(f,E,\delta)|<e$ and $L^*(f)=S_u(f,E)$. Similarly $L^*(f)=S(f,E)$.

7.2. If $f\geqq 0$, if $f\in R_A$ for $A\in A^*$, and if f is bounded on any $E\in\mathbf{A}$, then

(a) $f\in R_E$ for every $E\in\mathbf{A}'$,

(b) $\sup\{\int f_E\mid E\in\mathbf{A}'\}=\sup\{\int f_A\mid A\in\mathbf{A}^*\}$.

Proof. Since f is bounded on E and $L^{**}(1_E)<\infty$, $S_u(f,E)-S_u(f,A)<e/2$ for some $1_{A_1}\in C^*$, $1_{A_1}\leqq 1_E$. Dually $S_l(f,E)-S_l(f,A_2)<e/2$ ($1_{A_2}\in C^*$, $1_{A_2}\leqq 1_E$). The inequalities still hold if we replace A_1 and A_2 by $A=A_1\cup A_2$. Since $S_u(f,A)=S_l(f,A)$ we have $S_u(f,E)-S_l(f,E)<e$ for any e. Hence $f\in R_E$ and $\int f_E-\int f_A<e$ from which (b) follows.

7.3. $f\in C^*\rightarrow f\in R$ and $\int f=L^*(f)$.

Proof. Assume $f\geqq 0$. By 7.1, $f_A\in R_A$, that is, $f\in R_A$ for any $A\in\mathbf{A}^*$ and $\int f_A=L^*(f_A)$. Since f is bounded, $f\in R_E$ ($E\in\mathbf{A}'$) by 7.2(a). From 7.2(b) and 4.9 it follows that $\sup\{\int f_E\mid E\in\mathbf{A}'\}$ is equal to $L^*(f)$, hence finite, and equal to $\int f$ by 2.9. For any $f\in C^*$, 7.3 then follows by 2.91, 2.92.

7.4. $f\in R_A$ and $A\in\mathbf{A}^*\rightarrow f\in C^*$.

Proof. $f\in R_A$ means that f can be approximated from above and below by functions $\sum a_\nu 1_{A_\nu}$ where $A_\nu\in A^*$. Hence $f\in C^*$ by 4.7.

7.5. $f\in R\rightarrow f\in C^{**}$ and $L^{**}(f)=\int f$.

Proof. Assume $f\geqq 0$, then $f\in R\rightarrow f_A\in R_A$ for every $A\in\mathbf{A}^*$. By 7.4, $f_A\in C^*$ and hence $f\in C^{**}$.

Furthermore $\int f=\sup\{\int f_A\mid A\in\mathbf{A}^*\}=\sup\{L^*(f_A)\mid A\in\mathbf{A}'\}$ $=L^{**}(f)$ (2.9, 7.2, 7.1). For any $f\in R$, 7.5 follows from its truth for f^+, f^-.

The proof of our main theorem, 3.2, is now complete: That V is an R-measure was shown in §6, (1) and (2) follow from 7.3, (3) from 7.5 and the definition of L^{**}.

8. Some special cases.

A. 8.1. Assume that $f(x)=1\in C$ and $L(1)=1$. (This makes 3.1 (5), (10) and (11) redundant, (11) follows from the fact that $L(f\wedge a) \leqq L(a\cdot 1)=a$.) In this case $V(X)=1$ and 3.2 reduces to Bochner's theorem.

B. 8.2. DEFINITION. By an L-extension of V we shall mean a countably additive, complete measure U defined for sets of a countably additive, complemented family $\mathbf{B}$ such that $\mathbf{B}\supset\mathbf{A}$ and for $E\in\mathbf{A}$, $U(E)=V(E)$.

8.3. Replace 3.1 (11) by 3.1 (12): $\{f_n,\ g\in C,\ 0\leqq f_n\leqq g,\ \lim_n f_n(x)=0$ for all $x\}\rightarrow\lim_n L(f_n)=0$. Then (a) Theorem 3.2 still holds and in addition (b) V possesses an L-extension U such that (c) 3.2 (3) holds when "$g\in R$" is replaced by "g is measurable and integrable (U)."

PROOF. (a) 3.1 (12) implies 3.1 (11). For $f\in C$ put $f_n=\inf(f, 1/n)$. Then $\inf_{a>0} L(f\wedge a)\leqq\lim L(f_n^+)=0$.

(b) It is known[2] that any V with properties 1.1 (1)–(5) possesses an L-extension if and only if V has the additional property: $\{E_n\in\mathbf{A},\ E_n\geqq E_{n+1},\ V(E_1)<\infty,\ \bigcap_n E_n=\emptyset\}\rightarrow\lim_n V(E_n)=0$. That V has this property follows from

8.4. $\{f_n\in C^{**},\ f_n\geqq f_{n+1},$ for each x $\inf f_n(x)=0,\ L^{**}(f_1)<\infty\}$ $\rightarrow\lim L^{**}(f_n)=0$.

PROOF. Suppose that for all n, $L^{**}(f_n)\geqq e>0$. Since $L^{**}(f_n)<\infty$ there exists an $A\in\mathbf{A}^*$ such that $L^{**}(f_1-f_{1A})\leqq e/2$. Since $(1-1_A)f_n \leqq(1-1_A)f_1$ we also have $L^{**}(f_n-f_{nA})\leqq e/2$, and $L^{**}(f_{nA})\geqq e/2$ for all n. Hence we can find a $g_n\in C$ such that $0\leqq g_n\leqq f_{nA}$ and $L(g_n)\geqq e/3$. But evidently for each x, $\lim g_n(x)=0$. Since there exists a $g\in C$ such that $g\geqq f_1\cdot 1_A>g_n$, this contradicts 3.1 (12).

(c) To show that the analogue of 3.2 (3) holds for g measurable and integrable (U), we point out that (a) is sufficient to show that 3.2 (3) holds when g is the characteristic function of a set B measurable (U) with $U(B)<\infty$ and (b) if U is an L-extension of V then, given $e>0$, V contains an $E\in\mathbf{A}$ such that $V(E)>U(E)-e$. The result then follows from 3.2 (3).

(a) is a consequence of the ordinary Lebesgue theory while (b) results from the manner in which U is defined as an extension of V.[2]

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[2] This theorem is proved by Kolmogoroff (A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Berlin, 1933) for the case $V(X)=1$. When X is the sum of countably many sets of finite measure, the proof given by Jessen (B. Jessen, *Abstrakt maal- og integraltheorie*, 1, Matematisk Tidsskrift (B) (1934) p. 78) applies. The proof in the general case follows by a modification of that of Jessen.

A CLASS OF INTEGRAL EQUATIONS WHICH GENERALIZE ABEL'S EQUATION

EDMUND PINNEY

Introduction. A general class of integral equations is solved in §1. Specific examples are given in §3 which are shown in §5 to specialize to an equation from which Abel's may readily be derived. In this sense the results of this paper generalize Abel's integral equation solution. Another interesting special case is given in §6.

1. A general theorem.

Our development depends on this theorem:

THEOREM. (i) *Let the function* $g(x)$ *possess a continuous derivative for* $x > 0$, *and let* $\int^\infty g(t)t\,dt$ *converge. Let* $p(x)$, $q(x)$ *be functions for which*

$$(1)\qquad \int_0^\infty \int_0^\infty \left| g((x^2+s^2+t^2)^{1/2}) \right| \left| p(s) \right| \left| q(t) \right| ds dt \text{ exists,}$$

and for which the condition

$$(2)\qquad \int_0^{\pi/2} p(r\cos\theta)q(r\sin\theta)d\theta = 1$$

holds. Then

$$(3)\qquad \phi(x) = f(x) \equiv -\frac{1}{x}\frac{d}{dx}\int_0^\infty g((x^2+t^2)^{1/2})q(t)dt$$

is a solution of the integral equation

$$(4)\qquad g(x) = \int_0^\infty \phi((x^2+t^2)^{1/2})p(t)dt.$$

(ii) *If* $\phi(x)$ *is a solution of* (4) *having a continuous derivative for* $x > 0$, *such that* $\int^\infty \phi(t)t\,dt$ *converges and such that*

$$(5)\qquad \int_0^\infty \int_0^\infty \left| \phi((x^2+s^2+t^2)^{1/2}) \right| \left| p(s) \right| \left| q(t) \right| ds dt \text{ exists,}$$

then

$$\phi(x) = f(x).$$

PROOF. From (3),

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$$\begin{aligned} I &\equiv \int_0^\infty f((x^2+t^2)^{1/2})p(t)dt \\ &= -\frac{1}{x}\frac{d}{dx}\int_0^\infty p(t)dt\int_0^\infty g((x^2+t^2+s^2)^{1/2})q(s)ds. \end{aligned}$$

By (1) and Fubini's theorem this may be written

$$I = -\frac{1}{x}\frac{d}{dx}\int_0^\infty g((x^2+r^2)^{1/2})rdr\int_0^{\pi/2} p(r\cos\theta)q(r\sin\theta)d\theta.$$

From (2) and the conditions on $g(x)$,[1]

$$\begin{aligned} I &= -\frac{1}{x}\frac{d}{dx}\int_0^\infty g((x^2+r^2)^{1/2})rdr \\ &= -\int_0^\infty g'((x^2+r^2)^{1/2})\frac{rdr}{(x^2+r^2)^{1/2}} = -\left[g((x^2+r^2)^{1/2})\right]_0^\infty \\ &= g(x). \end{aligned}$$

Therefore $f(x)$ is a solution of (4).

If $\phi(x)$ is a solution of (4) as specified in (ii), then exactly as we showed that $I=g(x)$ we may show that $f(x)=\phi(x)$, completing the proof.

2. **The functions $p(x)$ and $q(x)$.** The application of this theorem requires the determination of pairs of functions $p(x)$, $q(x)$ satisfying (2). We shall now consider the set of such pairs for which $p(x)$, $q(x)$ are both continuous and for which

$$\int_0^\infty e^{-zt^2}|p(t)|dt \quad \text{and} \quad \int_0^\infty e^{-zt^2}|q(t)|dt \quad \text{converge, } z>0. \tag{6}$$

By (2), for $z>0$,

$$\int_0^\infty e^{-zr^2}\int_0^{\pi/2} p(r\cos\theta)q(r\sin\theta)d\theta rdr = \int_0^\infty e^{-zr^2}rdr.$$

By Fubini's theorem and (6) we may transform from polar to rectangular coordinates, obtaining

$$P(z)Q(z) = 1/2z, \qquad z>0, \tag{7}$$

where

$$P(z) = \int_0^\infty e^{-zt^2}p(t)dt, \qquad Q(z) = \int_0^\infty e^{-zt^2}q(t)dt. \tag{8}$$

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[1] Whittaker and Watson, *Modern analysis*, Cambridge University Press, §4.44.

By Lerch's theorem on null functions,[2] if $p(x)$ and $q(x)$ satisfy (7) and (8), they also satisfy (2).

We note that (7) is symmetrical in P and Q, so we conclude that p and q may be interchanged throughout the theorem of §1.

3. **Special values of $p(x)$ and $q(x)$.** We now concern ourselves with finding pairs of functions $p(t)$, $q(t)$ satisfying (7), (8) and which may be used in the theorem of §1. In this regard we note an extended Laguerre function definition,

$$L_\nu^\mu(z) = -\frac{\sin \pi\nu}{\pi}\Gamma(\mu+\nu+1)\sum_0^\infty {}_p \frac{\Gamma(p-\nu)}{\Gamma(p+\mu+1)}\frac{z^p}{p!}. \tag{9}$$

This reduces to the Laguerre polynomials when ν is a non-negative integer n. It is related to the Whittaker function[3] $M_{m,n}$ by the relation

$$L_\nu^\mu(z) = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} z^{-\mu/2-1/2} e^{z/2} M_{\mu/2+\nu+1/2,\mu/2}(z) \tag{10}$$

and to the Bessel function J_μ by

$$\lim_{s\to 0}\left(\frac{s}{\nu}\right)^\mu L_{\nu/s}^\mu(sz) = (\nu z)^{-\mu/2} J_\mu(2(\nu z)^{1/2}) \tag{11}$$

uniformly in νz for νz bounded.[4]

From (10) and §16.4 of Whittaker and Watson, for $|z| \gg 1$,

$$L_\nu^\mu(z) \sim -\frac{\sin \pi\nu}{\pi}\Gamma(\mu+\nu+1) z^{-\mu-\nu-1} e^z + \frac{e^{\pm\pi i\nu}}{\Gamma(\nu+1)} z^\nu,$$

where the upper or lower sign is taken according as $I(z) \lesseqgtr 0$.

In view of this we may expand the Laguerre functions and integrate term by term, getting, for $R(z) > R(k)$,

$$\begin{aligned} \int_0^\infty e^{-zt^2} t^{1-2\mu} L_\nu^{-\mu}(kt^2)\,dt &= \frac{\Gamma(1-\mu+\nu)}{2\Gamma(1+\nu)} z^{\mu-\nu-1}(z-k)^\nu, \quad R(\mu) < 1, \\ \int_0^\infty e^{-zt^2} t^{2\mu-1} L_{-\nu}^{\mu-1}(kt^2)\,dt &= \frac{\Gamma(\mu-\nu)}{2\Gamma(1-\nu)} z^{-\mu+\nu}(z-k)^{-\nu}, \quad R(\mu) > 0. \end{aligned} \tag{12}$$

Therefore, from (7), (8), we may take

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[2] Acta Math. vol. 27 (1903) pp. 339–352.

[3] Whittaker and Watson, *Modern analysis*, Cambridge University Press, chap. 16.

[4] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, New York, 1939, Theorem 8.1.3.

$$
\begin{aligned}
p(t) &= Ct^{1-2\mu}L_{\nu}^{-\mu}(kt^{2}),\\
q(t) &= \frac{2\nu}{C}\,\frac{\sin\pi(\mu-\nu)}{\sin\pi\nu}\,t^{2\mu-1}L_{-\nu}^{\mu-1}(kt^{2}),
\end{aligned}
\tag{13}
$$

where $0<R(\mu)<1$, and C, k are arbitrary except that $C\neq 0$.

(13) may be specialized in various ways. In particular from (11) and the fact that

$$
L_{\nu}^{\mu}(z) = \frac{\sin\pi\nu}{\sin\pi(\mu+\nu)}\,e^{z}L_{-\mu-\nu-1}^{\mu}(-z),
\tag{14}
$$

we may have

$$
\begin{aligned}
p(t) &= Ct^{1-\mu}J_{-\mu}(\alpha t),\\
q(t) &= (\alpha/C)t^{\mu}I_{\mu-1}(\alpha t),
\end{aligned}
\tag{15}
$$

for $0<R(\mu)<1$.

Letting $\nu\rightarrow 0$ in (13) we have, when $0<R(\mu)<1$,

$$
\begin{aligned}
p(t) &= t^{1-2\mu},\\
q(t) &= (2/\pi)\sin\pi\mu\;t^{2\mu-1}.
\end{aligned}
\tag{16}
$$

Abel's integral equation may be derived from that corresponding to this p and q.

When $\nu=n$, an integer not less than 0, the Laguerre function in $p(t)$ in (13) becomes a Laguerre polynomial. By (9) we may write for (13), for $0<R(\mu)<1$,

$$
\begin{aligned}
p(t) &= Ct^{1-2\mu}L_{\nu}^{-\mu}(kt^{2}),\\
q(t) &= -\frac{2}{C}\,\frac{\Gamma(\nu+1)}{\Gamma(\mu)\Gamma(\nu-\mu+1)}\,t^{2\mu-1}F(\nu;\mu;kt^{2}).
\end{aligned}
\tag{17}
$$

Therefore for $0<R(\mu)<1$,

$$
\begin{aligned}
p(t) &= Ct^{1-2\mu}L_{n}^{-\mu}(kt^{2}),\\
q(t) &= -\frac{2}{C}\,\frac{n!}{\Gamma(\mu)\Gamma(n-\mu+1)}\,t^{2\mu-1}F(n;\mu;kt^{2}).
\end{aligned}
\tag{18}
$$

A special case of interest is that in which $\mu=1/2$ in (15). Then we have

$$
\begin{aligned}
p(t) &= C\cos\alpha t,\\
q(t) &= (2/\pi C)\operatorname{ch}\alpha t.
\end{aligned}
\tag{19}
$$

4. **A counter example.** (1) is a severe restriction when $p(t)$, $q(t)$ have the general values given in (13), (15). Although the condition might be weakened somewhat it appears unlikely that it could be weakened very much. The following example, very kindly pointed out to me by Professor Harry Bateman, illustrates a case in which our theorem does not hold, despite the fact that $g(x)\to 0$ exponentially as $x\to\infty$.

Sonine[5] showed that for $a>c>0$, $0<n<p+1>3/2$,

$$\int_0^\infty J_{p-1}(c(x^2+t^2)^{1/2})(x^2+t^2)^{-p/2-1/2}J_{n-1}(at)t^n dt \\ = \frac{(c/2)^{p-1}}{\Gamma(p)}x^{n-1}K_{n-1}(ax).$$

Therefore when $0<n<1$ and

$$\phi(z) = J_{p-1}(cz)z^{-p-1},$$

and

$$g(z) = \frac{(c/2)^{p-1}}{\Gamma(p)}z^{n-1}K_{n-1}(az),$$

for the theorem to hold it is necessary that

$$\phi(x) = -\frac{a}{x}\frac{d}{dx}\left\{\frac{(c/2)^{p-1}}{\Gamma(p)}\int_0^\infty (x^2+t^2)^{n/2-1/2}K_{n-1}(a(x^2+t^2)^{1/2}) \right. \\ \left. \cdot I_{-n}(at)t^{1-n}dt\right\}.$$

This, however, is impossible since p appears only in the factor $(c/2)^{p-1}/\Gamma(p)$. Therefore the theorem fails. However it is easy to see that the integral in (1) does not exist.

5. **Abel's equation.** In this section we shall establish the relation between the integral equation corresponding to (16) and Abel's integral equation. The former and its solution are

$$\begin{aligned} g(x) &= \int_0^\infty \phi((x^2+t^2)^{1/2})t^{1-2\mu}dt, \\ \phi(x) &= -\frac{2}{\pi}\sin \pi\mu \frac{1}{x}\frac{d}{dx}\int_0^\infty g((x^2+t^2)^{1/2})t^{2\mu-1}dt, \end{aligned} \tag{20}$$

provided $0<R(\mu)<1$, $\int_0^\infty\int_0^\infty |g((x^2+t^2+s^2)^{1/2})| t^{1-2\mu}s^{2\mu-1}dsdt$ exists, and $\int^\infty g(t)tdt$ converges.

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[5] Math. Ann. vol. 16 (1880) pp. 50, 51.

Write

$$\Phi(x) = x^{\mu-2}\phi(x^{-1/2}),$$

$$G(x) = 2x^{-\mu}g(x^{-1/2}).$$

In (20) let

$$s = 1/(x^2 + t^2),$$

so

$$t^2 = x^2s^{-1}(x^{-2} - s), \qquad 2tdt = -s^{-2}ds.$$

Then

$$2g(x) = \int_0^{x^{-2}} \phi(s^{-1/2})s^{\mu-2}x^{-2\mu}\left(\frac{1}{x^2} - s\right)^{-\mu} ds$$

implies that

$$\phi(x) = -\frac{1}{\pi}\sin \pi\mu \frac{1}{x}\frac{d}{dx}\int_0^{x^{-}} g(s^{-1/2})s^{-\mu-1}x^{2\mu-2}\left(\frac{1}{x^2} - s\right)^{\mu-1}ds.$$

That is,

$$G(x) = \int_0^x \Phi(s)(x - s)^{-\mu}ds$$

implies that

$$\Phi(x) = \frac{1}{\pi}\sin \pi\mu\, x^{\mu}\frac{d}{dx}\left\{x^{1-\mu}\int_0^x G(s)(x - s)^{\mu-1}\frac{ds}{s}\right\}.$$

This is valid only if $G(0) = 0$, which we shall assume to be the case at present. Replace s by sx. Then

$$\begin{aligned}\Phi(x) &= \frac{1}{\pi}\sin \pi\mu\, x^{\mu}\frac{d}{dx}\int_0^1 G(sx)(1 - s)^{\mu-1}\frac{ds}{s}\\ &= \frac{1}{\pi}\sin \pi\mu\, x^{\mu}\int_0^1 G'(sx)(1 - s)^{\mu-1}ds,\end{aligned}$$

or

$$\Phi(x) = \frac{1}{\pi}\sin \pi\mu\int_0^x G'(s)(x - s)^{\mu-1}ds.$$

Suppose now that $G(0) \neq 0$. Assume $G(0)$ finite. Write $\Phi(x)$ as the solution, assumed to exist, of

$$G(x) = \int_0^x \Phi(s)(x - s)^{-\mu}ds.$$

Now $G_1(x) = G(x) - G(0)$ satisfies our previous conditions and therefore has a solution $\Phi_1(x)$ of the equation

$$G_1(x) = \int_0^x \Phi_1(s)(x-s)^{-\mu}ds$$

which is given by

$$\Phi_1(x) = \frac{1}{\pi}\sin \pi\mu \int_0^x G'(s)(x-s)^{\mu-1}ds.$$

Suppose we write $\Phi_1(x) = \Phi(x) - \sigma(x)$. Then

$$G(0) = \int_0^x \sigma(s)(x-s)^{-\mu}ds.$$

This is readily seen to have the solution

$$\sigma(x) = \frac{1}{\pi}\sin \pi\mu G(0)x^{\mu-1}.$$

Now

$$\Phi(x) = \Phi_1(x) + \sigma(x),$$

so the equation

$$G(x) = \int_0^x \Phi(s)(x-s)^{-\mu}ds, \qquad 0 < R(\mu) < 1,$$

has the solution

$$\Phi(x) = \frac{1}{\pi}\sin \pi\mu \left\{G(0)x^{\mu-1} + \int_0^x G'(s)(x-s)^{\mu-1}ds\right\}.$$

The existence of this solution is obvious. This result is Abel's equation. It can be transformed into its most general form by a linear transformation on x.

6. **An interesting special case.** We note that the equation corresponding to (19) may readily be transformed to

$$g(x) = \int_x^\infty \phi(u)\frac{\cos (\alpha(u-x)^{1/2})}{(u-x)^{1/2}}du \tag{21}$$

which implies a solution

$$\phi(x) = -\frac{1}{\pi}\frac{d}{dx}\int_x^\infty g(u)\frac{\mathrm{ch}\,(\alpha(u-x)^{1/2})}{(u-x)^{1/2}}du \tag{22}$$

provided $\int_0^\infty\int_0^\infty |g(x^2+s^2+t^2)|\,|\cos \alpha s|\,\mathrm{ch}\,\alpha t\, ds dt$ exists.

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A RELATION BETWEEN JACOBI AND LAGUERRE POLYNOMIALS

EARL D. RAINVILLE

1. **Introduction.** We shall obtain an apparently new expression for the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ as a series of products of Laguerre polynomials of orders α and β and of different arguments.

2. **A relation between the Legendre and Laguerre polynomials.** Because of the popularity of the Legendre and the simple Laguerre polynomials, we shall prove that special case of the relation between Jacobi and Laguerre polynomials. The method of proof carries through with no alteration in the general case and need not be repeated.

The generating function

$$e^tJ_0(2(ty)^{1/2}) = \sum_{n=0}^{\infty} L_n(y)\frac{t^n}{n!} \tag{1}$$

for the Laguerre polynomials may be found in the standard books. For the Legendre polynomials a special case of a well known result[1] of Bateman is

$$I_0((2t(x-1))^{1/2})I_0((2t(x+1))^{1/2}) = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{(n!)^2}\cdot \tag{2}$$

From (1) with $y=(1-x)/2$ it follows, since $J_0(iy)=I_0(y)$, that

$$e^tI_0((2t(x-1))^{1/2}) = \sum_{n=0}^{\infty} L_n\left(\frac{1-x}{2}\right)\frac{t^n}{n!}\cdot$$

Using (1) with $y=(1+x)/2$ and with t replaced by $(-t)$, we may conclude that

$$e^{-t}I_0((2t(x+1))^{1/2}) = \sum_{n=0}^{\infty} (-1)^nL_n\left(\frac{1+x}{2}\right)\frac{t^n}{n!}\cdot$$

Then, with the aid of the Cauchy product of two series, it may be seen that

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[1] H. Bateman, *A generalization of the Legendre polynomial*, Proc. London Math. Soc. (2) vol. 3 (1905) pp. 111–123. See his formula (8).

$$I_0((2t(x-1))^{1/2})I_0((2t(x+1))^{1/2}) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k C_{n,k} L_k\left(\frac{1+x}{2}\right)L_{n-k}\left(\frac{1-x}{2}\right)\frac{t^n}{n!},$$

where $C_{n,k}$ is the binomial coefficient.

Therefore, in view of (2), we have the desired result,

$$P_n(x) = n!\sum_{k=0}^{n}(-1)^k C_{n,k} L_k\left(\frac{1+x}{2}\right)L_{n-k}\left(\frac{1-x}{2}\right). \tag{3}$$

3. **A similar formula for the Jacobi polynomial.** From Bateman[2] again we may obtain the following generating function for the Jacobi polynomial:

$$[2^{\alpha+\beta}t^{-\alpha-\beta}(x-1)^{-\alpha}(x+1)^{-\beta}]^{1/2}I_\alpha((2t(x-1))^{1/2})I_\beta((2t(x+1))^{1/2}) = \sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}(x)\frac{t^n}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}.$$

For the Laguerre polynomial of order α it is known that

$$e^t(yt)^{-\alpha/2}J_\alpha(2(yt)^{1/2}) = \sum_{n=0}^{\infty}L_n^{(\alpha)}(y)\frac{t^n}{\Gamma(n+\alpha+1)}.$$

These two results may be used just as we used (1) and (2) and they lead us to

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}(-1)^k\frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(k+\beta+1)\Gamma(n-k+\alpha+1)} \cdot L_k^{(\beta)}\left(\frac{1+x}{2}\right)L_{n-k}^{(\alpha)}\left(\frac{1-x}{2}\right).$$

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[2] Bateman, loc. cit.

NOTES ON LEGENDRE POLYNOMIALS

EARL D. RAINVILLE

1. Introduction. We shall obtain three results. The first is a generating function for the Legendre polynomials that lies between the classical

$$(1 - 2xW + W^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) W^n$$

and a special case of a well known result of Bateman.[1] The second is an expression for $P_n(\cos\alpha)$ as a series in $P_k(\cos\beta)$ with α and β unrelated. Special cases of the latter are known. The third result is a relation in integral form between the Hermite and Legendre polynomials.

2. A generating function for $P_n(x)$. From Laplace's first integral,

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} [x + (x^2 - 1)^{1/2}\cos\beta]^n d\beta,$$

we find by direct expansion and the use of Wallis' formula the known[2] result

$$P_n(x) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{(1 - x^2)^k x^{n-2k}}{2^{2k}(k!)^2(n - 2k)!}, \tag{1}$$

where [ ] is the greatest integer symbol.

Examination of (1) in the light of the identity

$$\left(\sum_{n=0}^{\infty} a_n y^{2n}\right)\left(\sum_{n=0}^{\infty} b_n y^n\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} a_k b_{n-2k} y^n, \tag{2}$$

and of the power series for the Bessel function $J_0(y)$,

$$J_0(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{2^{2n}(n!)^2},$$

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[1] H. Bateman, *A generalization of the Legendre polynomial*, Proc. London Math. Soc. (2) vol. 3 (1905) pp. 111–123. I am much indebted to Professor Bateman for pointing out that the generating function in (3) may be found in the work of Catalan, F. H. Jackson, and others. This relation seems to have been less widely used than it deserves. The short, possibly new, derivation in §2 may contribute to an understanding of the material in §§3 and 4.

[2] E. W. Barnes, *On generalized Legendre functions*, Quarterly Journal of Pure and Applied Mathematics vol. 39 (1908) pp. 97–204. See p. 120.

reveals at once that

$$e^{tx}J_0(t(1-x^2)^{1/2}) = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}, \tag{3}$$

or

$$e^{t\cos\theta}J_0(t\sin\theta) = \sum_{n=0}^{\infty} P_n(\cos\theta)\frac{t^n}{n!}\cdot \tag{4}$$

3. **The polynomials $\phi_n(z)$.** In the study of the contribution to electron scattering of a freely rotating group within molecules comprising a jet of gas, Dr. Jerome Karle encountered[3] the polynomials

$$\phi_n(z) = \frac{1}{\pi}\int_0^{\pi}(1+z\cos\beta)^n d\beta.$$

With the aid of Laplace's first integral, we may write

$$\phi_n(z) = (1-z^2)^{n/2}P_n\left(\frac{1}{(1-z^2)^{1/2}}\right). \tag{5}$$

It happens that the $\phi_n(z)$ have a very simple generating function. For the modified Bessel function $I_0(y)$ it is known that

$$I_0(y) = \frac{1}{\pi}\int_0^{\pi} e^{y\cos\beta}d\beta.$$

Hence

$$e^tI_0(tz) = \frac{1}{\pi}\int_0^{\pi} e^{t(1+z\cos\beta)}d\beta,$$

or

$$e^tI_0(tz) = \sum_{n=0}^{\infty}\phi_n(z)\frac{t^n}{n!}\cdot \tag{6}$$

This generating function may also be obtained from equations (1), (3), and (5) above.

4. **A formula for $P_n(\cos\alpha)$.** We have shown elsewhere[4] that if

$$e^tG(xt) = \sum_{n=0}^{\infty} g_n(x)\frac{t^n}{n!},$$

then

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[3] Up to the present time computational difficulties independent of the $\phi_n(z)$ have prevented application of the results.

[4] *Certain generating functions and associated polynomials*, now in the hands of the editors of the Amer. Math. Monthly.

$$g_n(xy) = \sum_{k=0}^{n} C_{n,k} y^k (1-y)^{n-k} g_k(x).$$

Therefore, we may write

$$\phi_n(xy) = \sum_{k=0}^{n} C_{n,k} y^k (1-y)^{n-k} \phi_k(x). \tag{7}$$

From (5) it follows that $\phi_n(i \tan \alpha) = \sec^n \alpha P_n(\cos \alpha)$. Now, let us put $x = i \tan \beta$ and $y = \tan \alpha / \tan \beta$ in (7). It becomes

$$\sec^n \alpha P_n(\cos \alpha) = \sum_{k=0}^{n} C_{n,k} \frac{\tan^k \alpha}{\tan^n \beta} (\tan \beta - \tan \alpha)^{n-k} \sec^k \beta P_k(\cos \beta).$$

The above may be simplified by elementary means to the form

$$P_n(\cos \alpha) = \left(\frac{\sin \alpha}{\sin \beta}\right)^n \sum_{k=0}^{n} C_{n,k} \left[\frac{\sin (\beta - \alpha)}{\sin \alpha}\right]^{n-k} P_k(\cos \beta). \tag{8}$$

5. **Special cases of the above formula.** It is interesting to note certain special cases of (8), either because they yield well known results or because of their intrinsic neatness.

Using $\beta = 2\alpha$, we get

$$(2 \cos \alpha)^n P_n(\cos \alpha) = \sum_{k=0}^{n} C_{n,k} P_k(\cos 2\alpha).$$

Using $\alpha = \pi/2 + \beta$, we get

$$\sin^n \beta P_n(\sin \beta) = \sum_{k=0}^{n} (-1)^k C_{n,k} \cos^k \beta P_k(\cos \beta).$$

Finally, using $\beta = -\alpha$,

$$P_n(\cos \alpha) = \sum_{k=0}^{n} (-1)^k C_{n,k} (2 \cos \alpha)^{n-k} P_k(\cos \alpha),$$

or

$$P_n(x) = \sum_{k=0}^{n} (-1)^k C_{n,k} (2x)^{n-k} P_k(x),$$

which in symbolic notation is $P_n(x) \doteqdot \{2x - P(x)\}^n$.

6. **A formula for the Hermite polynomial.** Either the generating function (6) or the integral definition of the $\phi_n(z)$ may be used to obtain

$$\phi_n(z) = n! \sum_{k=0}^{[n/2]} \frac{z^{2k}}{2^{2k}(k!)^2(n-2k)!}.$$

The formula above naturally reminds one of the Hermite polynomial,

$$H_n(x) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2k)!}.$$

Indeed, we may write

$$\phi_n(2it^{1/2}) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{t^k}{(k!)^2(n-2k)!},$$

take the Laplace transform of both sides, and get

$$\int_0^\infty e^{-st}\phi_n(2it^{1/2})dt = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{s^{-k-1}}{k!(n-2k)!} = s^{-(n+2)/2}H_n(s^{1/2}/2).$$

Therefore,

$$H_n(x) = (2x)^{n+2}\int_0^\infty e^{-4x^2t}\phi_n(2it^{1/2})dt,$$

or, with $4t=\alpha$,

$$H_n(x) = 2^n x^{n+2}\int_0^\infty e^{-\alpha x^2}\phi_n(i\alpha^{1/2})d\alpha.$$

But, from (5) it follows that

$$\phi_n(i\alpha^{1/2}) = (1+\alpha)^{n/2}P_n\left(\frac{1}{(1+\alpha)^{1/2}}\right).$$

Hence, it is possible to write

$$H_n(x) = 2^n x^{n+2}\int_0^\infty e^{-\alpha x^2}(1+\alpha)^{n/2}P_n\left(\frac{1}{(1+\alpha)^{1/2}}\right)d\alpha,$$

or

$$H_n(x) = 2^{n+1}x^{n+2}e^{x^2}\int_0^1 e^{-x^2/\beta^2}\beta^{-n-3}P_n(\beta)d\beta.$$

This result can also be put in the form

$$H_n(x) = 2^{n+1}e^{x^2}\int_x^\infty e^{-t^2}t^{n+1}P_n\left(\frac{x}{t}\right)dt,$$

resembling that of Curzon[5] but apparently simpler.

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[5] H. E. J. Curzon, *On a connexion between the functions of Hermite and the functions of Legendre*, Proc. London Math. Soc. (2) vol. 12 (1913) pp. 236–259.

A CORRECTION TO MY BOOK ON TOPOLOGICAL GROUPS

ANDRÉ WEIL[1]

On p. 28 of my monograph on topological groups,[2] implicit use is made of the following proposition: "If G is a topological group, and g a closed invariant subgroup of G, then the factor-group G/g cannot have a dimension greater than that of G." This is a very plausible but hitherto unproved result; the assertions on the dimension of a projective limit (IGTA, p. 28) must therefore partake of its conjectural nature. However, the results on the dimension of compact groups and of locally compact abelian groups (IGTA, chap. 5, §25, and chap. 6, §29) can be justified by dealing with a special case of that proposition; to do so is the purpose of the present note. Since no satisfactory theory of dimension exists except for "separable metrizable" spaces,[3] it will be enough to consider the case of groups which are metrizable and separable. A compact group which satisfies these conditions will be called *enumerably compact* in this note; such a group has at most enumerably many irreducible linear representations; it follows from this, and from the results of IGTA, §25, that it is the projective limit of a *sequence* of compact Lie groups. The result to be proved is now as follows.

THEOREM. *Let G be an enumerably compact group. Let g_0 be a closed invariant subgroup of G such that G/g_0 is a Lie group. Let U_0 be a neighborhood of the unit-element in G/g_0, homeomorphic to the interior of a sphere; and let X be the set of all elements of G, the image of which in G/g_0 is in U_0. Then X is homeomorphic to the (topological) direct product $U_0 \times g_0$.*

Let ϕ be the canonical mapping of G on the factor-group $G_0 = G/g_0$, i.e., the mapping in which, to every element of G, corresponds its co-set with respect to g_0. As is well known, our theorem will be proved if we show that we can find in G a system of representatives U for U_0, i.e., a homeomorphic image $U = \psi(U_0)$ of U_0 in G such that the inverse ψ^{-1} of ψ coincides on U with ϕ. In order to do that, we write G

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[1] Guggenheim Fellow.

[2] *L'intégration dans les groupes topologiques et ses applications*, Actualités Scientifiques et Industrielles, Hermann, Paris, 1940; this book will be quoted as IGTA. I am much indebted to H. Samelson for having first pointed out to me the conjectural nature of the result in question, and also for the following correction to the same book: on p. 26, lines 8 and 15, the word "invariant" should be omitted.

[3] Cf. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941 (see the Appendix).

as projective limit of a sequence of Lie groups $G_n = G/g_n$, where the g_n, for $n = 0, 1, 2, \cdots$, are a sequence of closed invariant subgroups of G, beginning with the given subgroup g_0, such that $g_n \supset g_{n+1}$ for every n, and with no common element other than the unit-element of G. If we put $\gamma_n = g_{n+1}/g_n$, we can consider γ_n as a subgroup of $G_n = G/g_n$; it is a closed invariant subgroup of G_n, and we have (by IGTA, §3) $G_n = G_{n+1}/\gamma_{n+1}$; let ϕ_n be the canonical mapping of G_{n+1} on the factor-group $G_n = G_{n+1}/\gamma_{n+1}$. We now define by induction, for every n, a homeomorphic image U_n of U_0 in G_n as follows. Assuming that U_n has been defined, we may consider its inverse image by ϕ_n, in G_{n+1}, as a fibre-space with the base-space U_n and with fibres which are homeomorphic to γ_{n+1}; as U_n is homeomorphic to U_0, that is, to the interior of a sphere of a certain dimension, it follows from this, by Feldbau's theorem,[4] that this inverse image is homeomorphic to the topological product of U_n and of γ_{n+1}, and that we can find in it a continuous system of representatives, which we call U_{n+1}, for the base U_n. Then U_{n+1} is a homeomorphic image $U_{n+1} = \psi_n(U_n)$ of U_n in G_{n+1}, and the inverse ψ_n^{-1} of the homeomorphic mapping ψ_n of U_n on U_{n+1} is no other than ϕ_n. That being so, let x_0 be any point in U_0, and put $x_{n+1} = \psi_n(x_n)$ for $n = 0, 1, 2, \cdots$; as we have $x_n = \phi_n(x_{n+1})$ for every n, the sequence $(x_0, x_1, x_2, \cdots)$ defines a point x in the projective limit G of the sequence $G_0, G_1, G_2, \cdots$; if now we write $x = \psi(x_0)$, ψ is a mapping of U_0 in G, with the properties which were required for the proof of our theorem.

COROLLARY. *If a factor-group of an enumerably compact group G is a Lie group of dimension n, then G itself has at least the dimension n.*

The above proof suggests that the conjectural proposition mentioned at the beginning of this note is related to the following problem. Let G be a topological group, g a closed subgroup of G, and $H = G/g$ the homogeneous space determined by G and g (IGTA, §2), that is, the space of cosets of g in G; can G be considered as a fibre-space, with base H and with fibres homeomorphic to g, according to any one of the known definitions of fibre-spaces?[5] One may expect the answer to be in the affirmative, provided a point in H has a system of neighborhoods which are all homotopic to zero (that is, contractible to a point) in H.

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[4] J. Feldbau, *Sur la classification des espaces fibrés*, C. R. Acad. Sci. Paris vol. 208 (1939) p. 1621.

[5] Cf., for example, R. H. Fox, *On fibre-spaces*, Bull. Amer. Math. Soc. vol. 49 (1943) p. 555.

ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BY FEJÉR MEANS

A. ZYGMUND

1. **Continuous functions.** It has been proved by S. Bernstein that if $f(x)$ is periodic and of the class Lip α, $0<\alpha<1$, then the $(C, 1)$ means $\sigma_n(x)=\sigma_n(x;f)$ of the Fourier series of f satisfy the condition

$$\sigma_n(x) - f(x) = O(n^{-\alpha}), \tag{1.1}$$

uniformly in x. The result is false for $\alpha=1$. The place of (1.1) is then taken by

$$\sigma_n(x) - f(x) = O(\log n/n), \tag{1.2}$$

and, as simple examples show, the factor $\log n$ on the right cannot be removed (see, for example, A. Zygmund, *Trigonometrical series*, p. 62). It will be shown here that for power series the inequality (1.1) holds even for $\alpha=1$. More generally, we have the following theorem.

THEOREM 1. *Suppose that $f(x)$ is periodic, continuous, and that the Fourier series of f is of power series type,*

$$f(x) \sim \sum_{\nu=0}^{\infty} c_\nu e^{i\nu x}.$$

Then

$$| \sigma_{n-1}(x) - f(x) | \leqq A\omega(2\pi/n), \tag{1.3}$$

where $\omega(\delta)$ is the modulus of continuity of f and A is an absolute constant.

The proof is based on the following lemma.

LEMMA. *Suppose that*

$$g(x) \sim \sum_{-\infty}^{+\infty} \gamma_\nu e^{i\nu x} \tag{1.4}$$

satisfies $|g(x+h)-g(x)| \leqq M|h|$. Then

$$| \tilde{\sigma}_{n-1}(x) - \tilde{g}(x) | \leqq BM/n, \tag{1.5}$$

where $\tilde{g}(x)$ is the function conjugate to $g(x)$ and $\tilde{\sigma}_n(x)$ are the $(C, 1)$ means of the series conjugate to (1.4).

For the proof of the lemma we note that

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$$\tilde{g}(x) = -\frac{1}{\pi}\int_0^\pi [g(x+t) - g(x-t)]\,\frac{1}{2}\cot\frac{1}{2}t\,dt,$$

$$\tilde{\sigma}_{n-1}(x) = -\frac{1}{\pi}\int_0^\pi [g(x+t) - g(x-t)]$$

$$\cdot\left[\frac{1}{2}\cot\frac{1}{2}t - \frac{\sin nt}{n(2\sin(t/2))^2}\right]dt,$$

$$\tilde{g}(x) - \tilde{\sigma}_{n-1}(x) = \frac{1}{\pi}\int_0^\pi [g(x+t) - g(x-t)]\,\frac{\sin nt}{n(2\sin(t/2))^2}\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi/n} + \frac{1}{\pi}\int_{\pi/n}^{\pi} = P_n + Q_n,$$

say. Since $|\sin nt| \leqq n\sin t \leqq n(2\sin(t/2))$ for $0 \leqq t \leqq \pi$,

$$|P_n| \leqq \frac{1}{\pi}\int_0^{\pi/n} \frac{2Mt}{2\sin(t/2)}\,dt \leqq \frac{M}{\pi}\int_0^{\pi/n} \frac{tdt}{(2/\pi)t/2} = \frac{M\pi}{n}.$$

In order to estimate Q_n, we introduce the function

$$\Lambda_n(t) = \frac{1}{\pi n}\int_t^\pi \frac{\sin nu}{(2\sin(u/2))^2}\,du,$$

and integrate by parts. By the second mean value theorem,

$$|\Lambda_n(t)| \leqq \frac{2}{\pi n^2}\cdot\frac{1}{(2\sin(t/2))^2} < \frac{\pi}{2n^2t^2}.$$

The function g is absolutely continuous and $|g'(x)| \leqq M$ almost everywhere. Thus

$$|Q_n| \leqq \frac{1}{\pi}\left|\Big[(g(x+t) - g(x-t))\Lambda_n(t)\Big]_{\pi/n}^{\pi}\right|$$

$$+ \frac{1}{\pi}\left|\int_{\pi/n}^{\pi} [g'(x+t) + g'(x-t)]\Lambda_n(t)dt\right|$$

$$\leqq \frac{1}{\pi}\cdot M\cdot\frac{2\pi}{n}\cdot\frac{\pi}{2n^2(\pi/n)^2} + \frac{2M}{\pi}\int_{\pi/n}^{\pi} |\Lambda_n(t)|\,dt$$

$$< \frac{M}{\pi n} + \frac{M}{n^2}\int_{\pi/n}^{\infty}\frac{dt}{t^2} = \frac{2}{\pi}\,\frac{M}{n}.$$

This completes the proof of the lemma, with $B = \pi + 2/\pi$.

Suppose now that the Fourier series of f is of power series type so that $\tilde{f}=-if$. If $|f(x+h)-f(x)|\leqq M|h|$, then

$$\left|\sigma_{n-1}(x)-f(x)\right|=\left|\tilde{\sigma}_{n-1}(x)-\tilde{f}(x)\right|\leqq BM/n. \tag{1.6}$$

To complete the proof of Theorem 1, we introduce the function

$$f_h(x)=\frac{1}{2h}\int_{-h}^{h}f(x+t)dt=[F(x+h)-F(x-h)]/2h$$
$$\sim\sum_{\nu=0}^{\infty}c_\nu e^{i\nu x}\left(\frac{\sin\nu h}{\nu h}\right),$$

where $F(x)$ is a primitive of f. Hence df_h/dx exists, is continuous, and does not exceed $\omega(2h)/2h\leqq\omega(h)/h$ in absolute value. Moreover, the Fourier series of f_h is also of power series type. Now,

$$\begin{aligned}&\left|\sigma_{n-1}(x;f)-f(x)\right|\\&\leqq\left|\sigma_{n-1}(x;f)-\sigma_{n-1}(x;f_h)\right|+\left|\sigma_{n-1}(x;f_h)-f_h(x)\right|+\left|f_h(x)-f(x)\right|\\&=\alpha_n+\beta_n+\gamma_n,\end{aligned}$$

say, and

$$\gamma_n=\left|\frac{1}{2h}\int_{-h}^{h}[f(x+t)-f(x)]dt\right|\leqq\omega(h),$$

$$\beta_n\leqq B\frac{\omega(h)}{h}\cdot\frac{1}{n} \qquad \text{(by (1.6))},$$

$$\alpha_n=\left|\sigma_{n-1}(x;f-f_h)\right|\leqq\max_x\left|f-f_h\right|\leqq\omega(h).$$

If we set $h=2\pi/n$ and collect the results, we obtain (1.3) with $A=2+B/2\pi<4$.

2. **Additional remarks.** The foregoing proof of the lemma has certain disadvantages. First of all, it uses the result that a Lipschitz function is an indefinite integral, a fact which lies deeper than the assumptions of the lemma. Moreover, the argument does not work with the L^p metric. These difficulties are avoided by the following somewhat longer variant of the proof of the lemma. For the sake of brevity we do not compute the absolute constants C that occur in the proof.

Let P_n and Q_n have the same meaning as before, and let $\psi(x,t)=f(x+t)-f(x-t)$. Hence

$$|P_n|\leqq\left|\frac{1}{\pi}\int_0^{\pi/n}\psi(x,t)\frac{\sin nt}{n(2\sin(t/2))^2}dt\right|\leqq\int_0^{\pi/n}|\psi(x,t)|t^{-1}dt.$$

Let $R_n(t)=1/\pi n(2\sin(t/2))^2<1/nt^2$. Then, for $n\geqq1$,

$$
\begin{aligned}
Q_n &= \int_{\pi/n}^{\pi} \psi(x, t) R_n(t) \sin nt dt \\
&= -\int_{0}^{\pi(n-1)/n} \psi(x, t + \pi/n) R_n(t + \pi/n) \sin nt dt,
\end{aligned}
$$

$$
\begin{aligned}
2Q_n = &\int_{\pi/n}^{\pi(n-1)/n} \psi(x, t)\left[R_n(t) - R_n(t + \pi/n)\right] \sin nt dt \\
&+ \int_{\pi/n}^{\pi(n-1)/n} [\psi(x, t) - \psi(x, t + \pi/n)] R_n(t + \pi/n) \sin nt dt \\
&- \int_{0}^{\pi/n} \psi(x, t + \pi/n) R_n(t + \pi/n) \sin nt dt \\
&+ \int_{\pi(n-1)/n}^{\pi} \psi(x, t) R_n(t) \sin nt dt = I_n + J_n + K_n + L_n,
\end{aligned}
$$

say.

By the mean-value theorem

$$
\left| R_n(t) - R_n(t + \pi/n) \right| \leqq C n^{-2} t^{-3},
$$

so that

$$
\left| I_n \right| \leqq C n^{-2} \int_{\pi/n}^{\pi - \pi/n} \left| \psi(x, t) \right| t^{-3} dt \leqq C n^{-2} \int_{\pi/n}^{\pi} \left| \psi(x, t) \right| t^{-3} dt.
$$

Since $R_n(t+\pi/n) \leqq 1/nt^2$, and

$$
\begin{aligned}
\psi(x, t) - \psi(x, t + \pi/n) = \psi(x + t - \pi/2n, \pi/2n) \\
- \psi(x - t - \pi/2n, \pi/2n),
\end{aligned}
$$

we find

$$
\begin{aligned}
\left| J_n \right| \leqq C n^{-1} &\int_{\pi/n}^{\pi} \left| \psi(x + t - \pi/2n, \pi/2n) \right| t^{-2} dt \\
&+ C n^{-1} \int_{\pi/n}^{\pi} \left| \psi(x - t - \pi/2n, \pi/2n) \right| t^{-2} dt.
\end{aligned}
$$

Moreover, since $R_n(t+\pi/n) < Cn$ for $0 \leqq t \leqq \pi/n$,

$$
\left| K_n \right| \leqq C n \int_{0}^{\pi/n} \left| \psi(x, t + \pi/n) \right| dt.
$$

Finally,

$$
\left| L_n \right| \leqq C n^{-1} \int_{\pi(n-1)/n}^{\pi} \left| \psi(x, t) \right| dt = C n^{-1} \int_{0}^{\pi/n} \left| \psi(x + \pi, t) \right| dt.
$$

By assumption, $|\psi(x, u)| \leqq M|u|$, uniformly in x. From this we immediately deduce that each of the terms $|P_n|$, $|I_n|$, $|J_n|$, $|K_n|$, $|L_n|$ is less than or equal to CM/n, and (1.5) is proved.

Suppose now that instead of the inequality $|g(x+h)-g(x)| \leqq M|h|$ we have

$$\text{(2.1)}\quad M_p[g(x+h)-g(x)] = \left\{\int_0^{2\pi} |g(x+h)-g(x)|^p dx\right\}^{1/p} \leqq M|h|$$

for some $p \geqq 1$. Then Minkowski's inequality for integrals shows that $M_p[P_n]$, $M_p[I_n]$, $M_p[J_n]$, $M_p[K_n]$, $M_p[L_n]$ are all less than or equal to CM/n. For example,

$$M_p[P_n] \leqq \int_0^{\pi/n} M_p[\psi(x, t)]t^{-1}dt \leqq \int_0^{\pi/n} 2M dt = 2M\pi/n,$$

$$M_p[I_n] \leqq Cn^{-2}\int_{\pi/n}^{\pi} M_p[\psi(x, t)]t^{-3}dt$$

$$\leqq 2CMn^{-2}\int_{\pi/n}^{\infty} t^{-2}dt = CM/n,$$

and similarly in other cases. Thus, *under the hypothesis* (2.1),

$$M_p[\tilde{\sigma}_{n-1}(x) - \tilde{g}(x)] \leqq BM/n$$

where B is an absolute constant. By an argument similar to that by which Theorem 1 was deduced from the lemma, we obtain the following theorem.

THEOREM 2. *Suppose that the Fourier series of $f(x)$ is of the power series type. Then*

$$M_p[\sigma_{n-1}(x) - f(x)] \leqq A\omega_p(2\pi/n) \qquad (p \geqq 1)$$

where $\omega_p(\delta) = \sup_{|t|\leqq\delta} M_p[f(x+t)-f(x)]$.

Theorems 1 and 2 hold for (C, α) means, whatever $\alpha > 0$. The analogues for Abel means are immediate consequences of the Cauchy-Riemann equations.

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NOTE ON INTERPOLATION FOR A FUNCTION OF SEVERAL VARIABLES

HERBERT E. SALZER

The simplest interpolation formula for a function of ω variables $x, y, \cdots, z$ is the multiple Gregory-Newton formula, which approximates the function by a polynomial in $p, q, \cdots, r$ of total degree n, namely,

$$
\tag{1}
\begin{aligned}
&f(x + ph_1, y + qh_2, \cdots, z + rh_\omega) \\
&\quad = \sum_{i+j+\cdots+k=0}^{n} \binom{p}{i}\binom{q}{j}\cdots\binom{r}{k}\Delta_{x^i y^j \cdots z^k}^{i+j+\cdots+k} f(x, y, \cdots, z),
\end{aligned}
$$

where $x, y, \cdots, z$ denote the independent variables, h_m denotes the tabular intervals,

$$
\binom{p}{i} \text{ denotes } \frac{p(p-1)\cdots(p-i+1)}{i!}, \text{ with } \binom{p}{0} = 1,
$$

and $\Delta_{x^i y^j \cdots z^k}^{i+j+\cdots+k} f(x, y, \cdots, z)$ denotes the mixed partial advancing difference of $f(x, y, \cdots, z)$, of order i with respect to x, j with respect to y, and so on. The summation is for all sets of values of $i, j, \cdots, k$ such that $i+j+ \cdots +k$ goes from 0 to n. Using the notation $f_{s,t,\cdots,u}$ to denote $f(x+sh_1, y+th_2, \cdots, z+uh_\omega)$, it is apparent that the multiple Gregory-Newton formula involves all values $f_{s,t,\cdots,u}$ such that $s+t+ \cdots +u=0, 1, 2, \cdots, n$. Thus for the case of 2 dimensions the arguments are the $(n+1)(n+2)/2$ points forming a right triangle, vertex at (x, y), and for 3 dimensions the arguments are the $(n+1)(n+2)(n+3)/6$ points forming a solid tetrahedron, vertex at (x, y, z).

The purpose of the present note is to show that when (1) is expressed in the simpler form

$$
\tag{2}
f(x + ph_1, y + qh_2, \cdots, z + rh_\omega) = \sum_{s+t+\cdots+u=0}^{n} C_{s,t,\cdots,u} f_{s,t,\cdots,u},
$$

then we have

$$
\tag{3}
C_{s,t,\cdots,u} = \binom{n-p-q-\cdots-r}{n-s-t-\cdots-u}\binom{p}{s}\binom{q}{t}\cdots\binom{r}{u}.
$$

Thus (1) can be employed without the labor of finding all the mixed

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partial differences, which represents a very convenient simplification in the use of the multiple Gregory-Newton formula.

To prove[1] (3) consider the function

$$\binom{n-x-y-\cdots-z}{n-s_1-t_1-\cdots-u_1}\binom{x}{s_1}\binom{y}{t_1}\cdots\binom{z}{u_1},$$

where $s_1, t_1, \cdots, u_1$ are any set of non-negative integers whose sum is not greater than n. This function is a polynomial in $x, y, \cdots, z$ of total degree n and (2) holds exactly. Applying (2) for $x=y=\cdots=z=0$, $h_1=h_2=\cdots=h_\omega=1$, it is apparent that with the exception of $f_{s_1,t_1,\cdots,u_1}=1$, all the other quantities $f_{s,t,\cdots,u}$ vanish, because if some $s, t, \cdots$, or u is less than a respective $s_1, t_1, \cdots$, or u_1, or if every $s, t, \cdots, u$ is greater than or equal to a respective $s_1, t_1, \cdots, u_1$ with at least one greater than, then $f_{s,t,\cdots,u}$ will have a factor

$$\binom{a}{b}, \quad a \text{ and } b \text{ integers},$$

$b>a$, which is 0. This establishes (3).

Mathematical Tables Project,
National Bureau of Standards

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[1] This line of proof was suggested by Professor W. E. Milne. Another longer proof is by induction, making use of the properties of $\binom{p}{i}$ and Newton's backward-difference interpolation formula.

QUADRICS ASSOCIATED WITH A CURVE ON A SURFACE

V. G. GROVE

1. **Introduction.** Many of the important contributions to projective differential geometry of non-ruled surfaces are concerned with systems of quadrics associated with a point and a curve on the surface. Many of these quadrics belong to a certain family, a characterization of which is the main purpose of this paper.

Let the homogeneous projective coordinates (x^1, x^2, x^3, x^4) of a general point x on a non-ruled surface S be given as functions of the asymptotic parameters u, v, and let these functions be so normalized that they satisfy the Fubini canonical system of differential equations,

$$\begin{aligned} x_{uu} &= \theta_u x_u + \beta x_v + p x, \\ x_{vv} &= \gamma x_u + \theta_v x_v + q x, \qquad \theta = \log(\beta\gamma), \end{aligned}$$

wherein the coefficients satisfy certain integrability conditions [7].[1] The abbreviations

$$\phi = \partial \log(\beta\gamma^2)/\partial u, \qquad \psi = \partial \log(\beta^2\gamma)/\partial v$$

will be found useful.

Let C_λ, a curve on S through x, be considered as imbedded in a one-parameter family of curves defined by the differential equation

$$dv - \lambda du = 0.$$

Since the homogeneous coordinates of any point X may be written in the form

$$X = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv},$$

the coordinates of X referred to the tetrahedron x, x_u, x_v, x_{uv} may be taken as (x_1, x_2, x_3, x_4).

It is remarkable that many of the equations of quadrics associated with S and C_λ at x are of the form

$$x_2 x_3 + T x_4 = 0 \tag{1}$$

wherein

$$T = -x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4, \tag{2}$$

and

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$$(3) \qquad k_2 = l_2\beta/\lambda + m_2\gamma\lambda^2, \qquad k_3 = l_3\beta/\lambda^2 + m_3\gamma\lambda,$$

l_2, m_2, l_3, m_3 being constants and k_4 a parameter. In particular the quadrics of Darboux, of Moutard, of Davis [3], all of the quadrics derived by Wu [9], the conjugal quadrics [5], the asymptotic osculating quadrics, and the quadrics Hsiung [6] has associated with C_λ at x all belong to the system (1). We shall denote this family by $Q(l_2, m_2, l_3, m_3)$.

2. **A characterization of the family.** Let a line l be determined by the points $\rho\sigma$ with coordinates given by the expressions $\rho = x_u - bx$, $\sigma = x_v - ax$. The R_λ-associate of l, as defined by Bell [1], joins the points whose coordinates are

$$(4) \qquad \rho_\lambda = \rho + \beta x/\lambda, \qquad \sigma_\lambda = \sigma + \gamma\lambda x.$$

Bell [1] has called the one-parameter family of curves defined by

$$dv - \mu du = 0, \qquad \mu = -\beta/(\gamma\lambda^2)$$

the R_λ-derived curves, and has characterized them in terms of the R_λ-associate of l. The R_μ-associate of l joins the points

$$(5) \qquad \rho_\mu = \rho - \gamma\lambda^2 x, \qquad \sigma_\mu = \sigma - \beta x/\lambda^2.$$

From (4) and (5) we easily prove the following theorems.

The R_λ-associate of l coincides with the R_μ-associate of l if and only if C_λ is a curve of Darboux.

The R_λ-associate of l, l, and the R_μ-associate of l intersect the asymptotic tangents in points which with x are harmonic if and only if C_λ is a curve of Segre.

Now define points R, S, R_λ, S_λ by the cross ratio equations

$$(6) \qquad \begin{aligned} (x, \rho_\lambda, \rho_\mu, R) &= k, \qquad & (x, R, \rho, R_\lambda) &= K, \\ (x, \sigma_\lambda, \sigma_\mu, S) &= l, \qquad & (x, S, \sigma, S_\lambda) &= L, \end{aligned}$$

k, l, K, L being constants. One finds readily that the coordinates of R_λ, S_λ are given by the expressions

$$(7) \qquad R_\lambda = \rho + k_2 x, \qquad S_\lambda = \sigma + k_3 x$$

wherein k_2, k_3 are given by (3) and

$$(8) \qquad \begin{aligned} l_2 &= kK, \qquad & m_2 &= K(k-1), \\ l_3 &= L(l-1), \qquad & m_3 &= lL. \end{aligned}$$

We shall call the line joining R_λ, S_λ the $S_\lambda(l_2, m_2, l_3, m_3)$-associate of l.

The reciprocal l' of the line l joins x to the point with coordinates $x_{uv}-ax_u-bx_v$. It is easy to show that *the most general quadric having second order contact with S at x and with respect to which the polar line of l' is the $S_\lambda(l_2, m_2, l_3, m_3)$-associate of l is the quadric $Q(l_2, m_2, l_3, m_3)$.*

3. **The general transformation of Čech.** The coordinates of any point R on the tangent to C_λ at x may be written in the form

$$R = x_u + \lambda x_v + tx.$$

The polar plane of R with respect to $Q(l_2, m_2, l_3, m_3)$ has coordinates (u_1, u_2, u_3, u_4) defined by the formulas

$$\begin{gathered} u_1 = 0, \qquad u_2 = \lambda^2, \qquad u_3 = \lambda, \\ u_4 = -\lambda t + (l_2 + l_3)\beta + (m_2 + m_3)\gamma\lambda^3. \end{gathered} \tag{9}$$

If the local coordinates of R be written as $(x_1, x_2, x_3, 0)$, equations (9) assume the form

$$\begin{gathered} u_1 = 0, \qquad u_2 = x_2x_3^2, \qquad u_3 = x_2^2x_3, \\ u_4 = -x_1x_2x_3 + (l_2 + l_3)\beta x_2^3 + (m_2 + m_3)\gamma x_3^3 \end{gathered} \tag{10}$$

of the most general transformation of Čech [2].

Bell [1] has given a geometric characterization of this general transformation. Lane [7] has characterized this transformation for the special case $l_2+l_3=m_2+m_3$.

The quadric $Q_{-\lambda}(l_2, m_2, l_3, m_3)$ induces the transformation

$$\begin{gathered} u_1 = 0, \qquad u_2 = x_2x_3^2, \qquad u_3 = x_2^2x_3, \\ u_4 = -x_1x_2x_3 + (l_2 - l_3)\beta x_2^3 + (m_2 - m_3)\gamma x_3^3. \end{gathered} \tag{11}$$

The particular quadrics referred to above induce several interesting special transformations (10) and (11).

4. **The quadrics $Q_u(l_2, m_2, l_3, m_3)$, $Q_v(l_2, m_2, l_3, m_3)$.** We may characterize some of the quadrics in the family (1) in the following manner. The $S_\lambda(l_2, m_2, l_3, m_3)$-associate of the reciprocal of the projective normal joins the points defined by $R_\lambda=x_u+k_2x$, $S_\lambda=x_v+k_3x$, k_2, k_3 being defined by (3). The tangent to the locus of R_λ as x moves along C_λ and the point S_λ determine a plane π. The plane π intersects the projective normal in a point P. As x moves along C_λ the plane $x_3=0$ envelops a developable surface generated by a line which intersects the projective normal in a point Q. The plane determined by R_λ, S_λ and the harmonic conjugate of x with respect to P and Q has the equation $T=0$, T being defined by (2) and wherein

$$
\begin{aligned}
k_4 = \frac{1}{2}\Bigg\{ &- \frac{\beta}{\lambda^3}\Bigg[l_2\lambda' + (l_2^2 + l_2 l_3 + l_3)\beta \\
&+ \left[(1 - l_2)\,\frac{\beta_v}{\beta} + \theta_v\right]\lambda^2 + l_2\left(\theta_u - \frac{\beta_u}{\beta}\right)\lambda\Bigg] \\
&+ m_2\gamma\left[2\lambda' - (m_2 + m_3)\gamma\lambda^3 + \frac{\gamma_v}{\gamma}\lambda^2 + \left(\frac{\gamma_u}{\gamma} - \theta_u\right)\lambda\right] \\
&- \theta_{uv} - [(2l_2 + l_3)m_2 + m_3(1 + l_2) + 1]\beta\gamma\Bigg\}.
\end{aligned} \tag{12}
$$

The quadric having second order contact with S at x and passing through the lines $x_2=0$, $T=0$; $x_3=0$, $T=0$ has the equation (1), k_4 being given by (12). We shall denote this quadric by $Q_u(l_2, m_2, l_3, m_3)$.

In an analogous manner we may define a quadric $Q_v(l_2, m_2, l_3, m_3)$ with the equation (1) with k_4 given by the formula

$$
\begin{aligned}
k_4 = \frac{1}{2}\Bigg\{ &- \gamma\Bigg[- m_3\lambda' + (m_3^2 + m_2 m_3 + m_2)\gamma\lambda^3 \\
&+ \left[(1 - m_3)\frac{\gamma_u}{\gamma} + \theta_u\right]\lambda + m_3\left(\theta_v - \frac{\gamma_v}{\gamma}\right)\lambda^2\Bigg] \\
&+ \frac{l_3\beta}{\lambda^3}\left[- 2\lambda' - (l_2 + l_3)\beta + \frac{\beta_u}{\beta}\lambda + \left(\frac{\beta_v}{\beta} - \theta_v\right)\lambda^2\right] \\
&- \theta_{uv} - [(2m_3 + m_2)l_3 + l_2(1 + m_3) + 1]\beta\gamma\Bigg\}.
\end{aligned} \tag{13}
$$

5. **Applications.** It is easily verified that *the special quadrics* $Q_u(-1, 0, 1, 0)$, $Q_v(0, 1, 0, -1)$ *are the asymptotic osculating quadrics of the curve* C_λ *at* x. These quadrics may therefore be considered as generalizations of the asymptotic osculating quadrics. *The quadric* $Q_u(0, 0, 0, 0)$ *for a curve* C_λ *tangent to the asymptotic* $u=$const. (*or* $Q_v(0, 0, 0, 0)$ *for a curve* C_λ *tangent to* $v=$const.) *is the quadric of Lie.*

Let us call the quadrics $Q_u(0, 1, 0, -1)$, $Q_v(-1, 0, 1, 0)$ *the anti-asymptotic osculating quadrics*. They are given by (1) with the respective values of k_4:

$$
\begin{aligned}
k_4 = \frac{1}{2}\Bigg\{ &\gamma\left[2\lambda' + \frac{\gamma_v}{\gamma}\lambda^2 + \left(\frac{\gamma_u}{\gamma} - \theta_u\right)\lambda\right] \\
&- \frac{\beta}{\lambda}\left(\frac{\beta_v}{\beta} + \theta_v\right) - \theta_{uv}\Bigg\},
\end{aligned}
$$

$$k_4 = \frac{1}{2}\left\{\frac{\beta}{\lambda^3}\left[-2\lambda' + \frac{\beta_u}{\beta}\lambda + \left(\frac{\beta_v}{\beta} - \theta_v\right)\lambda^2\right] - \gamma\lambda\left(\frac{\gamma_u}{\gamma} + \theta_u\right) - \theta_{uv}\right\}.$$

It is easy to verify the following theorems.

The anti-asymptotic osculating quadrics (assumed distinct) intersect in the asymptotic tangents and in a conic whose plane passes through the projective normal if and only if C_λ *is a pangeodesic. These quadrics coincide if and only if* $\beta+\gamma\lambda^3=0$, *and* $\phi+\lambda\psi=0$; *that is, the curve* C_λ *must be tangent to a curve of Darboux, and that tangent must be the second canonical tangent of* S *at* x.

The conjugal quadrics $Q_u(0, k, k, 0)$, $Q_v(0, k, k, 0)$ have equations given by (1) with the respective values of k_4:

$$k_4 = \frac{1}{2}\left\{k\gamma\left[2\lambda' - k\gamma\lambda^3 + \frac{\gamma_v}{\gamma}\lambda^2 + \left(\frac{\gamma_u}{\gamma} - \theta_u\right)\lambda\right] - \frac{\beta}{\lambda^3}\left[k\beta + \left(\frac{\beta_v}{\beta} + \theta_v\right)\lambda^2\right] - \theta_{uv} - (1 + k^2)\beta\gamma\right\},$$

$$k_4 = \frac{1}{2}\left\{-k\frac{\beta}{\lambda^3}\left[2\lambda' + k\beta - \frac{\beta_u}{\beta}\lambda - \left(\frac{\beta_v}{\beta} - \theta_v\right)\lambda^2\right] - \gamma\left[k\gamma\lambda^3 + \left(\frac{\gamma_u}{\gamma} + \theta_u\right)\lambda\right] - \theta_{uv} - (1 + k^2)\beta\gamma\right\}.$$

These quadrics coincide if and only if

$$\begin{aligned} 2k(\beta + \gamma\lambda^3)\lambda' = {}& k(1 - k)\beta^2 + k\beta_u\lambda \\ & + \beta\lambda^2\left[(1 + k)\frac{\beta_v}{\beta} + (1 - k)\theta_v\right] \\ & - \gamma\lambda^4\left[(1 + k)\frac{\gamma_u}{\gamma} + (1 - k)\theta_u\right] \\ & - k\gamma_u\lambda^5 - k(1 - k)\gamma^2\lambda^6. \end{aligned} \tag{14}$$

The curves represented by (14) *are hypergeodesics if and only if* $\phi=\psi=0$, *that is,* S *is a coincidence surface.* In that case the hypergeodesics represented by (13) are given by

$$\lambda' = \frac{1}{2}(1 - k)\beta + \frac{1}{2}\frac{\beta_u}{\beta}\lambda - \frac{1}{2}\frac{\gamma_v}{\gamma}\lambda^2 - \frac{1}{2}(1 - k)\gamma\lambda^3.$$

The cusp-axis of these hypergeodesics is the projective normal. From (14) we see that *the quadrics* $Q_u(0, 1, 1, 0)$, $Q_v(0, 1, 1, 0)$ *of Davis coincide if and only if* C_λ *is a pan-geodesic. And the curves* C_λ *defined by* (14) *are pan-geodesics if and only if the quadrics are quadrics of Davis.*

As the point x moves along C_λ the asymptotic tangents generate ruled surfaces R_u, R_v. Hsiung [6] has shown the existence of a pair of quadrics associated with R_u and R_v. He cuts these surfaces by a plane through the points $P_u = x_u + Ax$, $P_v = x_v + Bx$. The locus of the conic having ordinary contact at P_u and second order contact at P_v with these sections is a quadric $Q(-1, 0, 0, -1)$ whose equation is (1) with k_4, given by the formula

$$\begin{aligned} k_4 = \frac{1}{2}\bigg\{ & \frac{\beta}{\lambda^3}\left[\lambda' - \lambda(\theta_u - \theta_v\lambda - \phi + 2\psi\lambda)\right] + \frac{\beta^2}{\lambda^3} \\ & + 2A\lambda(\beta + \gamma\lambda^3) + \beta\gamma - \theta_{uv}\bigg\}. \end{aligned}$$

This quadric of Hsiung coincides with $Q_u(-1, 0, 0, -1)$ *if and only if* P_u *is the point whose coordinates are given by the expression*

$$P_u = x_u - \beta x/\lambda. \tag{15}$$

Interchanging the roles of the asymptotic curves and ruled surfaces, a second point P_v with coordinates

$$P_v = x_v - \gamma\lambda x \tag{16}$$

is characterized.

The R_λ*-associate of the reciprocal of the projective normal, that reciprocal, the line* P_uP_v *defined by* (15) *and* (16), *and the tangent to the* R_λ*-derived curve through* x *are concurrent, and are moreover harmonic. The* R_λ*-associate of the line* P_uP_v *is the projective normal.*

It is known [7] that the asymptotic osculating quadric $Q_v(0, 1, 0, -1)$ reduces to the quadric of Wilczynski if C_λ is tangent to the asymptotic curve $v = \text{const.}$, and has an inflexion at x, and to the quadric of Fubini if C_λ is tangent to $v = \text{const.}$, and has the tangent plane to S at x as stationary osculating plane. These quadrics are respectively the special quadrics $h = 1$, $h = 1/3$ of the pencil

$$x_2x_3 + x_4\{-x_1 - (1/2)[\theta_{uv} + (1 - h)\beta\gamma]x_4\} = 0. \tag{17}$$

Lane [8] has characterized the invariant parameter h of the pencil (17) in terms of a cross ratio whose elements involve the quadric of Lie ($h = 0$) and the quadric of Wilczynski ($h = 1$). The definition (8) of m_3 enables us to describe the invariant parameter h without recourse to any special quadric of (17).

The quadric $Q_v(0, m_2, 0, m_3)$ for a curve C_λ tangent to $v=\text{const.}$ and having an inflexion at x $(\lambda=0, \lambda'+\beta=0)$ has the equation (17) with $h=-m_3$. The definitions (8) imply that $l=1$, $k=0$, $m_3=L$, $m_2=K$. From (6) we find that $R=\rho_\mu$, $S=\sigma_\lambda$. Then the points R_λ, S_λ determining the $S_\lambda(0, m_2, 0, m_3)$-associate of l are found from the cross ratio equations

$$(x, \rho_\mu, \rho, R_\lambda) = K, \quad (x, \sigma_\lambda, \sigma, S_\lambda) = L = -h.$$

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ON APPROXIMATE ISOMETRIES

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In a previous paper, a problem of mathematical "stability" for the case of the linear functional equation was studied.[1] It was shown that if a transformation $f(x)$ of a vector space E_1 into a Banach space E_2 satisfies the inequality $\|f(x+y)-f(x)-f(y)\|<\epsilon$ for some $\epsilon>0$ and all x and y in E_1, then there exists an additive transformation $\phi(x)$ of E_1 into E_2 such that $\|f(x)-\phi(x)\|<\epsilon$.

In the present paper we consider a stability problem for isometries. By an ϵ-isometry of one metric space E into another E' is meant a transformation $T(x)$ which changes distances by at most ϵ, where ϵ is some positive number; that is, $|\rho(x, y)-\rho(T(x), T(y))|<\epsilon$ for all x and y in E. Given an ϵ-isometry $T(x)$, our object is to establish the existence of a true isometry $U(x)$ which approximates $T(x)$; more precisely, to establish the existence of a constant $k>0$ depending only on the metric spaces E and E' such that $\rho(T(x), U(x))<k\epsilon$ for all x in E. In this paper this result will be proved for the case in which $E=E'$, where E is n-dimensional Euclidean space or Hilbert space (not necessarily separable). The case in which E is the space C of continuous functions will be treated in another paper.

The above problem of ϵ-isometries is related to the problem of constructing space models for sets in which distances between points are given only with a certain degree of exactness (measurements are possible only with a certain degree of precision). The question of the uniqueness of the idealized model corresponding to the given measurements and the extrapolation from the measurements to the model could be looked upon as a problem in determining a strict isometry from an approximate isometry.

In the case of certain simple metric spaces, for example the surface of the Euclidean sphere, this question can be answered in the affirmative, but it may be more difficult for other bounded manifolds. A simple but interesting example showing a case where the answer is *negative* has been worked out by R. Swain.

THEOREM 1. *Let E be a complete abstract Euclidean vector space.*[2]

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[1] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. vol. 27 (1941) pp. 222–224.

[2] A complete Euclidean vector space is a Banach space whose norm is generated by an inner product, (x, y). It includes real Hilbert space and n-dimensional Euclidean spaces as special cases.

Let $T(x)$ be an ϵ-isometry of E into itself such that $T(0)=0$. The limit $U(x)=\lim_{n\to\infty}(T(2^n x)/2^n)$ exists for every x in E and $U(x)$ is an isometric transformation.

Proof. Put $r=\|x\|$. Then $\big|\|T(x)\|-r\big|<\epsilon$ and $\big|\|T(x)-T(2x)\|-r\big|<\epsilon$. Put also $y_0=T(2x)/2$, so that $\big|r-\|y_0\|\big|<\epsilon/2$. Consider the intersection of the two spheres: $S_1=[y;\ \|y\|<r+\epsilon]$, $S_2=[y;\ \|y-2y_0\|<r+\epsilon]$. Now $T(x)$ belongs to this intersection, and for any point y of $S_1\cap S_2$ we have

$$2\|y-y_0\|^2=2\|y\|^2+2\|y_0\|^2-4(y,y_0);$$
$$\|y-2y_0\|^2=\|y\|^2+4\|y_0\|^2-4(y,y_0)<(r+\epsilon)^2$$

and $\|y\|^2<(r+\epsilon)^2$. It follows that

$$\begin{aligned}2\|y-y_0\|^2&<(r+\epsilon)^2+\|y\|^2-2\|y_0\|^2<2(r+\epsilon)^2-2\|y_0\|^2\\&<2(r+\epsilon)^2-2(r-\epsilon/2)^2=6\epsilon r+3\epsilon^2/2.\end{aligned}$$

Hence, $\|T(x)-T(2x)/2\|<2(\epsilon\|x\|)^{1/2}$ if $\|x\|\geqq\epsilon$, and $\|T(x)-T(2x)/2\|<2\epsilon$ in the contrary case.

Therefore, for all x in E the inequality

$$\|T(x/2)-T(x)/2\|<2^{-1/2}k(\|x\|)^{1/2}+2\epsilon \tag{1}$$

is satisfied, where $k=2\epsilon^{1/2}$. Now let us make the inductive assumption

$$\|T(2^{-n}x)-2^{-n}T(x)\|<2^{-n/2}k(\|x\|)^{1/2}\left(\sum_{i=0}^{n-1}2^{-i/2}\right)+(1-2^{-n})\cdot 4\epsilon. \tag{2}$$

The inequality (2) is true for $n=1$. Assuming it true for any particular value of n we shall prove it for $n+1$.

Dividing the inequality (2) by 2 we have

$$\begin{aligned}&\|T(2^{-n}x)/2-2^{-n-1}T(x)\|\\&\qquad<2^{-(n+1)/2}k(\|x\|)^{1/2}\left(\sum_{i=1}^{n}2^{-i/2}\right)+(1/2-2^{-n-1})\cdot 4\epsilon.\end{aligned}$$

Replacing x by $2^{-n}x$ in the inequality (1) we get

$$\|T(2^{-n-1}x)-T(2^{-n}x)/2\|<2^{-(n+1)/2}k(\|x\|)^{1/2}+2\epsilon.$$

On adding the last two inequalities we obtain

$$\begin{aligned}&\|T(2^{-n-1}x)-2^{-n-1}T(x)\|\\&\qquad<2^{-(n+1)/2}k(\|x\|)^{1/2}\left(\sum_{i=0}^{n}2^{-i/2}\right)+(1-2^{-n-1})\cdot 4\epsilon.\end{aligned}$$

This proves the induction. Therefore inequality (2) is true for all x

in E and for $n=1, 2, 3, \cdots$. If we put $a = k\sum_{i=0}^{\infty} 2^{-i/2}$, we have

$$\|T(2^{-n}x) - 2^{-n}T(x)\| < 2^{-n/2}a(\|x\|)^{1/2} + 4\epsilon.$$

Hence, if m and p are any positive integers,

$$\begin{aligned}&\|2^{-m}T(2^m x) - 2^{-m-p}T(2^{m+p}x)\| \\ &\qquad = 2^{-m}\|T(2^{m+p}x/2p) - 2^{-p}T(2^{m+p}x)\| < 2^{-m/2}a(\|x\|)^{1/2} + 2^{2-m}\epsilon,\end{aligned}$$

for all x in E.

Therefore since E is a complete space, the limit $U(x) = \lim_{n\to\infty} (T(2^n x)/2^n)$ exists for all x in E.

To prove that $U(x)$ is an isometry, let x and y be any two points of E. Divide the inequality

$$\left| \|T(2^n x) - T(2^n y)\| - 2^n\|x - y\| \right| < \epsilon$$

by 2^n and take the limit as $n \to \infty$. The result is $\|U(x) - U(y)\| = \|x - y\|$. This completes the proof of Theorem 1.

THEOREM 2. *Let T satisfy the hypotheses of Theorem* 1 *and let u and x be any points of E such that $\|u\| = 1$ and $(x, u) = 0$. Then $|(T(x), U(u))| \leq 3\epsilon$, where $U(x)$ is defined as in the statement of Theorem* 1.

PROOF. For an arbitrary integer n put $z = 2^n u$. Let y denote an arbitrary point of the sphere S_n of radius 2^n and center at z. Then $\|y - z\| = \|z\|$ and it follows that $(y, u) = 2^{-n-1}(y, y)$. Since T is an ϵ-isometry, $\|T(y) - T(z)\| = \eta(y, z) + \|T(z)\|$ where $|\eta(y, z)| < 2\epsilon$.

The last equality may be written

$$2(T(y), T(z)) = (T(y), T(y)) - 2\eta\|T(z)\| - \eta^2.$$

Dividing by 2^{n+1} and remembering that $z = 2^n u$, we obtain the equality

$$(T(y), 2^{-n}T(2^n u)) = \frac{1}{2^{n+1}}\left[(T(y), T(y)) - \eta^2\right] - \eta\left\|\frac{T(2^n u)}{2^n}\right\|. \tag{3}$$

Now let x be any point of the hyperplane $(x, u) = 0$.

Then $y = x + ru$, where $r = 2^n - (2^{2n} - \|x\|^2)^{1/2}$, is a point of the sphere S_n. For,

$$\begin{aligned}\|y - z\|^2 &= (y, y) - 2(y, z) + (z, z) \\ &= (x, x) + r^2 - 2(x, z) - 2(u, z)\cdot r + (z, z) \\ &= r^2 - 2^{n+1}r + \|x\|^2 + \|z\|^2 = \|z\|^2.\end{aligned}$$

Moreover, $\|y-x\| = r \to 0$ as $n \to \infty$. By Theorem 1, $t = \lim_{n\to\infty} 2^{-n}T(2^n u)$ exists and is a unit vector. Finally, for an arbitrary positive δ and n sufficiently large, one can easily establish the following inequalities by means of equality (3) and the above remarks:

$$\begin{aligned}
|(T(x), t)| &\leqq \left|\left(T(x), t - \frac{T(2^n u)}{2^n}\right)\right| + \left|\left(T(y), \frac{T(2^n u)}{2^n}\right)\right| \\
&\quad + \left|\left(T(x) - T(y), \frac{T(2^n u)}{2^n}\right)\right| \\
&< \|T(x)\| \cdot \left\| t - \frac{T(2^n u)}{2^n} \right\| + \frac{\delta}{2} + 2\epsilon \left\| \frac{T(2^n u)}{2^n} \right\| \\
&\quad + \|T(x) - T(y)\| \cdot \left\| \frac{T(2^n u)}{2^n} \right\| < \delta + 3\epsilon(1+\delta).
\end{aligned}$$

It follows that

$$|(T(x), U(u))| = |(T(x), t)| \leqq 3\epsilon.$$

THEOREM 3. *Let $T(x)$ satisfy again the hypotheses of Theorem 1, and let it take E into the whole of E. Then the transformation $U(x)$ also takes E into the whole of E.*

PROOF. For each point z of E, let $T^{-1}(z)$ denote any point whose T-image is z. Then $T^{-1}(z)$ is an ϵ-isometry of E. By Theorem 1, the limit $U^*(z) = \lim_{n\to\infty} (T^{-1}(2^n z)/2^n)$ exists, and U^* is an isometry of E. Now clearly

$$\begin{aligned}
\|2^n z - T(2^n U^*(z))\| &= \left\| T\left(2^n \frac{T^{-1}(2^n z)}{2^n}\right) - T(2^n U^*(z)) \right\| \\
&< 2^n \left\| \frac{T^{-1}(2^n z)}{2^n} - U^*(z) \right\| + \epsilon.
\end{aligned}$$

On dividing by 2^n and letting $n \to \infty$, we see, for each point z of E, that $z = UU^*(z)$. Therefore $U(E) = E$.

THEOREM 4. *Let E be a complete abstract Euclidean vector space. If $T(x)$ is an ϵ-isometry which takes E into the whole of E such that $T(0) = 0$, then the transformation $U(x) = \lim_{n\to\infty} (T(2^n x)/2^n)$ is an isometry of E into the whole of itself, and the inequality $\|T(x) - U(x)\| < 10\epsilon$ is satisfied for all x in E.*

PROOF. For a given point $x \neq 0$ let M denote the linear manifold orthogonal to x. By Theorem 3, U is an isometric transformation which takes E into the whole of E. Hence $U(M)$ is the linear manifold orthogonal to $U(x)$. Let w be the projection of $T(x)$ on $U(M)$. If $w=0$ put $t=0$. Otherwise put $t=w/\|w\|$. In either case (cf. Theorem 2), the inequality $|(T(x), t)| \leqq 3\epsilon$ is satisfied. Put $v=(1/\|x\|)\,U(x)$. Then v is a unit vector orthogonal to t and is coplanar with $T(x)$ and t.

Hence, by the pythagorean theorem we have the identity:

$$\|T(x) - U(x)\|^2 = (T(x), t)^2 + [\|x\| - (T(x), v)]^2. \tag{4}$$

Let $z_n = 2^n x$ and if the projection w_n of $T(z_n)$ on $U(M)$ is not zero, put $t_n = w_n/\|w_n\|$. Otherwise we shall put $t_n = 0$. In either case $(t_n, v) = 0$, and $|(T(z_n), t_n)| \leqq 3\epsilon$. If $\|T(z_n)\| < 3\epsilon$, it is obvious that $\|T(z_n)\| - |(T(z_n), v)| \leqq 3\epsilon$. If $\|T(z_n)\| \geqq 3\epsilon$, we have $0 \leqq \|T(z_n)\| - |(T(z_n), v)| = \|T(z_n)\| - (\|T(z_n)\|^2 - (T(z_n), t_n)^2)^{1/2} \leqq 3\epsilon$.

Hence the inequality:

$$\big|\, \|z_n\| - |(T(z_n), v)| \,\big| < 4\epsilon \tag{5}$$

is satisfied, since $\|z_n\| < \|T(z_n)\| + \epsilon$.

Two cases arise. If $(T(x), v) \geqq 0$, we put $n=0$ in the inequality (5) and use the identity (4) to obtain the inequality $\|T(x) - U(x)\| < 5\epsilon$. If $(T(x), v) < 0$, then for some integer $m \geqq 0$ we must have $(T(z_m), v) < 0$ and $((T(2z_m), v) \geqq 0$, since $(U(x), v)$ is positive and $U(x) = \lim_{n\to\infty} (T(z_n)/2^n)$. Hence, by inequality (2),

$$\|T(2z_m) - T(z_m)\| \geqq (T(2z_m), v) - (T(z_m), v) > 3\|z_m\| - 8\epsilon.$$

But we know that $\|T(2z_m) - T(z_m)\| < \|z_m\| + \epsilon$. Therefore,

$$\|x\| \leqq \|z_m\| < (9/2)\epsilon, \quad \text{and} \quad \|T(x) - U(x)\| < 2\|x\| + \epsilon \leqq 10\epsilon.$$

In order to prove the above theorem we had to assume that $T(x)$ takes E into itself. We now show that the theorem is not always true for ϵ-isometric transformations of one Euclidean space into *part* of another. Consider the transformation $T(x)$ of the real axis into a subset of the plane defined as follows: the coordinates x, y of $T(x)$ are $(x, 0)$ for $x \leqq 1$, and $(x, c \cdot \log x)$ for $x > 1$. It is easy to verify that T will be an ϵ-isometry if we choose c in such a way that $\epsilon > c^2 \cdot \max_{x>1}((\log x)^2/(2x-2))$.

On the other hand, $T(x)$ obviously cannot approximate an isometry in the sense of our theorem.

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PARALLELISM IN THE TENSOR ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

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In the extensions of classical differential geometry associated with ordinary differential equations the connection coefficients may be chosen arbitrarily, subject to certain laws of transformation under change of coordinates. But the calculations of differential invariants become decidedly simpler if the choice is made in a particular way, so as to reduce the number of "invariants of the connection" to a minimum, leaving only the intrinsic invariants of the system. For partial differential equations, no such intrinsic choice had hitherto been found, though something of the sort is badly needed inasmuch as the mass of calculations is far more complicated than for any system of ordinary differential equations.

Given the system of partial differential equations

$$
\begin{gathered}
\frac{\partial^{q+1} x^i}{\partial u^{\alpha_1} \cdots \partial u^{\alpha_{q+1}}} + \overset{i}{H}_{\alpha_1 \cdots \alpha_{q+1}}(u, x, \overset{i}{p}_{\alpha_1} \cdots \overset{i}{p}_{\alpha_1, \cdots, \alpha_q}) = 0; \\
\overset{i}{p}_{\alpha} = \frac{\partial x^i}{\partial u^\alpha}, \cdots, \overset{i}{p}_{\alpha_1 \cdots \alpha_q} = \frac{\partial \overset{i}{p}_{\alpha_1 \cdots \alpha_{q-1}}}{\partial u^{\alpha_q}}, \\
i, j, k, \cdots = 1, 2, \cdots, n; \alpha, \beta, \gamma, \cdots = 1, 2, \cdots, m,
\end{gathered}
\tag{1}
$$

the complete set of tensor operators and tensor differential invariants associated with the "path-space" of this system may be built up as follows. We assume that (a) the equations (1) have a manifold of solutions; (b) they are tensorial under nonsingular point-transformations in both the x and the u coordinates; (c) the equations both of the x-variation and the u-variation of (1) are tensor-invariant and (d) there exists a tensorial differentiation operator

$$
D_\alpha T^{\cdots}_{\cdots} = \partial_\alpha T^{\cdots}_{\cdots} + \gamma^{i}_{\alpha r} T^{r \cdots}_{\cdots} - \gamma^{r}_{\alpha j} T^{\cdots}_{r \cdots} + \Gamma^{\sigma}_{\alpha \rho} T^{\cdots \rho}_{\cdots} - \Gamma^{\rho}_{\alpha \sigma} T^{\cdots}_{\cdots \rho}
\tag{2}
$$

(one summation for every index of the tensor $T^{\cdots}_{\cdots}$), where

$$
\partial_\alpha = \frac{\partial}{\partial u^\alpha} + \overset{i}{p}_{\alpha} \frac{\partial}{\partial x^i} + \overset{i}{p}_{\alpha\beta} \frac{\partial}{\partial \overset{i}{p}_{\beta}} + \cdots - \overset{i}{H}_{\alpha\beta_1 \cdots \beta_q} \frac{\partial}{\partial \overset{i}{p}_{\beta_1 \cdots \beta_q}} .
$$

For the tensor invariance of this operator, it is necessary and sufficient that the connection coefficients $\gamma^{i}_{\alpha j}$, $\Gamma^{\alpha}_{\beta\nu}$ obey the transformation laws:

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$$(3)\quad \left\{\begin{aligned} \overset{s}{\gamma}_{\sigma j}\frac{\partial \bar{x}^i}{\partial x^s}\,\frac{\partial u^\sigma}{\partial \bar{u}^\alpha} &= \overset{i}{\bar{\gamma}}_{\alpha s}\frac{\partial \bar{x}^s}{\partial x^j} + \overset{s}{p}_\sigma\,\frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^s}\,\frac{\partial u^\sigma}{\partial \bar{u}^\alpha},\\ \overline{\Gamma}^{\sigma}_{\alpha\beta}\frac{\partial u^\nu}{\partial \bar{u}^\sigma} &= \overset{\nu}{\Gamma}_{\sigma\tau}\frac{\partial u^\sigma}{\partial \bar{u}^\alpha}\,\frac{\partial u^\tau}{\partial \bar{u}^\beta} + \frac{\partial^2 u^\nu}{\partial \bar{u}^\alpha \partial \bar{u}^\beta}\,. \end{aligned}\right.$$

Some fundamental tensor-invariants appear when we put the equations (1) and their equations of variation into an invariantive form by aid of the D operator, keeping in mind that p^i_α is itself a fundamental invariant. For the differential operators, we note that $\partial/\partial p^i_{\alpha_1\cdots\alpha_q}$ is always a tensorial operator, and alternate this with D_α, contracting for the index α. This gives, on discarding any additive tensorial terms, a new operator in which the leading differentiation is of one order lower than in the previous case. The process may be repeated till we reach an operator which has only one (covariant) Latin index, all Greek indices having dropped out. This gives the complete set of differential operators associated with the equations (1); further alternations then give linear combinations of the same operators with coefficients which are the complete set of differential invariants of the space. The process amounts essentially to calculating the Poisson brackets for a system of linear partial differential equations of the first order, or the alternants of a Lie group, further theoretical justification being unnecessary in view of Hilbert's fundamental theorem on invariants.

In actual practice, the process has been carried out for $q=1$ and $q=2$, but is difficult even in spite of the simplified notation; generally mistakes occur which cannot be rectified. Therefore, some special choice of connection coefficients would be desirable. This is given by the following theorem.

THEOREM 1. *The "best" intrinsic choice of connection (in the sense of reducing the amount of calculation) should be such that*

$$(4)\qquad \overset{i}{K}_{\alpha\beta j} \equiv \partial_\alpha \overset{i}{\gamma}_{\beta j} - \partial_\beta \overset{i}{\gamma}_{\alpha j} + \overset{\nu}{\gamma}_{\beta j}\overset{i}{\gamma}_{\alpha\nu} - \overset{\nu}{\gamma}_{\alpha j}\overset{i}{\gamma}_{\beta\nu} = 0.$$

The tensor $K^i_{\alpha\beta j}$ is therefore treated as an invariant of the connection here, merely for the sake of simplicity. But the vanishing of this invariant has more important consequences than just the reduction of bulk for our fundamental differential invariants. For this reason, I omit the actual calculations that prove Theorem 1 directly (particularly for the cases $q\geqq 2$) and take the further consequences as justification of the "best" choice.

The most important concept in classical differential geometry is the parallelism of Levi-Cività, defined originally for Riemannian spaces and immediately extended to paths with $m=1$. In that case, a vector λ^i of the path-space is said to admit parallel displacement along a curve if $D\lambda^i=0$ along the curve. In our general case the "path" has to be regarded as a surface. Suppose we start with n given components at an initial point and attempt, as in Riemannian geometry, to move the vector parallelly along a chosen path-base, here a "hypersurface," at least for $m<n$. We shall then have to solve

$$\frac{\partial \lambda^i(u)}{\partial u^\alpha} + f^i_{\alpha r}\lambda^r = 0; \qquad f^i_{\alpha j} = \gamma^i_{\alpha j} \text{ along the base.} \tag{5}$$

The conditions of integrability of (5) are, as usual,

$$\lambda^j \left\{ \frac{\partial f^i_{\beta j}}{\partial u^\alpha} - \frac{\partial f^i_{\alpha j}}{\partial u^\beta} + f^r_{\beta j} f^i_{\alpha r} - f^r_{\alpha j} f^i_{\beta r} \right\} = 0. \tag{6}$$

For proper unrestricted parallel displacement to be possible, the brackets must vanish for the given base. For a path-gometry, the least requirement is that displacement should be possible for *any path* as a base, seeing that the paths are necessary to define the operators ∂_α, and hence D_α, in (2). But, for a path, the braces in (6) are the tensor $K^i_{\alpha\beta j}$. This gives us the following theorem.

THEOREM 2. *To admit unrestricted parallel displacement of a vector along a path-base, it is necessary and sufficient that* $K^i_{\alpha\beta j}=0$ *thereon.*

This might have been regarded as an almost obvious restriction and the question then raised as to its importance, seeing that no such condition is needed for the $\Gamma^\alpha_{\beta\nu}$ coefficients of connection, that is, that parallelism along a path-base seems unnecessary for Greek-index vectors. This can be argued out at length by consideration of the difference between the dependent coordinates x and the parameters u that are our independent variables. But it seems more productive to prove:

THEOREM 3. *It is possible to make the connection coefficients* $\gamma^i_{\alpha j}=0$ *along any chosen path-base if and only if* $K^i_{\alpha\beta j}=0$ *throughout.*

This is the extension of the so-called theorem of Fermi to the path-spaces associated with partial differential equations. Its usefulness in shortening proofs and formal manipulations of differential geometry need not be explained. The actual proof is as follows. If it is possible to make $\gamma^i_{\alpha j}=0$ along *any* path, it then follows that $\partial_\nu\gamma^i_{\alpha j}=0$ also, as the operator is differentiation along the path and the connection coefficients vanish for some restricted but continuous region by hypothesis. Therefore, $K^i_{\alpha\beta j}=0$ for the chosen set of coordinates and the path in question. Now the vanishing of the tensor $K^i_{\alpha\beta j}$ does not depend upon the choice of coordinates. Furthermore, the path-space consists of the manifold swept out by the totality of paths, whence $K^i_{\alpha\beta j}=0$ throughout the space. For the converse, if $K^i_{\alpha\beta j}=0$, we can start with an ennuple of vectors $\lambda^i_{(j)}$ at a given point, and displace it along the path by parallel displacement. We know that by choosing det. $|\lambda^i_{(j)}|\neq 0$ initially the ennuple will remain a set of n independent vectors for some sufficiently restricted region. It only remains to set $\lambda^i_{(j)}=\partial x^i/\partial \bar{x}^j$ *along the base* of displacement, and this obviously admits infinitely many solutions for the transformations $\bar{x}\rightleftarrows x$, as the integrability condition is satisfied by hypothesis and we have imposed a set of initial conditions which depend upon the parameters u only. It follows at once from the transformation laws (3) that $\bar{\gamma}^i_{\alpha j}=0$ on the base *and this has been done by x-transformations alone.*

It may be remarked in conclusion that $K^i_{\alpha\beta j}=0$ identically whenever there is a single independent variable, $m=1$. This covers Riemannian geometry, its extensions to symmetric affine connections, and the case of path-spaces for *ordinary* differential equations of any order.

REFERENCES. For the basic ideas and notation for ordinary path-spaces, D. D. Kosambi, Math. Zeit. vol. 37 (1933) pp. 608–618; Quart. J. Math., Oxford Ser. vol. 6 (1935) pp. 1–12; for partial differential equations J. Indian Math. Soc. vol. 3 (1939) pp. 249–253. It may be remarked that the work of H. Hashimoto in calculating the invariants (J. Fac. Sci. Hokkaido Imp. Univ. vol. 8 (1940) pp. 163–172) is actually in error. The final results will be published in a forthcoming paper by V. Seetharaman, Proceedings of the Indian Academy of Sciences.

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NONCOMMUTATIVE VALUATIONS

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The topic of this paper is the extension of the basic facts of valuation theory to noncommutative systems.[1] The purpose of this generalization is twofold. First, the theory of valuations with commutative groups of values is placed in the framework of the theory of *l*-groups,[2] and secondly the general theory leads to the construction of a new class of infinite division algebras. These division algebras are of highly transcendental structure over their respective centers; moreover they may be considered, in special cases, as crossed transcendental extensions of other division algebras.

It is necessary to recall some facts on *l*-groups. A group Γ is called a simply ordered *l*-group if the following axioms are satisfied:

(I) There is defined a binary inclusion relation which is "homogeneous" in the sense that $\alpha \geqq \beta$ implies $\rho+\alpha+\sigma \geqq \beta+\beta+\sigma$ for all ρ, σ,

(II) Γ is a lattice with respect to the ordering relation, and

(III) given α, β, either $\alpha \geqq \beta$ or $\beta \geqq \alpha$.[3]

Furthermore $\alpha \geqq \beta$ means $\alpha \cup \beta = \alpha$. The totality of all positive elements of Γ is a semi-group and shall be denoted by Γ^+. The absolute value $|\alpha|$ of α is defined as $\alpha \cup -\alpha$. Hence $|\alpha|$ is equal to α or $-\alpha$ according as α lies in Γ^+ or the complement $\Gamma - \Gamma^+$.[4] Since Γ is simply ordered an *l*-ideal or isolated subgroup Δ which is defined by Birkhoff[5] to contain with each δ all ξ with $|\xi| < |\delta|$ may alternately be defined as follows. An isolated subgroup contains with each $\delta > 0$ all $\xi \in \Gamma^+$ satisfying $\xi < \delta$.

DEFINITION. *A one-valued function V on a division ring D upon an l-group Γ is called a valuation if the following postulates hold:*

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[1] For results on valuation theory see, for example, W. Krull, *Allgemeine Bewertungstheorie*, J. Reine Angew. Math. vol. 167 (1932) pp. 160–196; O. F. G. Schilling, *Arithmetic in fields of formal power series in several variables*, Ann. of Math. vol. 38 (1937) pp. 551–576; Saunders MacLane, *The uniqueness of the power series representation of certain fields with valuations*, Ann. of Math. vol. 39 (1938) pp. 370–382; A. Ostrowski, *Untersuchungen zur arithmetischen Theorie der Körper*, Math. Zeit. vol. 39 (1935) pp. 269–404; I. Kaplansky, *Maximal fields with valuations*, Duke Math. J. vol. 9 (1942) pp. 303–321.

[2] Garrett Birkhoff, *Lattice-ordered groups*, Ann. of Math. vol. 43 (1942) pp. 298–331.

[3] Birkhoff, loc. cit. pp. 299, 300, 312.

[4] Birkhoff, loc. cit. pp. 302, 308, 309.

[5] Birkhoff, loc. cit. pp. 309–311.

(i) *Each* $\alpha\in\Gamma$ *has the form* Va *for at least one element* $a\in D$.
(ii) $V(0)>\alpha$ *for all* $\alpha\in\Gamma^+$.
(iii) $V(ab)=V(a)+V(b)$.
(iv) $V(a+b)\geqq\min\,[V(a),\ V(b)]$.

REMARK 1. If Γ is noncommutative then D is certainly not a field. This follows, by contradiction, from (i) and (iii).

REMARK 2. All elements u of D with $V(u)=0$ form a normal subgroup U^* of D for which $D^*/U\cong\Gamma$. This holds since V is a homomorphism of D^* upon Γ.

As in the ordinary valuation theory it is now shown, using (iii) and (iv), that the totality $\mathfrak{O}$ of all $a\in D$ with $Va\geqq 0$ is a ring, the valuation ring of V.

LEMMA 1. *For* a, b *in* $\mathfrak{O}$, *the following statements are equivalent*:
(i) $a=c_1b$ *with* c_1 *in* $\mathfrak{O}$.
(ii) $a=bc_2$ *with* c_2 *in* $\mathfrak{O}$.
(iii) $V(a)\geqq V(b)$.

PROOF. Application of V to (i) or (ii) yields (iii). Conversely (iii) implies the existence of elements γ_1 and γ_2 in Γ^+ such that $V(a)=\gamma_1+V(b)$ and $V(a)=V(b)+\gamma_2$. Then there exist elements d_1 and d_2 with $V(d_1)=\gamma_1$ and $V(d_2)=\gamma_2$. Hence $a=u_1d_1b=bd_2u_2$ where u_1d_1 and d_2u_2 lie in $\mathfrak{O}$, u_1 and u_2 in U.

LEMMA 2. *Each ideal* $\mathfrak{A}$ *of* $\mathfrak{O}$ *is two-sided.*

PROOF. Suppose that $\mathfrak{A}$ is a left ideal, that is, $\mathfrak{O}\mathfrak{A}\subseteq\mathfrak{A}$. Let $V\mathfrak{A}$ be the set of all $V(a)$ with $a\in\mathfrak{A}$. This set is an upper class of Γ^+ as follows by Lemma 1. Now let $d=\sum_{j=1}^{N}a_jb_j$ be an arbitrary element of the set $\mathfrak{A}\mathfrak{O}$; $a_j\in\mathfrak{A}$, $b_j\in\mathfrak{O}$. Then $a_jb_j=b_j'a_j$, $b_j'\in\mathfrak{O}$ by Lemma 1. Consequently $d\in\mathfrak{A}$. Thus $\mathfrak{A}$ is a right ideal.

Now let $\mathfrak{P}$ be the ideal of elements $b\in\mathfrak{O}$ with $Vb>0$. Moreover, $\mathfrak{O}/\mathfrak{P}$ is a division ring $\mathbf{D}$ for the elements of $\mathbf{D}-0=\mathbf{D}^*$ can be represented by elements in the multiplicative group U. As usual $\mathfrak{P}$ is termed the prime ideal of V.

LEMMA 3. *The sets* $\mathfrak{O}$ *and* $\mathfrak{P}$ *are invariants for the group of inner automorphisms of* D.

PROOF. Let $d\in D^*$ and $a\in\mathfrak{O}$. Then $V(d^{-1}ad)=-V(d)+V(a)+V(d)\geqq 0$ by the invariance of Γ^+. Hence $d^{-1}ad\in\mathfrak{O}$. Next one finds $V(d^{-1}ad)>0$ for $a\in\mathfrak{P}$. The possibility $V(d^{-1}ad)=0$ is excluded for $-V(d)+V(a)+V(d)=0$ would imply $V(a)+V(d)=V(d)$, $V(a)=0$, in contradiction to the assumption on a.

LEMMA 4. *Let* $a, b \in D^*$, *then at least one of the pairs* $\{ab^{-1}, b^{-1}a\}$, $\{a^{-1}b, ba^{-1}\}$ *lies in* $\mathfrak{O}$.

PROOF. Suppose that $ab^{-1} \in \mathfrak{O}$. Then also $b^{-1}a \in \mathfrak{O}$ by Lemma 1. Now assume $ab^{-1} \notin \mathfrak{O}$. Then, by the definition of $\mathfrak{O}$, $0 > V(ab^{-1})$. Consequently $-V(a)+V(b) > 0$ and thus $a^{-1}b$ and ba^{-1} lie in $\mathfrak{O}$ by Lemma 1. Moreover, $b^{-1}a \notin \mathfrak{O}$ for otherwise $-V(b)+V(a) \geqq 0$ or $V(a)-V(b) \geqq 0$ contrary to the assumption on the elements a and b.

REMARK. If $d \in D$ then either $d \in \mathfrak{O}$ or $d = a^{-1}$ where $a \in \mathfrak{O}$. For the latter observe that $0 > V(d)$ implies $-V(d) > V(d^{-1}) + V(d) = V(1) = 0$.

LEMMA 5. *If* $\mathfrak{O}$ *is an invariant subring of* D *such that for any* $a \in D$, *either* a *or* a^{-1} *is in* $\mathfrak{O}$, *then* $\mathfrak{O}$ *is a valuation ring for some valuation of* D.

PROOF. The invariance of the ring $\mathfrak{O}$ implies the invariance of its group of units U. For if $u \in U$, then $d^{-1}ud$ and $d^{-1}u^{-1}d = (d^{-1}ud)^{-1}$ both lie in $\mathfrak{O}$ for every $d \in D$, that is, $d^{-1}ud \in U$. Now let $\mathfrak{P}$ be the complement $\mathfrak{O} - U$. The set $\mathfrak{P}$ is invariant under the group of inner transformations of D. To define the valuation V for which $\mathfrak{O}$ is the valuation ring set $V(a) = aU$ for $a \in D^*$ and $V(u) = 0$ for $u \in U$. The factor group D^*/U may then be considered as an additive group Γ. The group turns out to be a simply ordered l-group, if $\alpha = V(a) > \beta = V(b)$ in case ab^{-1} and $b^{-1}a$ lie in $\mathfrak{P}$.[6] As in the commutative theory it now follows that $\mathfrak{O}$ is the valuation ring for V.[7]

The preceding properties of the valuation V lead to another description of a valuation. By the homomorphism $a \rightarrow a \bmod \mathfrak{P} = H(a) = \mathbf{a} \in \mathbf{D}$ exactly the valuation ring $\mathfrak{O}$ is mapped upon $\mathbf{D}$. It is customary to agree that $H(d) = \infty$ if $d \notin \mathfrak{O}$. This can happen only if $H(d^{-1}) = 0$, for $d \notin \mathfrak{O}$ means $d = a^{-1}$ with $V(a) > 0$, that is, $a \in \mathfrak{P}$, and thus $H(a) = 0$. Finally $H(d^{-1}ad) \neq \infty$ if and only if $H(a) \neq \infty$ for inner automorphisms preserve non-negativeness.

THEOREM 1. *Let* H *be a homomorphism of a division ring* D *upon a division ring* $\mathbf{D}$ *and a symbol* ∞ *so that* (i) $H(a+b) = H(a) + H(b)$ *and* $H(ab) = H(a)H(b)$ *for any pair* $a, b \in D$ *with* $H(a) \neq \infty$, $H(b) \neq \infty$, (ii) $H(a) = \infty$ *if and only if* $H(a^{-1}) = 0$, *and* (iii) $H(d^{-1}ad) \neq \infty$ *for all* $d \in D^*$ *if and only if* $H(a) \neq \infty$. *Then* H *arises from a valuation* V *of* D.

PROOF. Denote by $H^{-1}(S)$ the inverse image of a subset $S \subseteq \mathbf{D}$. Then $H^{-1}(\mathbf{D}) = \mathfrak{O}$ is a ring with $H^{-1}(\mathbf{D}-0) = U$ for its subgroup of units. Certainly $\mathfrak{O}$ and U are invariant under the group of inner transformations. For the first assertion observe that $a\in\mathfrak{O}$ means $H(a)\neq\infty$. Thus $H(d^{-1}ad)\neq\infty$ for each $d\in D^*$ by assumption (iii), consequently $d^{-1}ad\in\mathfrak{O}$. In the second case observe $H(u)\neq 0$ for $u\in U$. If $d^{-1}ud\notin U$ then $H(d^{-1}ud)=0$, consequently $H(d^{-1}u^{-1}d)=\infty$, by assumption (ii). Consequently $H(u^{-1})=\infty$, by (iii), that is, $u\notin\mathfrak{O}$ contrary to the assumption on u. Now let $d\in D^*$, then either $H(d)\neq\infty$ or $H(d)=\infty$. By construction of $\mathfrak{O}$ we have $d\in\mathfrak{O}$ if $H(d)\neq\infty$ and, by (ii), $d^{-1}=a\in\mathfrak{O}$ if $H(d)=\infty$. Thus each element of D^* is a quotient of elements in $\mathfrak{O}$. Finally let $a, b\in D^*$, then at least one of the pairs $\{ab^{-1}, b^{-1}a\}$, $\{a^{-1}b, ba^{-1}\}$ lies in $\mathfrak{O}$. Without loss of generality we may assume $H(ab^{-1})\neq\infty$. Then $H(b^{-1}ab^{-1}b)=H(b^{-1}a)\neq\infty$ by assumption (iii). Therefore $\mathfrak{O}$ is a valuation ring by Lemma 5.

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[6] Observe, for example, that $V(a) \geqq 0$ if and only if $a \in \mathfrak{O}$, $d^{-1}ad \in \mathfrak{O}$ for $d \in D^*$ implies that Γ is homogeneous.

[7] Observe that for any a, b in D^* either $a^{-1}b$ or $b^{-1}a$ lies in $\mathfrak{O}$. Say the former holds, then $a(a^{-1}b)a^{-1} = ba^{-1}$ also lies in $\mathfrak{O}$.

Let Δ be an isolated subgroup of Γ. Then a proper restatement of the commutative proof may be used to show the following lemma.

LEMMA 6. *The prime ideals $\mathfrak{p}$ of the valuation ring $\mathfrak{O}$ are in 1-1 correspondence with the isolated subgroups Δ of Γ.*

COROLLARY. *The invariant isolated subgroups of Γ are in 1-1 correspondence with the prime ideals of $\mathfrak{O}$ which are invariant under all inner automorphisms of D.*

LEMMA 7. *Each invariant prime ideal $\mathfrak{p}$ of $\mathfrak{O}$ determines a quotient ring $\mathfrak{O}_\mathfrak{p}$ in which the extended ideal $\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ is a prime ideal whose residue class ring $\mathfrak{O}_\mathfrak{p}/\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ is a division ring.*

PROOF. Let $\mathfrak{O}_\mathfrak{p}$ be the set $\{ab^{-1}; a\in\mathfrak{O}, b\in\mathfrak{O}-\mathfrak{p}\}$. Then $\mathfrak{O}_\mathfrak{p}$ may alternately be defined as the set of quotients $b^{-1}a$, say $\mathfrak{O}'$. Observe $b^{-1}a=a^{-1}(ab^{-1})a=a(a^{-1}ba)^{-1}$ for $a\neq\mathfrak{O}$ with $a^{-1}ba\notin\mathfrak{O}-\mathfrak{p}$ for $\mathfrak{p}$ is an invariant prime ideal. Thus $\mathfrak{O}'\subseteq\mathfrak{O}_\mathfrak{p}$ and conversely. Next $\mathfrak{O}_\mathfrak{p}$ is an invariant subring of D, for $d^{-1}(ab^{-1})d=(d^{-1}ad)(d^{-1}bd)^{-1}$ where $d^{-1}ad\in\mathfrak{O}$, $d^{-1}bd\in\mathfrak{O}-\mathfrak{p}$ with $d\in D^*$ since $\mathfrak{O}$ and $\mathfrak{p}$ are invariant. It remains to show that $\mathfrak{O}_\mathfrak{p}$ is a ring. Let $a_1b_1^{-1}$, $a_2b_2^{-1}$ be two elements of $\mathfrak{O}_\mathfrak{p}$. Without loss of generality it may be assumed that $V(b_2)\leqq V(b_1)$, then $a_1b_1^{-1}+a_2b_2^{\cdot 1}=(a_1+a_2b_2^{-1}b_1)b_1^{-1}$ where $b_2^{-1}b_1\in\mathfrak{O}$. Consequently the sum lies in $\mathfrak{O}_\mathfrak{p}$. For the product observe $a_1b_1^{-1}\cdot a_2b_2^{-1}=a_1b_1^{-1}b_2^{*-1}a_2$ with $b_2^*\in\mathfrak{O}-\mathfrak{p}$ by the invariance of $\mathfrak{O}$ and $\mathfrak{p}$. Next $a_1(b_2^*b_1)^{-1}a_2=b_3^{-1}a_1a_2^*\in\mathfrak{O}_\mathfrak{p}$ with $b_3\in\mathfrak{O}-\mathfrak{p}$, $a_2^*\in\mathfrak{O}$, by the invariantive properties.

Consider next the extended set $\mathfrak{p}\mathfrak{O}_\mathfrak{p}=\{\sum_{j=1}^{N}p_ja_jb_j^{-1}$ with $p_j\in\mathfrak{p}$, $a_jb_j^{-1}\in\mathfrak{O}_\mathfrak{p}\}$. Then $p_ja_jb_j^{-1}=x_jb_j$ with $x_j=p_ja_jb_j^{-1}p_j^{-1}=(p_ja_jp_j^{-1})(p_j^{-1}b_jp_j)^{-1}$ with the first factor in $\mathfrak{O}$ and the second in $\mathfrak{O}-\mathfrak{p}$. Thus $\mathfrak{p}\mathfrak{O}_\mathfrak{p}=\mathfrak{O}_\mathfrak{p}\mathfrak{p}$. The general element $\sum p_ja_jb_j^{-1}$ may be written as $p_1\sum(p_1^{-1}p_j)a_jb_j^{-1}=p_1s_1$, $s_1\in\mathfrak{O}_\mathfrak{p}$, for it may be assumed without loss of generality that

$V(p_1) \leqq V(p_j), j=2, \cdots, N$. As an immediate consequence it follows that $\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ is an invariant set for $d^{-1}(p_1s_1)d=(d^{-1}p_1d)(d^{-1}s_1d)\in\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ since $\mathfrak{p}$ and $\mathfrak{O}_\mathfrak{p}$ were recognized to be invariant sets. Let p_1s_1, $p_2s_2\in\mathfrak{p}\mathfrak{O}_\mathfrak{p}$. Without loss of generality it may be assumed that $V(p_1)\leqq V(p_2)$. Then $p_1s_1+p_2s_2=p_1(s_1+p_1^{-1}p_2s_2)$ where $p_1^{-1}p_2\in\mathfrak{p}$, therefore $s_1+p_1^{-1}p_2s_2\in\mathfrak{O}_\mathfrak{p}$ and $p_1s_1+p_2s_2\in\mathfrak{p}\mathfrak{O}_\mathfrak{p}$. Finally $\mathfrak{p}\mathfrak{O}_\mathfrak{p}\cdot\mathfrak{O}_\mathfrak{p}\subseteq\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ and $\mathfrak{O}_\mathfrak{p}\cdot\mathfrak{O}_\mathfrak{p}\mathfrak{p}\subseteq\mathfrak{O}_\mathfrak{p}\mathfrak{p}$, that is, $\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ is a two-sided ideal. To show that $\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ is a prime ideal it suffices to prove that $a_1b_1^{-1}$, $a_2b_2^{-1}\in\mathfrak{O}_\mathfrak{p}-\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ implies $a_1b_1^{-1}\cdot a_2b_2^{-1}\in\mathfrak{O}_\mathfrak{p}-\mathfrak{p}\mathfrak{O}_\mathfrak{p}$. Observe a_1, $a_2\in\mathfrak{O}-\mathfrak{p}$, that is, $a_1a_2\in\mathfrak{O}-\mathfrak{p}$. Therefore $a_1b_1^{-1}\cdot a_2b_2^{-1}=a_1a_2(b_2\cdot a_2^{-1}b_1a_2)^{-1}$ where b_2, $a_2b_1a_2^{-1}\in\mathfrak{O}-\mathfrak{p}$ for $\mathfrak{p}$ is an invariant prime ideal of $\mathfrak{O}$. Consequently $a_1b_1^{-1}\cdot a_2b_2^{-1}\in\mathfrak{O}_\mathfrak{p}-\mathfrak{p}\mathfrak{O}_\mathfrak{p}$. Let $\mathfrak{O}_\mathfrak{p}/\mathfrak{p}\mathfrak{O}_\mathfrak{p}=D_0$, and suppose $c_0\in D_0^*$. Then there exist a, b so that $(ab^{-1})_0=c_0$ with $a\notin\mathfrak{p}$. Hence $(ab^{-1})^{-1}ba^{-1}\notin\mathfrak{p}\mathfrak{O}_\mathfrak{p}$, thus $(ba^{-1})_0\in D_0$ and $c_0(ba^{-1})_0=(1)_0$.

Now it is possible to carry over the results of the commutative case.

Lemma 8. *Each invariant prime ideal* $\mathfrak{p}$ *of* $\mathfrak{O}$ *with the associated isolated subgroup* Δ *of* Γ *determines a valuation* V_Δ *in* $D_\Delta=\mathfrak{O}_\mathfrak{p}/\mathfrak{p}\mathfrak{O}_\mathfrak{p}$ *whose valuation ring is* $\mathfrak{O}/\mathfrak{p}$ *and whose value group is* Δ. *More specifically,* $V_\Delta(a_0)=V(a)$ *where* $a \bmod \mathfrak{p}\mathfrak{O}_\mathfrak{p}=a_0\in D_\Delta$.

Lemma 9. *The quotient ring* $\mathfrak{O}_\mathfrak{p}$ *is a valuation ring of* D *whose value group is* Γ/Δ *letting* $V_{\Gamma/\Delta}(a)=V(a) \bmod \Delta$, *and whose associated residue algebra is* D_Δ.

The preceding lemmas may be combined so as to give the following theorem.

Theorem 2. *Relative to each isolated invariant subgroup* Δ *of* Γ *the valuation* V *can be split into* (i) *a valuation* $V_{\Gamma/\Delta}$ *of* D *and* (ii) *a valuation* V_Δ *of the residue algebra* D_Δ *for* $V_{\Gamma/\Delta}$ *with its value group equal to* Δ.

Remark. The prime ideals $\bar{\mathfrak{p}}_\nu$ of D_Δ in $\mathfrak{O}/\mathfrak{p}$ arise exactly as the homomorphic images of the prime ideals $\mathfrak{p}_\nu\subseteq\mathfrak{O}$ with $\mathfrak{p}_\nu\supset\mathfrak{p}$. Since every invariant subgroup of Δ is not an invariant subgroup of Γ it will in general not be true that an invariant prime $\bar{\mathfrak{p}}_\nu$ is the homomorphic image of an invariant prime ideal $\mathfrak{p}_\nu$ of $\mathfrak{O}$.

Theorem 3. *Suppose* D *has a valuation* V_1 *with value group* Γ_1 *and residue class algebra* D_1. *Then each valuation* V_2 *in* D_1 *with value group* Γ_2 *whose valuation ring has an invariant image in* D *determines a valuation* V *on* D *with value group* Γ *so that* Γ *contains an order isomorphic image of* Γ_2 *with* $\Gamma/\Gamma_2\cong\Gamma_1$. *Moreover the residue class ring of* D *with respect to* V *is equal to the residue class ring of* D_1 *with respect to* V_2.

The proof of the theorem involves a direct restatement of the proof for the parallel theorem involving a commutative field.

The preceding discussion of a division algebra with a valuation V may be utilized to establish the existence of a wide variety of algebras with prescribed value groups and algebras of residue classes. Suppose that $\bar{D}$ is a division algebra which is to be the algebra of residue classes for a valuation V with value group Γ. Noting that in a given algebra D the elements $d \in D^*$ induce by $a \pmod{\mathfrak{P}} \to d^{-1}ad \pmod{\mathfrak{P}}$, $a \in \mathfrak{O}$, automorphisms on the algebra of residues $\bar{D}$, one is led to the following construction. Assume that each $\gamma \in \Gamma^+$ induces an automorphism $\bar{a} \to \bar{a}^\gamma$ in $\bar{D}$ and let to γ be associated a symbol $t(\gamma)$. Let D be isomorphic to $\bar{D}$. Consider then a group extension of D^* by the group Γ with the defining relations

$$a \to a^\gamma = t(\gamma)^{-1}at(\gamma), \qquad t(0) = 1, \qquad t(\gamma)^{-1} = t(-\gamma),$$
$$t(\alpha)t(\beta) = f(\alpha, \beta)t(\alpha + \beta),$$

where the $f(\alpha, \beta)$ satisfy the customary relations for factor sets.[8] Now define D to be the set of all formal power series $A = \sum a_\nu t(\alpha_\nu)$ where $a_\nu \in D$ and $\{\alpha_\nu\}$ is a well ordered montonically increasing sequence in Γ with a finite first element. If $B = \sum b_\nu t(\beta_\nu)$ is another element of D then $A+B$ is to be the series obtained by adding the coefficients at identical marks. The product AB is to be defined by formal multiplication observing that $at(\alpha)bt(\beta) = ab^\alpha f(\alpha, \beta)t(\alpha+\beta)$. Thus the system D becomes a ring without divisors of zero. Next define $V(A) = \alpha_1 = V(t(\alpha_1))$ where $A = \sum a_\nu t(\alpha_\nu) = [\sum a_\nu t(\alpha_\nu)t(\alpha_1^{-1})]t(\alpha_1)$ $= [\sum a_\nu f(\alpha_\nu, -\alpha_1)t(\alpha_\nu - \alpha_1)]t(\alpha_1)$, $\alpha_\nu - \alpha_1 \geqq 0$. Then $V(a) = 0$ for $a \in D^*$. The function V satisfies the postulates for a valuation. Observe that $A = at(\alpha) + A_0$, $B = bt(\beta) + B_0$ with $V(A_0) > \alpha$, $V(B_0) > \beta$, respectively, imply $AB = ab^\alpha f(\alpha, \beta)t(\alpha+\beta) + C_0$ where $V(C_0) > \alpha+\beta$. Hence $V(AB) = \alpha+\beta$. Finally $V(A+B) \geqq \min [V(A), V(B)]$. For the proof one may assume without loss of generality that $\beta \geqq \alpha$. Then $A+B$ $= [a + A_0t(\alpha)^{-1} + bf(\beta, -\alpha)t(\beta-\alpha) + B_0t(\alpha)^{-1}]t(\alpha)$ which proves the triangle inequality. It remains to show that each element $A \in D^*$ has an inverse. Observe that each sequence $\{\sum_{j=0}^{n} C^j, n \to \infty, V(C) > 0\}$ has a limit $\sum_{j=0}^{\infty} C^j$ in the set D. Thus $(1+C)^{-1} = \sum_{j=0}(-1)^jC^j$ lies in D. Write $A = a(1+C)t(\alpha)$, $a \in D^*$, then $A^{-1} = t(-\alpha)\sum_{j=0}^{\infty}(-1)^jC^ja^{-1}$. It is shown directly that the valuation ring $\mathfrak{O} = \{\sum a_\nu t(\alpha_\nu), \alpha_\nu \geqq 0\}$

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[8] See, for example, H. Zassenhaus, *Lehrbuch der Gruppentheorie*, Leipzig, 1937; A. A. Albert, *Structure of algebras*, Amer. Math. Soc. Colloquium Publications, vol. 24, 1939. Factor sets were first used in the valuation theory by Kaplansky, loc. cit. pp. 315–317.

contains the prime ideal $\mathfrak{P}=\{\sum a_\nu t(\alpha_\nu),\ \alpha_\nu>0\}$ so that $\mathfrak{O}/\mathfrak{P}\cong D\cong\bar{D}$. Moreover $\mathfrak{O}$ and $\mathfrak{P}$ are invariant subsets of D.

DEFINITION 2. *A division algebra D is termed relatively complete with respect to a valuation V if each of its subfields K containing the center Z is relatively complete with respect to the induced valuation V_K.*

LEMMA 10. *A relatively complete division algebra of finite rank n^2 over its center Z has a commutative value group with respect to the given valuation.*

PROOF. Observe first that the valuation V induces a non-trivial valuation V_Z on Z. For let $d^m+z_1d^{m-1}+\cdots+z_m=0$, $z_j\in Z$, be the irreducible equation satisfied by an element $d\in D$ with $V(d)>0$. Then $V(z_m)=V(d)+V(d^{m-1}+z_1d^{m-2}+\cdots+z_{m-1})$. Thus $V(z_m)=V_Z(z_m)>0$ in case $V_Z(z_j)\geqq 0$, $j=1,\cdots,m-1$, by the triangle inequality for valuations. In case some $V_Z(z_j)<0$ nothing is to be proved. Since Z is, by hypothesis, relatively complete with respect to V_Z the usual theory of prolongation for valuations may be applied.[9] Set $V^*(a)=n^{-1}V_Z(Na)$ where N denotes the reduced norm of D/Z. Then $V^*(a)=V(a)$ for otherwise the subfields $Z(a)$, Z would be relatively complete with respect to two inequivalent valuations. Hence Z would be algebraically complete contrary to the hypothesis that D is a division algebra.[10] Next let $a, b\in D$, then $V(ab)=V^*(ab)=V^*(ba)=V(ba)$, that is, the value group Γ of V is abelian.

DEFINITION 3. *A relatively complete division algebra D is termed algebraic if the algebra $D(a, b)$ generated by any two elements $a, b\in D$ over Z has finite rank over its center $Z(a, b)$.*

THEOREM 4. *The value group of an algebraic relatively complete division algebra D is abelian.*

PROOF. Let α, β be two elements of Γ. Suppose that a, b are any two elements of D with $V(a)=\alpha$, $V(b)=\beta$. By hypothesis the algebra $D(a, b)$ has finite rank over the relatively complete field $Z(a, b)$. Hence, by Lemma 10, $V(ab)=V(ba)$, that is, $\alpha+\beta=\beta+\alpha$.

REMARK. The preceding theorem indicates that a division algebra with a noncommutative value group must contain elements which are transcendental over its center.

ILLUSTRATIVE EXAMPLE. Let Γ be the lexicographically ordered group of motions in the plane, that is, the group of all couples of real numbers (α, β) subject to the following law of combination: if (γ, δ) is a second pair then $(\alpha, \beta)+(\gamma, \delta)=(\alpha+\gamma, e^{\gamma}\beta+\delta)$. The set Γ^{+} of positive elements consists of all couples for which either $\alpha>0$ or $\alpha=0$ and $\beta>0$. Observe the following facts: (i) $-(\alpha,\beta)=(-\alpha, -\beta e^{-\alpha})$, (ii) the set $\{(0, \beta)\}$ is an invariant isolated subgroup Δ, (iii) $\Gamma/\Delta \cong\{\alpha\}$, and (iv) Γ has no proper center. Let F be a field and t a transcendental element over F. Consider the set D of all formal power series $\sum_{(\alpha,\beta)}a_{\alpha,\beta}t^{(\alpha,\beta)}$ where $a_{\alpha,\beta}\in F$ and the elements (α, β) form well ordered monotonically increasing sequences. Set $t^{(0,0)}=1$ and $t^{(\alpha,\beta)}t^{(\gamma,\delta)}=t^{\xi}$, $\xi=(\alpha, \beta)+(\gamma, \delta)$. Then D is a division algebra with Γ as a value group. The valuation ring $\mathfrak{O}$ consists of all series $\sum a_{\alpha,\beta}t^{(\alpha,\beta)}$ where all $(\alpha, \beta)\geqq(0, 0)$; the prime ideal $\mathfrak{P}$ contains all series with $(\alpha, \beta)>(0, 0)$. Both $\mathfrak{O}$ and $\mathfrak{P}$ are invariant subsets for the group of inner automorphisms of D^*, and $\mathfrak{O}/\mathfrak{P}\cong F$. Corresponding to Δ there is an invariant prime ideal $\mathfrak{p}$ in $\mathfrak{O}$ whose associated valuation ring maps homomorphically on the field of all formal series $\overline{F}_{\Delta} = \{\sum_{\beta}a_{\beta}\bar{t}^{(0,\beta)}\} = \{\sum a_{\beta}\bar{t}_1^{\beta}\}$. Select now in D the subfield $F_{\Delta}=\{\sum a_{\beta}t^{(0,\beta)}\} = \{\sum a_{\beta}t_1^{\beta}\}$. Next set $t^{(\alpha,0)}=t_2^{\alpha}$, then $t^{(\alpha,\beta)}=t_2^{\alpha}t_1^{\beta}$. The rule of combination in Γ implies $t_1^{\beta}t_2^{\alpha}=t_2^{\alpha}t_1^{\mu}$ with $\mu=e^{\alpha}\beta$. Since each element of D may now be expressed as $\sum_{-\infty<\alpha_\nu}A_{\alpha_\nu}t_2^{\alpha_\nu}$, $A_{\alpha_\nu}\in F_{\Delta}$, $\{\alpha_\nu\}$ increasing, it can be seen that D is a crossed extension of F_{Δ} by the set $\{t_2^{\alpha}\}$. The associated factor set is equal to unity for $t_2^{\alpha}t_2^{\gamma}=t_2^{\alpha+\gamma}=t_2^{\gamma+\alpha}=t_2^{\gamma}t_2^{\alpha}$. This interpretation of the algebra D as a transcendental crossed extension of the algebra F_{Δ} by means of an extension of the value group of F_{Δ} can be generalized in several directions. As in the construction of an algebra for a given algebra of residue classes $\overline{D}$ and a value group Γ one may introduce factor sets. Moreover statements can be made for an algebra D whose value group Γ possesses a chain of normal isolated subgroups Δ_j whose factor groups Γ_j/Γ_{j+1} are isomorphic to abelian ordered groups. To obtain explicit results it is useful to assume that (i) the successive algebras of residue classes D_{j+1} have isomorphic images $\phi_j(D_{j+1})$ in D_j, (ii) the isomorphism ϕ_j commutes with the group of inner automorphisms of D_j, and (iii) D_j is maximally complete with respect to the valuation having Γ_j/Γ_{j+1} for its value group.[11]

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[9] See, for example, O. F. G. Schilling, loc. cit. pp. 568–570 and Schilling, *Normal extensions of relatively complete fields*, Amer. J. Math. vol. 65 (1943) pp. 309–334.

[10] See Albert, loc. cit.

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[11] Observe in this connection the results of Kaplansky, loc. cit.

A NOTE ON THE REPLICAS OF NILPOTENT MATRICES

HSIO-FU TUAN

In a recent paper,[1] Chevalley proved the following theorem:

(A) *If Z is a nilpotent matrix over a field K of characteristic* 0, *the only replicas Z' of Z are the matrices $Z'=tZ$, $t\in K$.*[2]

For the proof of (A), he made use of a particular case of a theorem due to Ado and gave a proof for the results which he needed. In the present note, we shall give a direct simple proof of (A) and we shall in fact deduce it as an immediate consequence of the stronger theorem:

(B) *If Z and Z' are two nilpotent matrices over a field K of characteristic* 0, *and if $q(x)$ and $r(x)$ are two polynomials with coefficients in K and without constant terms such that $Z'=q(Z)$ and $Z'_{0,2}=r(Z_{0,2})$, then $Z'=tZ$, $t\in K$.*

We shall later establish corresponding results for fields K of prime characteristics, to be stated as theorems (C) and (D).

That (A) is implied by (B) follows immediately from the fact that if Z' is a replica of Z, then $Z'_{r,s}=p_{r,s}(Z_{r,s})$, where $p_{r,s}(x)$ are polynomials in K without constant terms.[3]

For the proof of (B), let n be the degree of Z and Z' and let m be the least nonnegative integer such that $Z^{m+1}=0$. Clearly $0\leqq m\leqq n-1$. The case $Z=0$ is trivial; we can therefore assume $1\leqq m\leqq n-1$. Let also l be the least nonnegative integer such that $(Z_{0,2})^{l+1}=0$. Clearly $Z_{0,2}$ is nilpotent and $1\leqq l\leqq n^2-1$. We shall see that $m\leqq l\leqq 2m$.

The matrix Z can be transformed by an (n, n) matrix T with coefficients in the algebraic closure $\overline{K}$ of K into the following form:

$$Z_1 = T^{-1}ZT = \begin{pmatrix} 0 & & & & \\ z_1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & z_{n-1} & 0 \end{pmatrix}, \tag{1}$$

where $z_1, \cdots, z_{n-1}$ are zeros and ones and not all zeros. Then for

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[1] Claude Chevalley, *On a kind of new relationship between matrices*, Amer. J. Math. vol. 65 (1943) pp. 521–531.

[2] Theorem 6, p. 530, loc. cit.

[3] Lemma 4, p. 529, loc. cit.

any integer i, $1 \leqq i \leqq m$, we have

$$(1)_i \qquad Z_1^i = T^{-1}Z^iT = \begin{pmatrix} 0 & & & & & \\ \vdots & \ddots & & & & \\ 0 & & \ddots & & & \\ z_{i,1} & \ddots & & \ddots & & \\ & \ddots & \ddots & & \ddots & \\ & & z_{i,j} & \ddots & & \ddots \\ & & & \ddots & \ddots & \\ & & & & z_{i,n-i} & \underbrace{0 \cdots 0}_{i} \end{pmatrix},$$

$$z_{i,j} = z_j \cdots z_{i+j-1}, \qquad j = 1, \cdots, n-i.$$

Hence if we write

$$(2) \qquad q(x) = q_1x + \cdots + q_mx^m, \; q_i \in K, \quad i = 1, \cdots, m,$$

we have

$$(3) \qquad Z_1' = T^{-1}Z'T = T^{-1}q(Z)T = q(T^{-1}ZT) = q(Z_1).$$

Denoting the (n, n) identity matrix by E, we have then[4]

$$(4)_1 \qquad (Z_1)_{0,2} = Z_1 \oplus Z_1 = Z_1 \otimes E + E \otimes Z_1$$

$$= \begin{pmatrix} 0 & & & & \\ z_1E & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & z_{n-1}E & 0 \end{pmatrix} + \begin{pmatrix} Z_1 & & & \\ & Z_1 & & \\ & & \ddots & \\ & & & Z_1 & \\ & & & & Z_1 \end{pmatrix}$$

$$= \begin{pmatrix} Z_1 & & & & \\ z_1E & Z_1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & Z_1 & \\ & & & z_{n-1}E & Z_1 \end{pmatrix},$$

hence for any positive integer i,[5]

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[4] We define $A \otimes B = (a_{ij}) \otimes B = (a_{ij}B)$. Observe that $(A \otimes B)(C \otimes D) = AB \otimes CD$. The heavy cross and heavy plus used in Chevalley's paper are here replaced by $\otimes$ and $\oplus$.

[5] In the following, * denotes the terms in which we are not interested.

$$((Z_1)_{0,2})^i = (Z_1 \otimes E + E \otimes Z_1)^i$$

$$= Z_1 \otimes E + \sum_{j=1}^{i-1} C_{i,j} Z_1^{i-j} \otimes Z_1^j + E \otimes Z_1^i$$

$$(4)_i \qquad = \begin{pmatrix} Z_1^i & & & & \\ & Z_1^i & & & \\ & & \ddots & & \\ & * & & Z_1^i & \\ & & & & Z_1^i \end{pmatrix}$$

($C_{i,j}$ being the binomial coefficients), and therefore $m \leqq l \leqq 2m$. We may write

$$(5) \qquad r(x) = r_1 x + \cdots + r_l x^l, \quad r_i \in K, \qquad i = 1, \cdots, l,$$

and then

$$(Z_1')_{0,2} = Z_1' \oplus Z_1' = Z_1' \otimes E + E \otimes Z_1'$$

$$= T^{-1}Z'T \otimes T^{-1}ET + T^{-1}ET \otimes T^{-1}Z'T$$

$$(6) \qquad = (T^{-1} \otimes T^{-1})(Z' \otimes E + E \otimes Z')(T \otimes T)$$

$$= (T \otimes T)^{-1} Z'_{0,2} (T \otimes T) = (T \otimes T)^{-1} r(Z_{0,2})(T \otimes T)$$

$$= r((T \otimes T)^{-1} Z_{0,2} (T \otimes T)) = r((Z_1)_{0,2}).$$

Consequently the same relations originally assumed for Z and Z' now hold for Z_1 and Z_1'. For simplicity in notations, we shall now just consider Z and Z' for Z_1 and Z_1' in the related formulas $(1)_i$, (3), $(4)_i$, (6).

Now, on the one hand,

$$Z'_{0,2} = Z' \oplus Z' = Z' \otimes E + E \otimes Z' = q(Z) \otimes E + E \otimes q(Z)$$

$$= E \otimes q(Z) + (q_1 Z + \cdots + q_m Z^m) \otimes E$$

$$= E \otimes q(Z) + Z \otimes q_1 E + \cdots + Z^m \otimes q_m E$$

$$(7) \qquad = \begin{pmatrix} q(Z) & & & & \\ q_1 z_1 E & q(Z) & & & \\ & q_2 z_2 E & q(Z) & & \\ & & \ddots & \ddots & \\ & * & & q(Z) & \\ & & & q_1 z_{n-1} E & q(Z) \end{pmatrix},$$

while, on the other hand,

$$
\begin{aligned}
Z'_{0,2} &= r(Z_{0,2}) = r_1 Z_{0,2} + r_2(Z_{0,2})^2 + \cdots + r_l(Z_{0,2})^l \\
&= r_1(Z \oplus Z) + r_2(Z \oplus Z)^2 + \cdots + r_l(Z \oplus Z)^l \\
&= r_1(Z \otimes E + E \otimes Z) + r_2(Z^2 \otimes E + 2Z \otimes Z + E \otimes Z^2) \\
&\quad + \cdots + r_l\left(Z^l \otimes E + \sum_{i=1}^{l-1} Z^{l-i} \otimes Z^i + E \otimes Z^l\right) \\
&= E \otimes (r_1 Z + r_2 Z^2 + \cdots + r_l Z^l) \\
&\quad + Z \otimes (r_1 E + 2r_2 Z + \cdots + l r_l Z^{l-1}) \\
&\quad + \cdots + Z^l \otimes r_l E \\
&= E \otimes r(Z) + Z \otimes r'(Z) + Z^2 \otimes (1/2) r''(Z) \\
&\quad + \cdots + Z^l \otimes (1/l!) r^{(l)}(Z) \\
&= \begin{bmatrix}
r(Z) & & & & & \\
z_1 r'(Z) & r(Z) & & & & \\
& z_2 r'(Z) & r(Z) & & & \\
& & \ddots & \ddots & & \\
& & & \ddots & r(Z) & \\
& & & & z_{n-1} r'(Z) & r(Z)
\end{bmatrix},
\end{aligned}
\tag{8}
$$

where $r'(x), r''(x), \cdots, r^{(l)}(x)$ are the successive derivatives of $r(x)$.

In (7) and (8), comparing the terms (which are (n, n) matrices) on the main diagonal and on the first parallel just below, we obtain

$$q(Z) = r(Z), \tag{9}$$

$$q_1 z_i E = z_i r'(Z); \qquad i = 1, \cdots, n-1. \tag{10}$$

(9) gives

$$q_1 Z + \cdots + q_m Z^m = r_1 Z + \cdots + r_l Z^l, \qquad l \geqq m,$$

and hence

$$q_j = r_j, \qquad j = 1, \cdots, m. \tag{11}$$

(10) gives

$$q_1 z_i E = z_i(r_1 E + 2r_2 Z + \cdots + l r_l Z^{l-1}), \qquad i = 1, \cdots, n-1,$$

by (11),

$$z_i(2q_2 E + \cdots + m q_m Z^{m-1} + \cdots) = 0, \qquad i = 1, \cdots, n-1,$$

as not all z_i (for $i = 1, \cdots, n-1$) are zero, hence

$$2q_2 E + \cdots + m q_m Z^{m-1} + \cdots = 0,$$

and consequently

$$2q_2 = \cdots = mq_m = 0. \tag{12}$$

Now K is of characteristic 0; we can therefore conclude

$$q_2 = \cdots = q_m = 0. \tag{13}$$

It then follows that

$$Z' = q_1 Z = tZ \tag{14}$$

with $q_1 = t \in K$, as is to be proved.

Let us now suppose that K is of prime characteristic, say p. If $p > m$, which is certainly the case if $p \geqq n$, then we can still infer (13) from (12) and hence still have (14) as before. For the cases $p \leqq m$, from (12) we can only infer that

$$q_i = 0, \qquad 2 \leqq i \leqq m, \qquad p \nmid i, \tag{15}$$

and hence we can only conclude that Z' is necessarily of the form

$$Z' = q_1 Z + q_p Z^p + q_{2p} Z^{2p} + \cdots + q_{m'p} Z^{m'p}, \tag{16}$$

where $m' = [m/p]$, the largest integer not greater than m/p. In fact, the conclusions (13) and (14) are then no longer true in general.

We shall first prove that $Z' = Z^\alpha$ with $\alpha = p^a$ (a being any non-negative integer) is a replica of Z. This follows from the facts:

$$\begin{aligned}
Z'_{r,s} = (Z^\alpha)_{r,s} &= \underbrace{(Z^\alpha) \oplus \cdots \oplus (Z^\alpha)}_{r} \oplus \underbrace{(Z^\alpha) \oplus \cdots \oplus (Z^\alpha)}_{s} \\
&= (-{}^tZ^\alpha) \oplus \cdots \oplus (-{}^tZ^\alpha) \oplus Z^\alpha \oplus \cdots \oplus Z^\alpha \\
&= -({}^tZ^\alpha) \otimes \cdots \otimes E \otimes E \otimes \cdots \otimes E - \cdots \\
&\quad - E \otimes \cdots \otimes ({}^tZ^\alpha) \otimes E \otimes \cdots \otimes E \\
&\quad + E \otimes \cdots \otimes E \otimes Z^\alpha \otimes \cdots \otimes E + \cdots \\
&\quad + E \otimes \cdots \otimes E \otimes E \otimes \cdots \otimes Z^\alpha \\
&= (-{}^tZ \otimes \cdots \otimes E \otimes E \otimes \cdots \otimes E - \cdots \\
&\quad - E \otimes \cdots \otimes {}^tZ \otimes E \otimes \cdots \otimes E \\
&\quad + E \otimes \cdots \otimes E \otimes Z \otimes \cdots \otimes E + \cdots \\
&\quad + E \otimes \cdots \otimes E \otimes E \otimes \cdots \otimes Z)^\alpha \\
&= (Z_{r,s})^\alpha,
\end{aligned} \tag{17}$$

because $(-1)^\alpha = \pm 1 \equiv -1 \pmod{p}$ for $p=2$ and $(-1)^\alpha = -1$ for $p \neq 2$.

We shall now prove that conversely if $Z'=q(Z)=\sum_{i=1}^{m}q_iZ^i$ with $q_i\in K$ is a replica of Z, then we have

$$(18)\quad Z' = q(Z) = \sum_{j=1}^{m_0} q_{i(j)}Z^{i(j)}, \qquad i(j) = p^j, \quad i(m_0) \leqq m < i(m_0+1).$$

We shall show more strongly that only then $Z'_{1,1}=s(Z_{1,1})$, where $s(x)$ is a polynomial with coefficients in K and without the constant term.

We can assume that Z is of the form (1); because as before,

$$(Z_1')_{1,1} = (T^{-1}Z'T)_{1,1} = ({}^tT^{-1}\otimes T)^{-1}Z'_{1,1}({}^tT^{-1}\otimes T),$$

hence $Z'_{1,1}=r(Z_{1,1})$ implies also

$$(19)\qquad \begin{aligned}(Z_1')_{1,1} &= ({}^tT^{-1}\otimes T)^{-1}s(Z_{1,1})({}^tT^{-1}\otimes T)\\ &= s(({}^tT^{-1}\otimes T)^{-1}Z_{1,1}({}^tT^{-1}\otimes T)) = s((Z_1)_{1,1}).\end{aligned}$$

Then, for any positive integer i, $1\leqq i\leqq m$, we have

$$(Z')^i = (-{}^tZ)^i = (-1)^i\,{}^tZ_i$$

$$(20)_i\qquad = \left(\begin{array}{ccccccc} 0 & \cdots & 0 & z_{i,1} & & & \\ & \ddots & & \ddots & \ddots & & \\ & & \ddots & & \ddots & z_{i,j} & \\ & & & \ddots & & \ddots & \ddots \\ & & & & \ddots & & z_{i,n-i} \\ & & & & & \ddots & 0 \\ & & & & & & \vdots \\ & & & & & & 0 \end{array}\right), \begin{array}{l} \\ \\ \\ \\ \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\}i \end{array}$$

$$z_{i,j} = (-1)^iz_j\cdots z_{i+j-1}, \qquad\qquad j = 1,\cdots,n-i,$$

and further

$$\begin{aligned}(Z_{1,1})^i &= (Z'\oplus Z)^i = (Z'\otimes E + E\otimes Z)^i = (-{}^tZ\otimes E + E\otimes Z)^i\\ &= (-1)^i\,{}^tZ^i\otimes E + \sum_{j=1}^{i-1}(-1)^{i-j}C_{i,j}Z^{i-j}\otimes Z^j + E\otimes Z^i\end{aligned}$$

$$(21)_i\qquad = \begin{pmatrix} Z & -z_1E & & & \\ & Z & \ddots & & \\ & & \ddots & \ddots & \\ & & & Z & -z_{n-1}E \\ & & & & Z \end{pmatrix}^i = \begin{pmatrix} Z^i & & & \\ & Z^i & & * \\ & & \ddots & \\ & & & Z^i \\ & & & & Z^i \end{pmatrix},$$

hence it follows as before that the least nonnegative integer k such that $(Z_{1,1})^{k+1}=0$ satisfies $m\leqq k\leqq 2m$. We can therefore write

$$\begin{aligned} s(x) &= s_1x + \cdots + s_kx^k \qquad (s_i \in K) \\ &= (s_1x+\cdots+s_mx^m)+(s_{m+1}x^{m+1}+\cdots+s_kx^k) \qquad (22)\\ &= s'(x)+s''(x). \end{aligned}$$

We have

$$\begin{aligned} Z'_{1,1} &= Z' \oplus Z = (-{}^tZ') \oplus Z = -{}^tZ' \otimes E + E \otimes Z' \\ &= -q({}^tZ)\otimes E + E\otimes q(Z) \qquad (23)\\ &= E\otimes q(Z) + \sum_{j=1}^{m} {}^tZ^j \otimes (-q_jE), \end{aligned}$$

while, on the other hand,

$$\begin{aligned} Z'_{1,1} &= \sum_{i=1}^{k} s_i(Z_{1,1})^i = \sum_{i=1}^{k} s_i(-{}^tZ\otimes E + E\otimes Z)^i \\ &= \sum_{i=1}^{k} s_i \sum_{j=\max(0,i-m)}^{\min(i,m)} C_{i,j}(-{}^tZ)^j \otimes Z^{i-j} \\ &= \sum_{j=0}^{m} {}^tZ^j \otimes (-1)^j\left(\sum_{i=\max(1,j)}^{\min(k,m+j)} s_iC_{i,j}Z^{i-j}\right) \qquad (24)\\ &= E\otimes \sum_{i=1}^{m} s_iZ_i + \sum_{j=1}^{m} {}^tZ^j \otimes (-1)^j\left(\sum_{i=j}^{\min(k,m+j)} s_iC_{i,j}Z^{i-j}\right) \\ &= E\otimes s'(Z) + \sum_{j=1}^{m} {}^tZ^j \otimes s_j(Z), \end{aligned}$$

where $s_j(x)$ are polynomials with coefficients in K for $j=1, \cdots, m$ (observe $\max(0, i-m)\leqq \min(i, k-m)\leqq \min(i, m)$). Writing the matrix $Z'_{1,1}$ as given by (23) and (24) in the form of two compound (n, n) matrices whose elements are again (n, n) matrices and comparing the terms on their main diagonals and on their m first parallels above the main diagonals, we can then conclude that first

$$q(Z) = \sum_{i=1}^{m} q_iZ^i = \sum_{i=1}^{m} s_iZ^i = s(Z), \qquad (25)$$

$$q_i = s_i, \qquad i = 1, \cdots, m; \qquad (26)$$

and that also

$$z_{j,h}(-q_jE) = z_{j,h}s_j(Z), \qquad j = 1, \cdots, m,\ h = 1, \cdots, n-j,$$

as $Z'^j\neq 0$ for $j=1, \cdots, m$, so for each j we have at least one h, say $h(j)$ (=one of $1, \cdots, n-j$), such that $z_{j,h(j)}\neq 0$, then $z_{j,h(j)}=\pm 1$, and hence

$$-q_jE = s_j(Z) = (-1)^j \sum_{i=j}^{\min(k,m+j)} s_iC_{i,j}Z^{i-j}$$

$$= (-1)^j s_jE + (-1)^j \sum_{i=j+1}^{\min(k,m+j)} s_iC_{i,j}Z^{i-j},^{5}$$

$$0 = (1+(-1)^j)s_jE + (-1)^j \sum_{i=j+1}^{\min(k,m+j)} s_iC_{i,j}Z^{i-j}, \tag{27}$$

$$(1+(-1)^j)s_j = 0, \qquad s_iC_{i,j} = 0, \qquad j = 1, \cdots, m; \quad i = j+1, \cdots, \min(k, m+j)^{6} \tag{28}$$

(observe $\min(k, m+j)\geqq m$), in particular

$$s_iC_{i,j} = 0, \qquad i = 2, \cdots, m; j = 1, \cdots i-1. \tag{29}$$

Now it is easily seen that if $i=p^ji'$ with $p\nmid i'$, then $p\nmid C_{i,i/i'}$,[7] hence it follows immediately that if $i'\neq 1$, namely, $i\neq p^j$ for $j=1, \cdots, m_0$ with $p^{m_0}\leqq m<p^{m_0+1}$, then $q_i=s_i=0$ for $i=2, \cdots, m$, as was to be proved.

Summarizing our results for fields K of characteristic $p\neq 0$, we have the following theorems:

(C) *If Z and Z' are two nilpotent matrices over a field K of characteristic $p\neq 0$, and if $q(x)$, $r(x)$ and $s(x)$ are three polynomials with coefficients in K and without constant terms such that $Z'=q(Z)$, $Z'_{0,2}=r(Z_{0,2})$ and $Z'_{1,1}=s(Z_{1,1})$, then we have*

$$Z' = q(Z) = \sum_{j=0}^{m_0} t_jZ^{i(j)}, \qquad i(j) = p^j, \qquad t_j \in K. \tag{30}$$

(D) *If Z is a nilpotent matrix over a field K of characteristic $p\neq 0$, the only replicas Z' of Z are the matrices* (30).

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[6] If $k=m$, then for $j=m$, there is no i, $j+1\leqq i\leqq \min(k, m+j)$.

[7] Consider $p^j=p^j\cdots 1$ and $i\cdots(i-p^j+1)=(p^j+p^j(i'-1))\cdots(1+p^j(i'-1))$. As $p^v\|u$ (namely $p^v|u$ and $p^{v+1}\nmid u$) for $u=p^j, \cdots, 1$ implies $p^v\|(u+p^j(i'-1))$ because $v\leqq j$, so $p\nmid C_{i,i/i'}$.

A NEW APPLICATION OF THE SCHUR DERIVATE

GORDON OVERHOLTZER

Fermat's theorem in elementary number theory states that if p is a rational prime, a an integer,

$$a^p \equiv a \pmod{p}.$$

Hence

$$a^{p^{n+1}} \equiv a^{p^n} \pmod{p^{n+1}}$$

or

$$(0, 1) \qquad (a^{p^{n+1}} - a^{p^n})/p^{n+1}$$

is a rational, hence a p-adic, integer.

By introducing as the derivate, Δa_n, of a sequence $\{a_n\}$ with respect to the number p the expression

$$(0, 2) \qquad \Delta a_n = (a_{n+1} - a_n)/p^{n+1},$$

I. Schur[1] in 1933 generalized Fermat's theorem. The Fermat theorem states that the first Schur derivate of the sequence $\{a^q\}$ with $q = p^n$, (0, 1), is integral. Schur proved the generalization that, if a is prime to p, not only the first derivate, but the higher Schur derivates up to the $(p-1)$st are integral (in the p-adic or rational sense). Zorn[2] in 1936 extended this result by proving that all Schur derivates of $\{a^q\}$ with $q = p^n$ are p-adically bounded, hence convergent, and discussing the p-adic function, $\lim_{n\to\infty} \Delta^m a^q$, where $q = p^n$.

It is a fact[3] of elementary number theory that the sum of the kth (k a positive or negative integer or zero) powers of the rational integers less than and prime to p^n (p a rational prime, n a positive integer) is divisible by p^n if $p-1$ does not divide k or by p^{n-1} if $p-1$ divides k. The quotient of the division of such a sum by p^n,

$$(0, 3) \qquad S[n, x^k] = \sum_{i=1}^{q}{}' i^k/p^n,$$

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[1] Preuss. Akad. Wiss. Sitzungsber. (1933) p. 145.

[2] Ann. of Math. vol. 38 (1937) pp. 451–464.

[3] For a recent proof see H. Gupta, Proceedings of the Indian Academy of Sciences, Section A, vol. 13 (1944) pp. 85–86. His theorem is stated for even k, but the evenness of k is not used. Note that all concepts used are defined for negative k and the same proof holds. Classic results in number theory are Wolstenholme's theorem and Leudesdorf's generalization which yield divisibility by p^{2n} for $k = -1$, $p > 3$.

(where $q=p^n$ and the prime on the summation sign indicates summation over i, $(i, p)=1$) is then a p-adic integer or has at most a denominator of p.

The present paper studies the p-adic properties of the sequence $\{S[n, x^k]\}$. It is proved that the sequence is p-adically convergent. A p-adic function, $\lim_{n\to\infty} S[n, x^k]$, may be defined for rational integers, and an elementary expression for this function for positive k given in terms of Bernoullian numbers. All the Schur derivates of this sequence, $\Delta^m S[n, x^k]$, are p-adically bounded, hence p-adically convergent. The limit of $\Delta^m S[n, x^k]$ is given in terms of the p-adic function, $\lim_{n\to\infty} S[n, x^{k-m}]$.

The quotients

$$Q[n, x^k] = \sum_{i=1}^{q}{}' i^k/p^{2n}, \tag{0, 4}$$

where $q=p^n$, are p-adically bounded for k odd. Moreover $\{Q[n, x^k]\}$ is p-adically convergent for k odd. The function $\lim_{n\to\infty} Q[n, x^k]$ (k odd) is given in terms of $\lim_{n\to\infty} S[n, x^{k-1}]$.

If we write

$$T[n, g(x)] = \sum_{i=1}^{q}{}' g(i), \qquad S[n, g(x)] = T[n, g(x)]/p^n, \tag{0, 5}$$

where $q=p^n$, it is evident that the results concerning $\{S[n, x^k]\}$ extend to $\{S[n, g(x)]\}$ if $g(x)$ is a polynomial. It is proved also that these results extend to $S[n, f(x)]$ where $f(x)$ is the power series

$$f(x) = \sum_{j=e}^{\infty} a_j x^j$$

and the valuation of a_j approaches zero as j approaches ∞. Moreover

$$\lim_{n\to\infty} S[n, f(x)] = \sum_{j=e}^{\infty} a_j \lim_{n\to\infty} S[n, x^j].$$

The results for $\{S[n, x^k]\}$ are deduced as consequences of general formulas for $\Delta^m S[n, g(x)]$, Schur derivates of the sequence of sums of the values of $g(x)$ for x less than and prime to p^n divided by p^n. The sums are over a special set of values, a reduced residue set modulo p^n; sums over other sets of values of x might be studied. For example, the function might be summed over integers congruent to c modulo p and less than p^n. The formulas of this paper could be used in this special case by setting $g(x)=0$ in the other residue classes. In fact, the methods of this paper treat the values belonging to a particular residue class modulo p together as a part of the whole sum.

1. Power series representations of the Schur derivates. It will be convenient first to consider the sum of the values of a function, $g(x)$, for the integers less than and prime to p^n, p a rational prime. The results of this section will be applied to the special case $g(x) = x^k$ and theorems concerning the p-adic convergence of $\{S[n, x^k]\}$ and the Schur derivates of this sequence will be proved.

Assume that $g(x)$ is a function defined for p-adic integers and $g(x)$ can be represented by power series[4]

$$g(x) = \sum_{j=0}^{\infty} g^{(j)}(a)(x-a)^j/j!, \qquad a = 1, 2, \cdots, p-1,$$

p-adically convergent if $\phi(x-a) < 1$, ϕ the valuation function. Note that the power series about different a are independent. A consequence of this analyticity condition is that $g(x)$ can be developed about any point in the circle of convergence, and the series will converge for any point in the original circle to the same limit. Since each of the power series can be differentiated term by term, $g^{(k)}(x)$ is a function satisfying the analyticity conditions imposed on $g(x)$ in this paragraph.

Assume also that $S[n, g^{(k)}(x)]$ is p-adically bounded uniformly, that is, if $\phi(p) = \delta$,

$$\phi\{S[n, g^{(k)}(x)]\} \leqq \delta^N, \qquad n = 1, 2, 3, \cdots, \tag{1, 1}$$

where N is independent of k.

The power series of $g(x)$ will be used to express $S[n+1, g(x)]$ and consequently $\Delta S[n, g(x)]$ as an infinite series in $S[n, g^{(\alpha)}(x)]$. Now

$$T[n+1, g(x)] = \sum_{i=1}^{r}{}' g(i) = \sum_{j=1}^{q}{}' \sum_{\nu=0}^{p-1} g(j + \nu p^n),$$

where $r = p^{n+1}$ and $q = p^n$,

$$T[n+1, g(x)] = \sum_{j=1}^{q}{}' \sum_{\nu=0}^{p-1} \{g(j) + (\nu p^n) g'(j) + (\nu p^n)^2 g''(j)/2! + \cdots \}.$$

Summing out on ν and writing $b(k) = \sum_{\nu=0}^{p-1} \nu^k = T[1, x^k]$, we obtain

---

[4] For a discussion of the power series of $g(x)$ and proof of the properties cited, see Schöbe, *Beiträge zur Funktionentheorie in nichtarchimedisch bewerteten Körpern*, Inaugural Dissertation, Halle, 1930, pp. 17–19. However, we use only such of his theorems as are capable of an easy proof.

$$
\begin{aligned}
T[n+1, g(x)] &= \sum_{j=1}^{q}{}'[pg(j) + b(1)p^n g'(j) + b(2)p^{2n}g''(j)/2! + \cdots] \\
&= p\sum_{j=1}^{q}{}'g(j) + b(1)p^n\sum_{j=1}^{q}{}'g'(j) \\
&\qquad + b(2)p^{2n}\sum_{j=1}^{q}{}'g''(j)/2! + \cdots \\
&= pT[n, g(x)] + b(1)p^nT[n, g'(x)] \\
&\qquad + b(2)p^{2n}T[n, g''(j)]/2! + \cdots ,
\end{aligned}
$$

$$
(1, 2) \qquad T[n+1, g(x)] - pT[n, g(x)] = \sum_{\rho=1}^{\infty} \frac{b(\rho)p^{\rho n}}{\rho!} T[n, g^{(\rho)}(x)].
$$

Divide (1, 2) by p^{n+1} and obtain

$$
(1, 3) \qquad S[n+1, g(x)] - S[n, g(x)] = \frac{1}{p}\sum_{\rho=1}^{\infty} \frac{b(\rho)p^{\rho n}}{\rho!} S[n, g^{(\rho)}(x)].
$$

Substitution of (1, 3) into the definition of $\Delta S[n, g(x)]$ gives

$$
\Delta S[n, g(x)] = \frac{1}{p}\sum_{\rho=1}^{\infty} \frac{b(\rho)p^{\rho n}}{\rho!} S[n, g^{(\rho)}(x)]/p^{n+1},
$$

$$
(1, 4) \qquad \Delta S[n, g(x)] = \frac{1}{p^2}\sum_{\rho=1}^{\infty} \frac{b(\rho)p^{n(\rho-1)}}{\rho!} S[n, g^{(\rho)}(x)].
$$

Write

$$
(1, 5) \qquad L'(\rho) = p^{-2}b(\rho)/\rho!.
$$

Since $g^{(k)}(x)$ is a function satisfying all the conditions imposed on $g(x)$, (1, 4) becomes

$$
(1, 6) \qquad S[n, g^{(k)}(x)] = \sum_{\rho=1}^{\infty} L'(\rho)p^{n(\rho-1)}S[n, g^{(k+\rho)}(x)].
$$

Note that the coefficients $L'(\rho)$ depend only on the summation index.

An estimate of the valuation of $L'(\rho)$ will be required for the investigation of the higher derivates. From (1, 5)

$$
\phi[L'(\rho)] = \phi[p^{-2}b(\rho)/\rho!] = \phi(p^{-2})\phi[b(\rho)]\phi[1/\rho!].
$$

If $\rho = \sum_{i=0}^{e} u_i p^i$, $0 \leqq u_i < p$, the power of p in $\rho!$ is $(\rho - \sum_{i=0}^{e} u_i)/(p-1)$ and is, therefore, at most $\rho/(p-1)$. Hence

$$
(1, 7) \qquad \phi[L'(\rho)] < \delta^{-2}\delta^{-\rho/(p-1)} = \delta^{-2-\rho/(p-1)}.
$$

In general, an expression for $\Delta^{r+1}S[n, g(x)]$ as an infinite series in $S[n, g^{(\alpha)}(x)]$ may be obtained from a series for $\Delta^r S[n, g(x)]$ by substitution into the definition of $\Delta^{r+1}S[n, g(x)]$, algebraic manipulation, and rearrangement of a double series. A recursion formula for the coefficients of the series for $\Delta^{r+1}S[n, g(x)]$ in terms of the coefficients of $\Delta^r S[n, g(x)]$, $\Delta S[n, g(x)]$ results. Justification of the rearrangement of the double series requires an estimate of the valuation of the coefficients of $\Delta^r S[n, g(x)]$ and $\Delta S[n, g(x)]$. This estimate of the valuation of the coefficients is established by induction, using the recursion formula for the coefficients. To prove the possibility of representation of $\Delta^r S[n, g(x)]$ as an infinite series in $S[n, g^{(\alpha)}(x)]$ in the manner described, the results concerning the recursion formula and the valuation of the coefficients will be stated before it is evident how they are obtained.

(1, 8) THEOREM. *If $g(x)$ is a function defined for p-adic integers, and $g(x)$ may be expanded into power series about $a=1, 2, \cdots, p-1$, p-adically convergent if $\phi(x-a)<1$, and $S[n, g^{(k)}(x)]$ is uniformly p-adically bounded, then there exist coefficients $L^{(m)}(\alpha)$ independent of n and $g(x)$ such that*

$$(1, 9) \qquad \Delta^m S[n, g(x)] = \sum_{\alpha=m}^{\infty} L^{(m)}(\alpha) p^{n(\alpha-m)} S[n, g^{(\alpha)}(x)],$$

$$L'(\gamma) = \frac{p^{-2}b(\gamma)}{\gamma!},$$

$$(1, 10) \qquad L^{(m+1)}(\gamma) = \sum\nolimits^{*m} p^{\alpha-m} L^{(m)}(\alpha) L'(\beta) + (p^{\gamma-m-1} - p^{-1}) L^{(m)}(\gamma),^{5}$$

$$(1, 11) \qquad \phi[L^{(m)}(\alpha)] < \delta^{-2m-\alpha/(p-1)}.$$

The proof is by induction on m. For $m=1$, (1, 9) and (1, 11) become (1, 4) and (1, 7). The first formula of (1, 10) is (1, 5).

Assume (1, 9) and (1, 11) hold for $m \leqq r$. The truth of (1, 9) and (1, 11) for $m=1, r$ implies (1, 9) for $m=r+1$ and the recursion formula of (1, 10) for $m=r$. The recursion formula of (1, 10) for $m=r$ implies (1, 11) for $m=r+1$. Substitution of (1, 9) for $m=r$ into the definition of $\Delta^{r+1}S[n, g(x)]$ gives

---

[5] The composite superscript "$*m$" after a "$\sum$" shall indicate the summation condition $\alpha+\beta=\gamma$, $\beta \geqq 1$, $\alpha \geqq m$. The notations "$\sum^{*r}$" and "max^{*r}" are to be interpreted correspondingly.

$$
\begin{aligned}
\Delta^{r+1}S[n,\,g(x)] &= \{\Delta^{r}S[n+1,\,g(x)] - \Delta^{r}S[n,\,g(x)]\}/p^{n+1}\\
&= \left\{\sum_{\alpha=r}^{\infty} L^{(r)}(\alpha)p^{(n+1)(\alpha-r)}S[n+1,\,g^{(\alpha)}(x)]\right.\\
&\qquad \left.-\sum_{\alpha=r}^{\infty} L^{(r)}(\alpha)p^{n(\alpha-r)}S[n,\,g^{(\alpha)}(x)]\right\}/p^{n+1}\\
&= \sum_{\alpha=r}^{\infty} L^{(r)}(\alpha)\{p^{(n+1)(\alpha-r)}S[n+1,\,g^{(\alpha)}(x)]\\
&\qquad - p^{(n+1)(\alpha-r)}S[n,\,g^{(\alpha)}(x)] + p^{(n+1)(\alpha-r)}S[n,\,g^{(\alpha)}(x)]\\
&\qquad - p^{n(\alpha-r)}S[n,\,g^{(\alpha)}(x)]\}/p^{n+1}\\
&= \sum_{\alpha=r}^{\infty} L^{(r)}(\alpha)\{p^{(n+1)(\alpha-r)}\Delta S[n,\,g^{(\alpha)}(x)]\\
&\qquad + (p^{(n+1)(\alpha-r)} - p^{n(\alpha-r)})S[n,\,g^{(\alpha)}(x)]/p^{n+1}\}\\
&= \sum_{\alpha=r}^{\infty} L^{(r)}(\alpha)\left\{p^{(n+1)(\alpha-r)}\sum_{\beta=1}^{\infty} L'(\beta)p^{n(\beta-1)}S[n,\,g^{(\alpha+\beta)}(x)]\right.\\
&\qquad \left.+ (p^{(n+1)(\alpha-r)} - p^{n(\alpha-r)})S[n,\,g^{(\alpha)}(x)]/p^{n+1}\right\}.
\end{aligned}
$$

The coefficient of $S[n,\,g^{(\gamma)}(x)]$ in $\Delta^{r+1}S[n,\,g(x)]$, after a formal rearrangement of terms, which will be justified later, is

$$
\begin{aligned}
&\textstyle\sum^{*r} L^{(r)}(\alpha)p^{(n+1)(\alpha-r)}L'(\beta)p^{n(\beta-1)} + (p^{(n+1)(\gamma-r)} - p^{n(\gamma-r)})L^{(r)}(\gamma)/p^{n+1}\\
&= \textstyle\sum^{*r} p^{n(\alpha+\beta-r-1)+\alpha-r}L^{(r)}(\alpha)L'(\beta) + (p^{(n+1)(\gamma-r-1)} - p^{n(\gamma-r-1)-1})L^{(r)}(\gamma)\\
&= p^{n[\gamma-(r+1)]}[\textstyle\sum^{*r} p^{\alpha-r}L^{(r)}(\alpha)L'(\beta) + (p^{\gamma-r-1} - p^{-1})L^{(r)}(\gamma)].
\end{aligned}
$$

Writing

$$(1,\,12)\qquad L^{(r+1)}(\gamma) = \textstyle\sum^{*r} p^{\alpha-r}L^{(r)}(\alpha)L'(\beta) + (p^{\gamma-r-1} - p^{-1})L^{(r)}(\gamma),$$

which is (1, 10) for $m=r$, we obtain

$$(1,\,13)\qquad \Delta^{r+1}S[n,\,g(x)] = \sum_{\gamma=r+1}^{\infty} L^{(r+1)}(\gamma)p^{n[\gamma-(r+1)]}S[n,\,g^{(\gamma)}(x)],$$

since from (1, 10) for $m=r$, $L^{(r+1)}(i)=0$, $i=1, 2, \cdots, r$.

The validity of the rearrangement remains to be established. It suffices to show that the iterated series

$$\sum_{\alpha=r}^{\infty} L^{(r)}(\alpha)p^{(n+1)(\alpha-r)}\sum_{\beta=1}^{\infty} L'(\beta)p^{n(\beta-1)}S[n,\,g^{(\alpha+\beta)}(x)]$$

may be summed as

$$\sum_{\gamma=r+1}^{\infty}\left\{\sum{}^{*r}L^{(r)}(\alpha)p^{(n+1)(\alpha-r)}L'(\beta)p^{n(\beta-1)}S[n, g^{(\alpha+\beta)}(x)]\right\}.$$

It is sufficient[6] to show that the valuation of the general term $p^{n(\alpha+\beta-r-1)+\alpha-r}L^{(r)}(\alpha)L'(\beta)S[n, g^{(\alpha+\beta)}(x)]$ becomes arbitrarily small for $\alpha+\beta$ sufficiently large. Using (1, 1) and our induction assumption (1, 11) for $m=1, r$,

$$\begin{aligned}
&\phi\left\{p^{n(\alpha+\beta-r-1)+\alpha-r}L^{(r)}(\alpha)L'(\beta)S[n, g^{(\alpha+\beta)}(x)]\right\}\\
&\qquad\leqq \delta^{n(\alpha+\beta-r-1)+\alpha-r}\,\delta^{-2r-\alpha/(p-1)}\,\delta^{-2-\beta/(p-1)}\,\delta^{N}\\
&\qquad< \delta^{(\alpha+\beta)[n-1/(p-1)]-n(r+1)-r-2r-2+N}\\
&\qquad< \delta^{\epsilon}
\end{aligned}$$

if $\alpha+\beta>(\epsilon+nr+n+3r+2-N)/[n-1/(p-1)]$. Hence the rearrangement is valid and (1, 10) for $m=r$, (1, 12), and (1, 9) for $m=r+1$, (1, 13) have been established.

Finally it must be established that (1, 11) is hereditary. By (1, 10) for $m=r$,

$$\begin{aligned}
&\phi[L^{(r+1)}(\gamma)]\\
&\quad\leqq \max\left\{\phi\left[\sum{}^{*r}p^{(\alpha-r)}L^{(r)}(\alpha)L'(\beta)\right], \phi\left[(p^{\gamma-r-1}-p^{-1})L^{(r)}(\gamma)\right]\right\}\\
&\quad\leqq \max\left\{\max{}^{*r}\left(\delta^{\alpha-r}\delta^{-2r-\alpha/(p-1)}\delta^{-2-\beta/(p-1)}, \delta^{-1}\delta^{-2r-\gamma/(p-1)}\right\}\right.\\
&\quad= \max\left\{\max_{\alpha\geqq r}\left(\delta^{\alpha-r-2(r+1)-\gamma/(p-1)}\right), \delta^{-2r-1-\gamma/(p-1)}\right\}\\
&\quad= \delta^{-2(r+1)-\gamma/(p-1)}
\end{aligned}$$

which is (1, 11) for $m=r+1$. The induction is complete.

2. **The Schur derivates of x^k.** In this section the results of the previous section are applied to the special case of $g(x)=x^k$ (k a rational integer; positive, negative, or zero). Note that $g(x)=x^k$ is a function satisfying the conditions imposed on $g(x)$ in the preceding section. The function is defined for p-adic integers. The function x^k may be developed in power series by the binomial theorem. Finally

$$\phi\{S[n, x^k]\} \leqq \delta^{-1}, \tag{2, 1}$$

since $\sum\nolimits_{i=1}^{\prime q} i^k$, where $q=p^n$, is divisible by p^n if $p-1$ does not divide k or by p^{n-1} if $p-1$ divides k.

The formula (1, 9) and the estimate of the valuation (1, 11) make it possible to establish bounds for each Schur derivate $\Delta^m S[n, x^k]$.

---

[6] This rearrangement is an application of a more general theorem of Schöbe, loc. cit. p. 16.

$$\begin{aligned}\phi(\Delta^m S[n, x^k]) &= \phi\left\{\sum_{\alpha=m}^{\infty} L^{(m)}(\alpha) p^{n(\alpha-m)} S[n, (x^k)^{(\alpha)}]\right\}\\ &\leqq \max_{\alpha\geqq m} \phi\{L^{(m)}(\alpha) p^{n(\alpha-m)} S[n, (x^k)^{(\alpha)}]\}\\ &\leqq \max_{\alpha\geqq m} \delta^{-2m-\alpha/(p-1)+n(\alpha-m)-1}\\ &= \max_{\alpha\geqq m} \delta^{\alpha[n-1/(p-1)]-m(n+2)-1}.\end{aligned}$$

Since the exponent of δ in the valuation of the terms of $\Delta^m S[n, x^k]$ is a nondecreasing function of α, the first term $(\alpha = m)$ has the maximum valuation. Hence

$$\phi(\Delta^m S[n, x^k]) \leqq \delta^{-2m-m/(p-1)-1}. \tag{2, 2}$$

Since the exponent of δ in (2, 2) is independent of n, it follows that

(2, 3) THEOREM. *All the Schur derivates $\Delta^m S[n, x^k]$ are p-adically bounded.*

If the mth derivate is bounded, the $(m-1)$st derivate is p-adically convergent. An immediate consequence of (2, 3) is

(2, 4) THEOREM. *All the Schur derivates $\Delta^m S[n, x^k]$ are p-adically convergent.*

In particular, since the first derivate $\Delta S[n, x^k]$ is bounded,

(2, 5) THEOREM. *$S[n, x^k]$ is p-adically convergent.*

The formula (1, 9) may also be used to obtain expressions for the p-adic limits of the Schur derivates. From the argument used in establishing (2, 3), it may be observed that all terms of $\Delta^m S[n, x^k] = \sum_{\alpha=m}^{\infty} L^{(m)}(\alpha) p^{n(\alpha-m)} S[n, (x^k)^{(\alpha)}]$ after the first have power of p at least

$$\begin{aligned}(m + 1)[n - 1/(p - 1)] - m(n + 2) - 1&\\ &= n - 2m - (m + 1)/(p - 1) - 1.\end{aligned}$$

Hence

$$\Delta^m S[n, x^k] = L^{(m)}(m) S[n, (x^k)^{(m)}] + p^{n-2m-(m+1)/(p-1)-1} R_n, \tag{2, 6}$$

where R_n is a p-adic integer. As n approaches ∞, the second term of the right member of (2, 6) approaches zero. Hence

$$\begin{aligned}\lim_{n\to\infty}\Delta^m S[n, x^k] &= \lim_{n\to\infty} L^{(m)}(m)S[n, (x^k)^{(m)}]\\ &= k(k-1)\cdots(k-m+1)L^{(m)}(m)\lim_{n\to\infty} S[n, x^{k-m}].\end{aligned}\tag{2, 7}$$

From the Bernoulli numbers, an elementary expression for $\lim_{n\to\infty} S[n, x^k]$ for positive k may be derived. Summation over numbers relatively prime to p may be accomplished by summing over all integers and subtracting the sum over multiples of p. Then, setting $q=p^n$ and $s=p^{n-1}$,

$$T[n, x^k] = \sum_{i=1}^{q}{}' i^k = \sum_{i=1}^{q} i^k - \sum_{i=1}^{s} (pi)^k = \sum_{i=1}^{q-1} i^k - p^k\sum_{i=1}^{s-1} i^k.$$

For positive even k, introducing the Bernoulli numbers, B_i,

$$\begin{aligned}T[n, x^k] &= (p^n)^{k+1}/(k+1) - (p^n)^k/2 + \cdots + (-1)^{k/2-1}B_{k/2}p^n\\ &\quad - p^k[(p^{n-1})^{k+1}/(k+1) - (p^{n-1})^k/2 + \cdots\\ &\qquad\qquad + (-1)^{k/2-1}B_{k/2}p^{n-1}]\\ &= (-1)^{k/2-1}B_{k/2}p^n - p^k(-1)^{k/2-1}B_{k/2}p^{n-1} + S_n,\end{aligned}$$

where S_n has power of p at least $2(n-1)$ minus the maximum exponent of p in the denominators of the coefficients of $(p^n)^2, \cdots, (p^n)^{k+1}$, say $2n-c_k$, where c_k is independent of n. Then

$$S[n, x^k] \equiv (-1)^{k/2-1}B_{k/2}(1-p^{k-1}) (\operatorname{mod} p^{n-c_k}).$$

Hence, for k even,

$$\lim_{n\to\infty} S[n, x^k] = (-1)^{k/2-1}B_{k/2}(1-p^{k-1}).\tag{2, 8}$$

For positive odd k,

$$\begin{aligned}T[n, x^k] &= (p^n)^{k+1}/(k+1) - (p^n)^k/2 + \cdots\\ &\quad + (-1)^{(k-3)/2}kB_{(k-1)/2}(p^n)^2/2\\ &\quad - p^k[(p^{n-1})^{k+1}/(k+1) - (p^{n-1})^k/2 + \cdots\\ &\qquad\qquad + (-1)^{(k-3)/2}kB_{(k-1)/2}(p^{n-1})^2/2].\end{aligned}$$

$T[n, x^k]$ has power of p at least $2(n-1)$ minus the maximum exponent of p in the denominators of the coefficients of $(p^n)^{k+1}, \cdots, (p^n)^2$, say $2n-c_k$, c_k independent of n. Hence

$$T[n, x^k] \equiv 0 \ (\operatorname{mod} p^{2n-c_k}).\tag{2, 9}$$

Then

$$S[n, x^k] \equiv 0 \pmod{p^{n-ck}}$$

and, for k odd,

$$\lim_{n\to\infty} S[n, x^k] = 0. \tag{2, 10}$$

3. **The quotients Q.** The congruence (2, 9) shows that, for positive odd k,

$$Q[n, x^k] = T[n, x^k]/p^{2n} \tag{3, 1}$$

is p-adically bounded, with

$$\phi(Q[n, x^k]) \leqq \delta^{-ck}.$$

The possibility of a theory concerning the p-adic properties of $\{Q[n, x^k]\}$ for odd k is suggested. We mention the following result.

(3, 2) Theorem. $Q[n, x^k] \equiv kS[n, x^{k-1}]/2 \pmod{p^n}$, *$k$ odd and p an odd prime.*

We prove this for positive k as follows: Since k is odd,

$$\begin{aligned}
\sum_{i=1}^{q}{}' i^k &= \sum_{i=1}^{q}{}'(p^n - i)^k \\
&= \sum_{i=1}^{q}{}' \sum_{\nu=0}^{k} \binom{k}{\nu} (p^n)^\nu (-i)^{k-\nu} \\
&= \sum_{i=1}^{q}{}' \{(-i)^k + p^n k(-i)^{k-1} \\
&\qquad\qquad + p^{2n}k(k-1)(-i)^{k-2}/2 + \cdots + (p^n)^k\} \\
&\equiv -\sum_{i=1}^{q}{}' i^k + p^n k \sum_{i=1}^{q}{}'(-i)^{k-1} \pmod{p^{3n}},
\end{aligned}$$

where $q = p^n$, since if k is odd, $k-2$ is not divisible by $p-1$, p an odd prime, and $\sum_{i=1}^{\prime q}(-i)^{k-2}$ is divisible by p^n. Hence

$$2T[n, x^k] \equiv p^n k T[n, x^{k-1}] \pmod{p^{3n}}.$$

Division by $2p^{2n}$ yields (3, 2).

Theorem (3, 2) yields immediately that, k odd,

$$Q[n, x^k] = kS[n, x^{k-1}]/2 + p^n I, \tag{3, 3}$$

I a p-adic integer. Hence $\{Q[n, x^k]\}$ is p-adically convergent, since each term of the right member of (3, 3) is convergent.

(3, 4) THEOREM. *For k odd, p an odd prime, $Q[n, x^k]$ is p-adically convergent, and*

$$\lim_{n\to\infty} Q[n, x^k] = k \lim_{n\to\infty} S[n, x^{k-1}]/2.$$

By (2, 8), for positive odd k,

$$\lim_{n\to\infty} Q[n, x^k] = k(-1)^{(k-1)/2-1}B_{(k-1)/2}(1 - p^{k-2})/2. \tag{3, 5}$$

4. **Extension to functions defined by power series.** The results of Theorems (2, 3), (2, 4), and (2, 5) extend immediately to $\Delta^m S[n, g(x)]$, if $g(x)$ is a polynomial. The object of this section is to extend these results to the case where $g(x)$ is a power series

$$f(x) = \sum_{i=e}^{\infty} a_i x_i, \tag{4, 1}$$

where e is any integer. A necessary and sufficient condition that $f(x)$ be defined for p-adic integers is that $\phi(a_i)\to 0$ as $i\to\infty$. If the coefficients a_i satisfy this condition, $f(x)$ is a function satisfying all the conditions imposed on $g(x)$ in §1. Now

$$\begin{aligned} S[n, f(x)] &= \sum_{i=1}^{q}{}' f(i)/p^n = \sum_{i=1}^{q}{}' \sum_{j=e}^{\infty} a_j i^j/p^n \\ &= \sum_{j=e}^{\infty} a_j \sum_{i=1}^{q}{}' i^j/p^n = \sum_{j=e}^{\infty} a_j S[n, x^j], \end{aligned}$$

where $q = p^n$. We wish to show that the limit can be taken term by term, that is,

$$\lim_{n\to\infty} S[n, f(x)] = \lim_{n\to\infty} \sum_{j=e}^{\infty} a_j S[n, x^j] = \sum_{j=e}^{\infty} a_j \lim_{n\to\infty} S[n, x^j]. \tag{4, 2}$$

To establish (4, 2), it is sufficient to show that

(4, 3) $\lim_{n\to\infty} a_j S[n, x^j]$ exists,

(4, 4) $\sum_{j=e}^{\infty} a_j S[n, x^j]$ converges uniformly in n, that is, for each $\epsilon > 0$, there exists an N such that $\phi(a_j S[n, x^j]) < \epsilon$ for $j \geqq N$, N independent of n.

The condition (4, 3) is satisfied by virtue of Theorem (2, 5) and the condition (4, 4) is satisfied from the condition on the coefficients a_j and the fact that $\phi(S[n, x^j]) \leqq \delta^{-1}$. Hence we have

(4, 5) THEOREM. *$\{S[n, f(x)]\}$ is p-adically convergent and*

$$\lim_{n\to\infty} S[n, f(x)] = \sum_{j=0}^{\infty} a_j \lim_{n\to\infty} S[n, x^j]. \tag{4, 6}$$

5. **Sums of kth powers.** This section will consider the sum of the kth powers of the positive rational integers congruent to one modulo p and less than p^n. Write

$$T_1[n, x^k] = \sum_{i=1,\, i\equiv p \pmod{p}}^{q} i^k, \tag{5, 1}$$

where $q=p^n$. In this case k may be chosen a p-adic integer.

Since the residue class one is a subgroup of the cyclic group of a reduced residue set modulo p^n, it is cyclic and there exists a primitive root, r, generating the residue class modulo p^n. Then

$$\begin{aligned} T_1[n, x^k] &\equiv r^k + r^{2k} + \cdots + r^{p^{n-1}k} \pmod{p^n} \\ &\equiv r^k(1 - r^{kp^{n-1}})/(1 - r^k) \pmod{p^n}. \end{aligned}$$

If $k=p^\mu\lambda$, $(\lambda, p)=1$, then the power of p in $1-r^{kp^{n-1}}$ is $\mu+n$ and the power of p in $1-r^k$ is $\mu+1$. Hence

$$T_1[n, x^k] \equiv 0 \pmod{p^{n-1}}. \tag{5, 2}$$

If we write

$$S_1[n, x^k] = T_1[n, x^k]/p^{n-1}, \tag{5, 3}$$

$S_1[n, x^k]$ is a p-adic integer.

For other residue classes define, $(i_0, p)=1$,

$$T_{i_0}[n, x^k] = \sum_{i=1,\, i\equiv i_0 \pmod{p}}^{q} i^k, \tag{5, 4}$$

where $q=p^n$. Then

$$\begin{aligned} T_{i_0}[n, x^k] &= \sum_{\nu=0}^{p^{n-1}-1} (i_0+\nu p)^k = i_0^k \sum_{\nu=0}^{p^{n-1}-1} \left(1 + \frac{\nu}{i_0} p\right)^k \\ &\equiv i_0^k \sum_{\nu=0}^{p^{n-1}-1} (1+\nu p)^k \pmod{p^n} \equiv i_0^k T_1[n, x^k], \end{aligned}$$

$$T_{i_0}[n, x^k] \equiv 0 \pmod{p^{n-1}}, \tag{5, 5}$$

since ν and ν/i_0 run through the same residue classes modulo p^n. Note that $T_{i_0}[n, x^k]$ is exactly divisible by p^{n-1}, that is, not divisible by p^n.

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ON THE CAPACITY OF A CONDENSER

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1. Introduction. (a) Any student of electric phenomena is familiar with the *Leyden jar*. In its simplest form it consists of two circular cylinders, with a common axis, charged with electricity. If there is a difference in the potentials of the cylinders, then electric energy, which can be recovered at any time in the form of electric discharge, is stored up in the jar.

Let us denote the radii of the cylinders by a and b, $a<b$. Disregarding the effect of the ends, the potential u of the electric field between the cylinders depends only on the distance r from the axis, and so it must have the form

$$u = A \log r + B, \tag{1}$$

where A and B are constants. On the cylinders, which are equipotential surfaces, we have

$$u_a = A \log a + B, \qquad u_b = A \log b + B. \tag{2}$$

The density σ_a of the charge on the cylinder $r=a$ is

$$\sigma_a = -\frac{\epsilon}{4\pi}\left(\frac{\partial u}{\partial r}\right)_{r=a} = -\frac{\epsilon A}{4\pi a}, \tag{3}$$

where ϵ denotes the dielectric constant of the field. The density σ_b of the charge on the other cylinder will be

$$\sigma_b = \frac{\epsilon}{4\pi}\left(\frac{\partial u}{\partial r}\right)_{r=b} = \frac{\epsilon A}{4\pi b}. \tag{4}$$

Hence the charge on the cylinders, per unit length, will be $2\pi a\sigma_a = -\epsilon A/2$ and $2\pi b\sigma_b = \epsilon A/2$, respectively. On the other hand the potential difference is

$$u_b - u_a = A \log (b/a). \tag{5}$$

The ratio

$$\frac{\text{charge (per unit length)}}{\text{potential difference}} = \frac{\epsilon A/2}{A \log (b/a)} = \frac{\epsilon}{2 \log (b/a)} \tag{6}$$

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is called the *capacity* (per unit length) of the jar. It measures the charge which is necessary in order to produce the potential difference 1.

(b) Another very elementary problem of electrostatics, related to the Leyden jar, is that of a *spherical condenser*. We consider two concentric, thin metallic spheres with radii a and b, respectively, $a < b$. The potential u of the field between these spheres depends only on the distance r from the center; hence it must have the form

$$u = A/r + B, \tag{7}$$

where A and B are constants. On the spheres, which are equipotential surfaces, we have

$$u_a = A/a + B, \qquad u_b = A/b + B. \tag{8}$$

The densities of the charges are in this case

$$\sigma_a = -\frac{\epsilon}{4\pi}\left(\frac{\partial u}{\partial r}\right)_{r=a} = \frac{\epsilon A}{4\pi a^2}, \qquad \sigma_b = \frac{\epsilon}{4\pi}\left(\frac{\partial u}{\partial r}\right)_{r=b} = -\frac{\epsilon A}{4\pi b^2}, \tag{9}$$

so that the total charges on the single spheres are $\pm\epsilon A$. The potential difference is

$$u_a - u_b = A(a^{-1} - b^{-1}). \tag{10}$$

The ratio

$$\frac{\text{charge}}{\text{potential difference}} = \frac{\epsilon A}{A(a^{-1} - b^{-1})} = \frac{\epsilon}{a^{-1} - b^{-1}} \tag{11}$$

is called the *capacity* of the spherical condenser. In the limiting case when $b \to \infty$, we find for the capacity the expression ϵa. Hence assuming $\epsilon = 1$, we find that the capacity of a sphere "with respect to an infinitely large sphere" is equal to the radius of the sphere. In general, the capacity has the dimension of a length.

2. **Capacity of a condenser.** (a) We consider two closed surfaces s_0 and s_1 such that s_1 is contained in the interior of s_0. We say that s_0 and s_1 form a "condenser" and we refer to the closed set of points exterior to s_1 and interior to s_0 as the "field." There is a point-function $u(p)$, harmonic in the field, such that $u(p) = 0$ on s_0 and $u(p) = 1$ on s_1. This function is uniquely determined. We call it the *potential function of the condenser*. It is the potential of certain electric charges, properly distributed on the surfaces s_0 and s_1.

The density of this charge is (assuming that the dielectric constant is 1)

$$-\frac{1}{4\pi}\frac{\partial u}{\partial n}, \tag{1}$$

where n is the normal directed into the interior of the field, that is, n is the interior normal on s_0 and the exterior one on s_1. Hence the total charges will be

$$-\frac{1}{4\pi}\iint\frac{\partial u}{\partial n}dS \quad \text{and} \quad -\frac{1}{4\pi}\iint\frac{\partial u}{\partial n}dS, \tag{2}$$

respectively, the positive directions of the normals being determined as above.

Since u is a harmonic function, these integrals are equal, except their sign. We form generally

$$C = \frac{1}{4\pi}\iint\frac{\partial u}{\partial n}dS, \tag{3}$$

where the integration is extended over an arbitrary closed surface lying in the interior of s_0 and containing s_1; the normal is the interior one. This integral is independent of the surface of the integration. It is a positive quantity, called the *capacity of the condenser*. The charges (2) on s_0 and s_1 are $-C$ and $+C$, respectively. The potential difference is in this case 1 so that this definition is a generalization of that given in 1(b).

(b) According to the theory of harmonic functions the potential function $u(p)$ and the capacity C of the condenser can also be obtained in the following way. We consider an arbitrary function $\psi(p)$ with continuous derivatives, defined in the field such that $\psi=0$ on s_0 and $\psi=1$ on s_1. We form Dirichlet's integral

$$W(\psi) = \frac{1}{4\pi}\iiint(\operatorname{grad}\psi)^2dV, \tag{4}$$

where the integration is extended over the field. Then the minimum of $W(\psi)$ for the set of all functions mentioned is attained when $\psi(p)=u(p)$, and the minimum of $W(\psi)$ is equal to C. (See Kellogg [8, p. 279].[1])

An alternate form of the integral $W(\psi)$ is

$$W(\psi) = \iiint\psi\rho dV + \iint\psi\sigma dS, \tag{5}$$

where

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[1] Numbers in brackets refer to the Bibliography at the end of this paper.

$$\rho = -\frac{\Delta\psi}{4\pi}, \qquad \sigma = -\frac{1}{4\pi}\frac{\partial\psi}{\partial n} \tag{6}$$

and the second integral in (5) is extended over s_0+s_1. The normal is directed into the interior of the field. The quantities (6) are the volume and surface density of a distribution of charges to which the potential ψ corresponds. Here we assume that ψ has continuous second derivatives. (See Kellogg [8, p. 278].) The expression $W(\psi)/2$ can be interpreted as the potential energy of this distribution.

In the next sections we consider two important limiting cases of condensers.

3. **Green's function of the exterior of a surface.** (a) Let s be a closed surface. Green's function $G(p)$ of the exterior of s (with respect to the point of infinity) is characterized by the following properties:

1. $G(p)$ is a harmonic function at every finite point in the exterior of s.
2. If r denotes the distance of p from a fixed point, $\lim_{r\to\infty} rG(p)$ exists.
3. $G(p)=1$ if p is on s.

The function G is uniquely determined.

We recognize that this function is the limit of the potential function $u_\rho(p)$ of a condenser made up by $s_1=s$ and by a sphere s_0 with radius ρ as $\rho\to\infty$. The field is in this case the exterior of s. The capacity C of this condenser can be represented as in §2 (3), that is,

$$C = \frac{1}{4\pi}\iint \frac{\partial G}{\partial n}\, dS, \tag{1}$$

where the integration is extended over an arbitrary surface containing s.

In what follows, we refer to the function $G(p)$ and the constant C as the *potential function and the capacity of the surface s* (of the solid bounded by s, of the conductor bounded by s), respectively. The function $G(p)$ is the potential of certain electric charges on s which are in electrostatic equilibrium because s is, in view of property 3, an equipotential surface. (Kellogg [8, p. 176].) The distribution of these charges is called the *natural distribution.* This is in general an inhomogeneous distribution. (In the case of a sphere it is homogeneous.)

The potential function just defined should not be confused with the potential of homogeneous charges on s. Also we make a clear distinction between the potential function and capacity of a condenser and of a surface (solid, conductor).

(b) Introducing polar coordinates r, θ, ϕ for the variable point p, we have for large r

$$(2)\qquad G(p) = \frac{Y_0(\theta, \phi)}{r} + \frac{Y_1(\theta, \phi)}{r^2} + \cdots + \frac{Y_n(\theta, \phi)}{r^{n+1}} + \cdots ,$$

where $Y_n(\theta, \phi)$ is a surface harmonic of order n. In view of the definition (1), we see that $Y_0(\theta, \phi)$ must be equal to C. Hence the capacity of a surface is the first coefficient in the expansion of the potential function $G(p)$ for large r.

An alternate definition of the potential function and capacity, similar to that in 2 (b), is possible.

Another characterization of the capacity and the natural distribution is the following (Riemann-Weber [28, p. 772]). We consider all positive charges on s with the total charge 1, defined by the density function $\mu(p)$, p on s:

$$(3)\qquad \mu(p) \geqq 0, \qquad \iint \mu(p)dS = 1.$$

Then the integral

$$(4)\qquad \iiiint \frac{\mu(p_1)\mu(p_2)}{r_{12}}\, dS_1 dS_2,$$

where p_1 and p_2 run independently on s (r_{12} is the distance of p_1 from p_2, dS_1 and dS_2 are the area elements at p_1 and p_2, respectively), will be a minimum if $\mu(p)$ is the density of the natural distribution. The minimum of this integral is $1/C$.

For certain reasons which will become clear later, we call the capacity C also the *outer radius* of the surface s (of the solid bounded by s). In the case of a sphere it coincides with the ordinary radius.

4. **Green's function of the interior of a surface.** (a) Let s be a closed surface, q an arbitrary fixed point in the interior of s. Green's function $G(p; q)$ of the interior of s with respect to q is characterized by the following properties:

1. $G(p; q)$ is a harmonic function in the interior of s except at q.
2. If r denotes the distance of p from q, we have $G(p; q) = 1/r - H(p; q)$, where $H(p; q)$ is a harmonic function of p throughout the interior of s.
3. $G = 0$ if p is on s.

The function G is uniquely determined by s and q.

We show that Green's function is closely related to the potential function of a properly defined condenser. We denote the value of H at the point $p=q$ by H_0; since $H=1/r$ on s, we see that $H_0>0$ and H_0^{-1} is a mean-value of the distances r of the points of s from q.

Let ϵ be a positive number, $\epsilon H_0<1$. On the sphere

$$r = \epsilon - H_0\epsilon^2 \tag{1}$$

we have

$$G = 1/(\epsilon - H_0\epsilon^2) - H_0 + O(\epsilon) = 1/\epsilon + O(\epsilon), \tag{2}$$

where the term $O(\epsilon)$ divided by ϵ remains bounded as $\epsilon\to 0$. Let $u_\epsilon(p)$ be the potential function of the condenser formed by the sphere (1) and the surface s. Then the harmonic function

$$u_\epsilon(p)/\epsilon - G(p; q) \tag{3}$$

is 0 on s and equal to $1/\epsilon - G=O(\epsilon)$ on the sphere (1). Hence by the maximum-minimum principle for harmonic functions we have

$$u_\epsilon(p)/\epsilon - G(p; q) = O(\epsilon) \tag{4}$$

in the whole condenser. Consequently

$$\lim_{\epsilon\to 0} (u_\epsilon(p)/\epsilon) = G(p; q) \tag{5}$$

at every point p in the interior of s different from q.

(b) We call the positive constant H_0^{-1} the *inner radius* of s with respect to the point q. As pointed out before, the inner radius is a certain mean-value of the distances of the points of s from q. In the case of a sphere it coincides with the ordinary radius.

If we subject the surface s to an inversion ($r'=1/r$) with respect to q, it will be transformed into a surface s'. It can be shown (cf. Kellogg [8, p. 232]) that the exterior radius of s' is the reciprocal of the inner radius of s with respect to q.

5. **Generalization.** The concept of the capacity or outer radius of a surface (of a solid bounded by a surface) can be generalized. Let us consider a finite number of mutually exclusive solids each bounded by a closed surface. The domain exterior to the set of these solids is connected. We can find again a function $G(p)$ harmonic in this domain for which $\lim_{r\to\infty} rG(p)$ exists and for which the boundary values are 1 on all the boundary surfaces. We call $G(p)$ the potential function of this set of solids (or of the set of surfaces). The capacity or outer radius of this set is defined as in §3 (1) by taking as a surface of integration any closed surface enclosing the whole set in question. The alternate definitions can also be generalized.

By a limiting process this definition can be further generalized and extended to an arbitrary closed set. The capacity or outer radius is an important monotonic set-function. Concerning its relation to certain problems of the Tchebychev type, cf. Pólya-Szegö [24].

Similar generalizations are possible for the inner radius.

6. **Generalized condenser.** Let us consider the mutually exclusive closed surfaces $s_1, s_2, \cdots, s_n$, and let the closed surface s_0 contain all these surfaces. The domain having the surfaces $s_0, s_1, \cdots, s_n$ as its boundary is called again the field.

We define a harmonic function $u_i(p)$ in the field assuming the boundary values

$$u_i(p) = \begin{cases} 1 \text{ on } s_i, \\ 0 \text{ on } s_k, \quad k \neq i; \quad i, k = 0, 1, \cdots, n. \end{cases} \tag{1}$$

We introduce the integrals

$$C_{ik} = -\frac{1}{4\pi}\iint_{s_k} \frac{\partial u_i}{\partial n} dS, \tag{2}$$

where the integration is extended over the surface s_k and the normal is directed into the interior of the field. Obviously C_{ik} is positive if $i=k$ and is negative if $i \neq k$. We call the coefficients C_{ik} the *capacity coefficients.*

By Green's identity we conclude

$$C_{i0} + C_{i1} + \cdots + C_{in} = 0, \qquad i = 0, 1, \cdots, n. \tag{3}$$

Also

$$u_0(p) + u_1(p) + \cdots + u_n(p) = 1, \tag{4}$$

since the function on the left-hand side of (4) has the boundary value 1 on all surfaces s_i.

By Green's identity we find further

$$\begin{aligned} &\frac{1}{4\pi}\iiint \operatorname{grad} u_i \cdot \operatorname{grad} u_k dV \\ &\qquad = -\frac{1}{4\pi}\iint u_k \frac{\partial u_i}{\partial n} dS - \frac{1}{4\pi}\iiint u_k \Delta u_i dV. \end{aligned} \tag{5}$$

The volume integrals are extended over the field, the surface integrals over the complete boundary. Since $u_k=0$ on all the surfaces except on s_k, and $\Delta u_i=0$, we find

$$\frac{1}{4\pi}\iiint \operatorname{grad} u_i \cdot \operatorname{grad} u_k dV = C_{ik}. \tag{6}$$

This new representation implies the *symmetry relation*

$$C_{ik} = C_{ki}. \tag{7}$$

Another remarkable corollary of (6) is the following identity involving the arbitrary real numbers $\alpha_0, \alpha_1, \cdots, \alpha_n$:

$$\sum_{i=0}^{n}\sum_{k=0}^{n} C_{ik}\alpha_i\alpha_k = \frac{1}{4\pi}\iiint\left(\sum_{i=0}^{n}\alpha_i \operatorname{grad} u_i\right)^2 dV. \tag{8}$$

Thus the quadratic form on the left-hand side of (8) is *non-negative.* It is easy to show that the form arising if i and k run from 1 to n is *positive.*

If we write

$$u(p) = \alpha_0 u_0(p) + \alpha_1 u_1(p) + \cdots + \alpha_n u_n(p), \tag{9}$$

the function $u(p)$ will be harmonic in the field and $u=\alpha_k$ on s_k; that is, α_k is the constant value of the potential u on the conductor s_k. The "charge" of u on s_k is

$$\beta_k = -\frac{1}{4\pi}\iint_{s_k}\frac{\partial u}{\partial n}dS = \alpha_0 C_{0k} + \alpha_1 C_{1k} + \cdots + \alpha_n C_{nk}. \tag{10}$$

Obviously

$$\beta_0 + \beta_1 + \cdots + \beta_n = 0. \tag{11}$$

If the charges β_k are prescribed such that they satisfy the relation (11), the system (10) of linear equations can be solved for the α_k. Doing this, we can disregard the equation corresponding to $k=0$ because it is a consequence of the others. They can be written as follows:

$$\begin{gathered}(\alpha_1 - \alpha_0)C_{1k} + (\alpha_2 - \alpha_0)C_{2k} + \cdots + (\alpha_n - \alpha_0)C_{nk} = \beta_k, \\ k = 1, 2, \cdots, n.\end{gathered} \tag{12}$$

The determinant of this system is different from zero and so

$$\alpha_i - \alpha_0 = D_{i1}\beta_1 + D_{i2}\beta_2 + \cdots + D_{in}\beta_n, \qquad i = 1, 2, \cdots, n, \tag{13}$$

where the constants D_{ik} are uniquely determined. They are called the *potential coefficients.* Physical considerations show that they are all positive. The mathematical background of this fact is elucidated by a theorem of Stieltjes [30, p. 82].

The simplest non-trivial example of a generalized condenser is the case $n=2$ when s_1 and s_2 are two mutually exclusive spheres and s_0 is a sphere with infinitely large radius. (Cf. §16.)

7. **Cylindrical (two-dimensional) problems.** (a) As a generalization of the Leyden jar we consider two infinite cylinders with parallel generators, one containing the other. We intersect these cylinders perpendicularly to the generators and denote by l_0 the exterior and by l_1 the interior contour of the cross section. Let u denote the harmonic function in the field between the two cylinders which assumes the constant value 0 on the exterior and the constant value 1 on the interior cylinder. This function is constant along any line in the field parallel to the generators. It is a two-dimensional harmonic function.

The density of the charge on the cylinders is

$$-\frac{\epsilon}{4\pi}\frac{\partial u}{\partial n}, \tag{1}$$

where ϵ is the dielectric constant and n denotes the normal directed into the field. The total charge on the cylinders, per unit length, will be

$$-\frac{\epsilon}{4\pi}\int\frac{\partial u}{\partial n}\,dL, \tag{2}$$

where the integration is extended along l_0 and l_1, respectively, and the normal is defined as before.

(b) We assume $\epsilon=1$ and from now on we consider the problem in the plane. Let l_0 and l_1 be two closed curves, l_1 in the interior of l_0; we say that l_0 and l_1 form a "two-dimensional condenser." The closed set between the curves is called the field. The potential function $u(p)$ of this condenser is a two-dimensional harmonic function in the field, satisfying the relations $u(p)=0$ on l_0 and $u(p)=1$ on l_1. The "two-dimensional capacity" c of this condenser is defined by

$$c=\frac{1}{2\pi}\int\frac{\partial u}{\partial n}\,dL, \tag{3}$$

where the integration is extended over an arbitrary closed curve lying in the interior of l_0 and containing l_1; the normal is the interior one. Since u is a harmonic function, this integral is independent of the line of integration. The capacity c is always positive. The charges (2) are $\pm c/2$. If c is the "two-dimensional capacity" of the two-dimensional condenser defined above, the capacity (per unit length) of the cylindrical condenser in space will be $c/2$. In case of the Leyden jar, discussed in §1, we have ($\epsilon = 1$)

$$u = (\log b - \log r)/(\log b - \log a), \qquad c = 1/\log (b/a). \tag{4}$$

As in space, an alternate definition is possible by means of Dirichlet's integral.

(c) Let l be a closed curve in the plane. Green's function $G(p)$ of the exterior of l (with respect to the point of infinity) is characterized by the following properties:

1. $G(p)$ is a harmonic function at every finite point in the exterior of l.

2. If r denotes the distance of p from a fixed point, we have for large r

$$\begin{aligned} G(p) = \log \frac{1}{r} - \log \frac{1}{\bar{r}} + \frac{a_1 \cos \phi + b_1 \sin \phi}{r} \\ + \frac{a_2 \cos 2\phi + b_2 \sin 2\phi}{r^2} + \cdots, \end{aligned} \tag{5}$$

where $\bar{r}$ is a certain positive constant and $a_1, b_1, a_2, b_2, \cdots$ are real constants.

3. $G(p) = 0$ if p is on l.

The function $G(p)$ is uniquely determined.

We refer to the constant $\bar{r}$ as the *outer radius* or *transfinite diameter* of l (of the domain bounded by l).[2]

Let us consider the two-dimensional condenser made up by $l_1 = l$ and by a circle l_0 of radius ρ. Denoting its potential by $u_\rho(p)$ and capacity by c_ρ, we can show by an argument similar to that in §4 that

$$\begin{aligned} (u_\rho(p) - 1) \log (\rho/\bar{r}) - G(p) = O(1/\rho), \\ \lim_{\rho \to \infty} (u_\rho(p) - 1) \log \rho = G(p), \end{aligned} \tag{6}$$

provided p is in the exterior of l. Moreover

$$c_\rho \log (\rho/\bar{r}) - 1 = O(1/\rho), \qquad \lim_{\rho \to \infty} c_\rho \log \rho = 1. \tag{7}$$

The definition of the outer radius can be extended to an arbitrary closed set. It is a monotonic set-function. Concerning its relation to certain problems of the Tchebychev type, cf. Pólya-Szegö [24].

In order to avoid confusion, we point out that the term outer radius of a plane set can be taken in the "two-dimensional" or "three-dimensional" sense. For instance, the outer radius of a circular disk of radius 1 is equal to 1 in the first sense and to $2/\pi$ in the second sense.

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[2] In Pólya-Szegö [24] this constant was designated also as the capacity constant. In the present exposition we prefer not to use this term.

(d) Let l be again a closed curve in the plane, q an arbitrary fixed point in the interior of l. Green's function $G(p; q)$ of the interior of l with respect to q is characterized by the following properties:

1. $G(p; q)$ is a harmonic function in the interior of l except for $p=q$.
2. If r denotes the distance of p from q, we have

$$G(p; q) = \log (1/r) - H(p; q), \tag{8}$$

where $H(p; q)$ is a harmonic function of p throughout the interior of l.

3. $G=0$ if p is on l.

The function G is uniquely determined by l and q.

This concept is again closely related to the potential of a properly defined condenser. We denote the value of H at the point $p=q$ by H_0 and write $H_0=\log (1/r_q)$. Then the positive constant r_q is called the *inner radius* of l with respect to q. Since $H=\log (1/r)$ on l, we see that $H_0=\log (1/r_q)$ must be a mean-value of the values $\log (1/r)$, hence r_q must be a mean-value of the distances r of the points of l from q.

8. **Conformal mapping.** The concepts of outer and inner radii in the plane are closely related to the conformal mapping.

(a) Let l be a closed curve in the plane. The exterior of l can be mapped conformally onto the exterior of a circle such that the point at infinity and the line element (that is, the magnitude and direction of the line-element) at infinity are preserved. *The radius of this circle is uniquely determined by l. It is identical with the outer radius $\bar{r}$ of l.*

If we introduce complex numbers z in the plane of l and complex numbers w in the plane of the circle, the mapping can be represented in the form $w=f(z)$ where $f(z)$ is regular in the finite z-plane outside of l and has a simple pole at $z=\infty$. It maps the exterior of l onto the domain $|w|>\bar{r}$. For large $|z|$ we have

$$w = f(z) = z + c_0 + c_{-1}z^{-1} + c_{-2}z^{-2} + \cdots, \tag{1}$$

where the coefficients $c_0, c_{-1}, \cdots$ are uniquely determined complex numbers.

Green's function $G(p)$ can be represented in terms of the map function $f(z)$. If the point p corresponds to the complex number z, we have

$$G(p) = \log (\bar{r}/|f(z)|). \tag{2}$$

(b) Let l again be a closed curve, and q a fixed point in the interior of l. The interior of l can be mapped conformally onto the interior of a circle with center at q such that the point q and the line-element at q

are preserved. *The radius of this circle is uniquely determined by l and q. It is identical with the inner radius r_q of l with respect to q.*

If we introduce complex numbers z in the plane of l and complex numbers w in the plane of the circle, and if q is the point $z=w=0$, this mapping can be represented in the form $w=g(z)$, where $g(z)$ is regular in the interior of l, $g(0)=0$, $g'(0)=1$. It maps the interior of l onto the circle $|w|<r_q$. For sufficiently small $|z|$ we have

$$w = g(z) = z + c_2z^2 + c_3z^3 + \cdots, \tag{3}$$

where the coefficients $c_2, c_3, \cdots$ are uniquely determined complex numbers.

Green's function $G(p; q)$ can be represented in terms of this map function $g(z)$. If p is the point z, we have

$$G(p; q) = \log (r_q/|g(z)|). \tag{4}$$

9. **Symmetrization in space.** (a) The capacity or outer radius $C=C(S)$ of a solid S is an interesting function which can be compared with other important set-functions, like the volume $V(S)$, the area $A(S)$ or the diameter $D(S)$. In case S is convex, the integral $M(S)$ of the mean curvature can also be considered. It is obvious that $C(S)$ is "monotonic" and that in case of a sphere $C(S)$ is equal to the radius of this sphere.

In the theory of convex sets various inequalities are established between the quantities mentioned [**2**, p. 84]. These results are centered around the classical isoperimetric inequality:

$$A^3 - 36\pi V^2 \geqq 0, \tag{1}$$

with equality only for the sphere. This inequality holds also for nonconvex solids. Thus: *Of all solids with given area, the sphere has the maximum volume.*

Another important inequality, holding for convex solids, is the following:

$$M^2 - 4\pi A \geqq 0, \tag{2}$$

with equality only for the sphere. Thus: *Of all convex solids with given area, the sphere has the minimum integral of the mean curvature.*

Another way of expressing these inequalities is the following. Let $R_V=[3V/(4\pi)]^{1/3}$ be the radius of the sphere with a volume equal to that of the given solid S. Let $R_A=[A/(4\pi)]^{1/2}$ and $R_M=M/(4\pi)$ have the analogous meaning taking the area and the integral of the mean curvature instead of the volume. Then

$$R_M \geqq R_A \geqq R_V. \tag{3}$$

In the first inequality S must be convex, in the second S may be arbitrary.

(b) In the study of the isoperimetric inequality (1) and in related problems the idea of *symmetrization* is fundamental.

Let us consider a plane ϵ, a straight line e, and a point E. The symmetrization of the solid S with respect to the plane ϵ is defined as a geometric transformation changing the solid S into a solid S' symmetric with respect to ϵ and characterized by the following properties. Any straight line perpendicular to ϵ and intersecting one of the solids S and S' intersects also the other and both intersections have the same length. The intersection with S may consist of several segments whose total length has to be taken; the intersection with S' is either a segment or a point.

Symmetrization with respect to the line e changes S into a solid of revolution S'' with axis e, defined by the following property. Any plane perpendicular to e and intersecting one of the solids S and S'' intersects also the other and both intersections have the same area; the intersection with S'' is either a circular disk or a point.

Symmetrization with respect to the point E changes the solid S into a sphere S''' with center at E; S and S''' have the same volume.

The first kind of symmetrization, repeated for a suitably chosen sequence of planes, generates the other two kinds of symmetrization.

All three kinds of symmetrization preserve the volume and diminish (or leave unchanged) the area of the solid. This is a fundamental result; see, for instance, Bonnesen-Fenchel [2, pp. 69–72]. That part of the result which is concerned with the symmetrization with respect to a point is the classical isoperimetric property of the sphere expressed by the inequality (1).

(c) Let us consider a condenser made up by the closed surfaces s_0 and s_1. We symmetrize the condenser by symmetrizing the two solids bounded by s_0 and s_1 simultaneously, with respect to the same plane or line or point.

All three kinds of symmetrization diminish (or leave unchanged) the capacity of the condenser. That part of this statement which is concerned with the symmetrization with respect to a point was discovered by Poincaré [20, pp. 17–22]. A different proof was suggested by Faber [4] and the first complete proof was given by Szegö [32]. The two other parts are due to Pólya-Szegö [25]. Poincaré's theorem can be expressed in the form

$$C \geqq R_V. \tag{4}$$

From the previous general result, by proper limiting processes, we conclude [25]:

All three kinds of symmetrization decrease (or leave unchanged) the outer radius (capacity) of a surface; they increase (or leave unchanged) the inner radius of a surface with respect to an interior point.[3]

As a corollary we note:

The inner radius of a surface with respect to an interior point is never greater than the outer radius.

Another interesting corollary is the following [25]:

The outer radius of a cylindrical solid is never less than the outer radius of a right circular cylinder of the same height whose base has the same area as that of the given solid.

An analogous extremal property holds for the right circular cone. A limiting case of the above statement arises when the height of the cylinder becomes 0. Since the outer radius of a circular disk of radius a is $2a/\pi$ (§12 (11)), we have:

The outer radius of a disk (in the three-dimensional sense) is never less than $2/\pi$ times the radius of a circle which has the same area as the given disk.

10. **Symmetrization in the plane.** The analogous considerations in the plane are rather obvious.

In the plane we have two kinds of symmetrization, one with respect to a straight line e and another with respect to a point E.

Let l be a closed curve in the plane and A the area enclosed by l, L the length of l, $\bar{r}$ the outer radius of l (in the two-dimensional sense), r_q the inner radius of l with respect to a point q in the interior of l, C the outer radius (capacity) of the disk bounded by l (in the three-dimensional sense). Then we have:

Both kinds of symmetrization increase (or leave unchanged) r_q,[4] *preserve A and diminish (or leave unchanged) L, $\bar{r}$, and C.*

Let us consider first the symmetrization with respect to a line. The part of the above statement concerning A is trivial, and that concerning L is classical (Bonnesen-Fenchel [2, p. 72]). The other parts are due to Pólya-Szegö [25].

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[3] In this case the plane of symmetrization must go through the point q (similarly for the symmetrization with respect to a line or a point).

[4] See footnote 3.

As to the symmetrization with respect to a point, the statement about A is trivial and that concerning L is equivalent to the isoperimetric property of the circle. The statements about r_q and $\bar{r}$ are equivalent to the area theorems of Bieberbach (see Pólya-Szegö [**23**, vol. 2, p. 21, problem 125, p. 22, problem 126]). Concerning extensions of these statements to multiply connected domains, see Pólya [**22**] and Szegö [**31**]. The part concerning C is due to Pólya-Szegö [**25**]. (Cf. the last theorem in §9.)

We note the following inequalities between the quantities mentioned:

$$(1) \qquad r_q \leqq \left(\frac{A}{\pi}\right)^{1/2} \leqq \frac{\pi}{2} C, \qquad \left(\frac{A}{\pi}\right)^{1/2} \leqq \bar{r} \leqq \frac{L}{2\pi}.$$

They all follow from the above statement except the last one which is of a different nature (see, for instance, Pólya-Szegö [**23**, vol. 2, p. 21, problem 124]). As a special corollary we mention:

$$(2) \qquad r_q \leqq \bar{r}.$$

11. **Parallel surfaces.** (a) By the diameter D of a surface we mean the greatest distance between any two of its points.

Considering all closed surfaces s with given diameter, the capacity C of s is a maximum if and only if s is a sphere.

Concerning this extremum property of the sphere, cf. Maxwell [**11**, p. 117] and Szegö [**33**]. It can be expressed in the form of an inequality:

$$(1) \qquad D \geqq 2C.$$

The analogous inequality for the inner radius follows from here since this radius is never greater than C (cf. §9).

It suffices to prove (1) for convex surfaces. The proof given by Szegö is based on the use of parallel surfaces. If we denote by s_h the surface parallel to s in the distance h and by $C(h)$ the capacity of s_h, the derivative $C'(h)$ can be computed. From this not only (1) but the following more precise inequality follows:

Of all closed convex surfaces s with given integral M of the mean curvature, the capacity of s is a maximum if and only if s is a sphere.

This is equivalent to the inequality

$$(2) \qquad M \geqq 4\pi C,$$

which is indeed more precise than (1) since $2\pi D \geqq M$.

Recapitulating, we found the following inequalities for C, V, D, M:

$$D/2 \geqq R_M \geqq C \geqq R_V. \tag{3}$$

(b) The following inequalities for the capacity of a *condenser* are due to Pólya-Szegö [25].

Let the inner surface s_1 of a condenser of capacity C be convex and the outer surface s_0 be parallel to s_1. Let h be the distance between s_0 and s_1, A the area of the surface s_1 and M the integral of the mean curvature of s_1. Then

$$A/(4\pi h) < C < A/(4\pi h) + M/(4\pi). \tag{4}$$

For concentric spheres the second inequality becomes an equation.

The lower bound is a customary approximation for C in case h is small; see for instance Maxwell [11, p. 150]. The upper bound is suggested by certain approximations due to Clausius [3, p. 43]. Concerning the limiting case $h \to \infty$, see (2). An improvement of the upper bound is the following inequality (Pólya-Szegö [25]):

$$\begin{aligned} &\frac{1}{C} \geqq \frac{2\pi}{\lambda M} \log \left(\frac{(1-\lambda)M + 4\pi h}{(1+\lambda)M + 4\pi h} \cdot \frac{1+\lambda}{1-\lambda} \right), \\ &\lambda = (1 - 4\pi A M^{-2})^{1/2}. \end{aligned} \tag{5}$$

(c) Finally we point out the inequalities corresponding to (4) in the two-dimensional case.

Let l_1 be a closed convex curve in the plane and l_0 an arbitrary outer curve parallel to l_1. We denote by c the capacity of the two-dimensional condenser bounded by l_0 and l_1. If L_0 is the length of l_0 and L_1 the length of l_1, we have

$$(L_0/L_1 - 1)^{-1} < c < \{\log (L_0/L_1)\}^{-1}$$

unless l_1 is a circle.

12. **Example 1. Ellipsoid.** (a) Let a, b, c be three constants, $a > b > c > 0$. We consider the quadrics

$$\frac{x^2}{a^2 + \rho} + \frac{y^2}{b^2 + \rho} + \frac{z^2}{c^2 + \rho} = 1, \tag{1}$$

where ρ is a real parameter. For any point (x, y, z) not in a coordinate plane, this equation has three distinct real roots in ρ which we denote by λ, μ, ν:

$$-a^2 < \nu < -b^2 < \mu < -c^2 < \lambda. \tag{2}$$

They are called the *ellipsoidal coordinates*. (See, for instance, Kellogg [8, p. 186].) On the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \tag{3}$$

we have $\lambda = 0$. By means of these coordinates various condenser and conductor problems related to ellipsoids can be discussed in a very elegant way. We consider the problem of electrostatic equilibrium on the ellipsoid (3) (see, for instance, Kellogg [8, pp. 188–191]). If the charge on the ellipsoid is 1, the potential at an outside point $p(x, y, z)$ will be

$$u(p) = \frac{1}{2}\int_{\lambda}^{\infty} \frac{dt}{\{(a^2+t)(b^2+t)(c^2+t)\}^{1/2}}, \tag{4}$$

where λ has the meaning above, $\lambda \geqq 0$. For the capacity C we find

$$\frac{1}{C} = \frac{1}{2}\int_{0}^{\infty} \frac{dt}{\{(a^2+t)(b^2+t)(c^2+t)\}^{1/2}}. \tag{5}$$

The density of the charge on the ellipsoid is

$$\sigma = d/4\pi abc, \tag{6}$$

where d is the distance from the center of the tangent plane at the point (x, y, z).

The problem of a condenser bounded by two confocal ellipsoids can also be treated.

(b) We denote by α, β, γ the numerical eccentricities of the principal sections of the ellipsoid (3):

$$\alpha^2 = \frac{b^2 - c^2}{b^2}, \qquad \beta^2 = \frac{a^2 - c^2}{a^2}, \qquad \gamma^2 = \frac{a^2 - b^2}{a^2}. \tag{7}$$

Then we have the following remarkable representations (Pólyz-Szegö [25]):

$$\begin{aligned} \frac{1}{C} &= \frac{1}{a}\sum_{n=0}^{\infty} \frac{(\beta\gamma)^n}{2n+1} P_n\left(\frac{\beta^2+\gamma^2}{2\beta\gamma}\right) \\ &= \frac{c}{ab}\sum_{n=0}^{\infty} \frac{2\cdot 4 \cdots 2n}{1\cdot 3 \cdots (2n+1)} (\alpha\beta)^n P_n\left(\frac{\alpha^2+\beta^2}{2\alpha\beta}\right), \end{aligned} \tag{8}$$

where $P_n(x)$ denotes Legendre's polynomial.

In the special case of a prolate ellipsoid we have $b = c$, $\alpha = 0$, $\beta = \gamma$, and

$$C^{-1} = (2a\beta)^{-1} \log \frac{1+\beta}{1-\beta}. \tag{9}$$

In the special case of an oblate ellipsoid we have $a=b$, $\alpha=\beta$, $\gamma=0$, and

$$C^{-1} = (a\beta)^{-1} \arcsin \beta. \tag{10}$$

Another interesting special case is that of an elliptical disk. Then $c=0$ and we have

$$\frac{1}{C} = \int_0^{\pi/2} \frac{d\phi}{(a^2 \cos^2 \phi + b^2 \sin^2 \phi)^{1/2}}, \tag{11}$$

that is, C is the arithmetic-geometric mean of the semi-axes a, b multiplied by $2/\pi$. In the case of a circular disk, $a=b$, we find $C=2a/\pi$.

(c) In the special case of the ellipsoid (3), the inequalities (3) of §11 have been refined by Pólya-Szegö [25] (cf. also Pólya-Szegö [21], Watson [36], Szegö [34]) in the following way:

$$\begin{aligned} R_V = (abc)^{1/3} &< \frac{(bc)^{1/2} + (ca)^{1/2} + (ab)^{1/2}}{3} < C \\ &< \frac{a+b+c}{3} < R_M < a. \end{aligned} \tag{12}$$

As to the radius R_A, it turns out that C and R_A are in general not "comparable." But we have (Pólya-Szegö [25])

$$C \gtrless R_A \tag{13}$$

for an ellipsoid of revolution, according as it is prolate or oblate. It is interesting to observe that for such an ellipsoid

$$C/R_A = 1 \pm \frac{2}{945}\beta^6 + \cdots, \tag{14}$$

where the signs ± correspond to prolate and oblate ellipsoids, respectively. The neglected terms contain higher powers of β. (See Aichi [1], Lord Rayleigh [26] and Russell [29].)

From (11) we conclude that the capacity of an elliptical disk with semi-axes a, b does not change if we replace a, b by $(a+b)/2$ and $(ab)^{1/2}$ (cf. Kellogg [8, p. 61]). In an analogous way, Pólya-Szegö [25] investigated certain transformations, related to the Landen transformation of elliptic integrals, which carry ellipsoids into ellipsoids preserving the capacity.

13. Peripolar (toroidal) and dipolar coordinates. (a) Let $a>0$. In the complex $\sigma+it$ plane we introduce the coordinates η, θ of a point p (σ, t) by the following equation:

$$\sigma + it = a\frac{1 + e^{-\eta+i\theta}}{1 - e^{-\eta+i\theta}}, \tag{1}$$

or

$$\sigma = a(\cosh\eta - \cos\theta)^{-1}\sinh\eta, \quad t = a(\cosh\eta - \cos\theta)^{-1}\sin\theta. \tag{2}$$

The geometric meanings of η and θ are obvious. Let A and B be the two points $-a$ and $+a$ on the real σ-axis; then

$$\eta = \log(Ap/Bp), \qquad \theta = \text{angle } (ApB). \tag{3}$$

Of course θ is determined only mod 2π. For the points in the upper half-plane ($t>0$) we take $0<\theta<\pi$, mod 2π, and for the points in the lower half-plane ($t<0$) we take $\pi<\theta<2\pi$, mod 2π. For the point at infinity we have $\theta=0$, mod 2π, on the line AB we have $\theta=\pi$, mod 2π, and on the part of the real σ-axis outside AB we have $\theta=0$, mod 2π.

(b) Let us consider a rectangular coordinate system in space and a point p with coordinates x, y, z. We define the *peripolar* (*toroidal*) coordinates η, θ, ϕ of p in the following way (Riemann [27], C. Neumann [15], Mehler [12]). We put a plane through p and the z-axis which forms the azimuthal angle ϕ with the xz-plane. (This plane is undetermined when p is on the z-axis.) In this plane we introduce the coordinates σ, t; η, θ as before, writing $x=\sigma\cos\phi$, $y=\sigma\sin\phi$, $z=t$; thus

$$\begin{aligned} x &= a(\cosh\eta - \cos\theta)^{-1}\sinh\eta\cos\phi, \\ y &= a(\cosh\eta - \cos\theta)^{-1}\sinh\eta\sin\phi, \\ z &= a(\cosh\eta - \cos\theta)^{-1}\sin\theta. \end{aligned} \tag{4}$$

The range of these coordinates is $\eta\geqq 0$, $0\leqq\theta<2\pi$, $0\leqq\phi<2\pi$.

(c) We define the *dipolar* coordinates (C. Neumann [18]) of the point p in a similar manner by putting a plane through p and the x-axis which forms the azimuthal angle ϕ with the xy-plane. In this plane we introduce σ, t; η, θ as before, writing $x=\sigma$, $y=t\cos\phi$, $z=t\sin\phi$; we find in this case

$$\begin{aligned} x &= a(\cosh\eta - \cos\theta)^{-1}\sinh\eta, \\ y &= a(\cosh\eta - \cos\theta)^{-1}\sin\theta\cos\phi, \\ z &= a(\cosh\eta - \cos\theta)^{-1}\sin\theta\sin\phi. \end{aligned} \tag{5}$$

The range of these co-ordinates is $-\infty<\eta<\infty$, $0<\theta\leqq\pi$, $0\leqq\phi<2\pi$.

We apply the same symbols η, θ, ϕ for both kinds of coordinates; no confusion is possible since in each definite problem only one kind of coordinates is used.

14. **Example 2. Anchor ring (torus).** In peripolar coordinates an anchor ring is defined by the condition $\eta=\eta_0>0$. In the exterior space we have $0<\eta<\eta_0$. The potential function of the ring is (Hicks [6]):

$$(1)\quad u(p)=\frac{2^{3/2}}{\pi}(\cosh\eta-\cos\theta)^{1/2}\sum_{n=0}^{\infty}{}'\frac{Q_{n-1/2}(\cosh\eta_0)}{P_{n-1/2}(\cosh\eta_0)}\cos n\theta P_{n-1/2}(\cosh\eta).$$

The symbol $\sum'$ indicates that the term $n=0$ has to be multiplied by 1/2. The functions[5]

$$(2)\quad \begin{aligned} P_{n-1/2}(\cosh\eta) &= e^{-(n+1/2)\eta}F(1/2,\, n+1/2;\, 1;\, 1-e^{-2\eta}),\\ Q_{n-1/2}(\cosh\eta) &\\ &= \pi\frac{1\cdot3\cdots(2n-1)}{2\cdot4\cdots2n}e^{-(n+1/2)\eta}F(1/2,\, n+1/2;\, n+1;\, e^{-2\eta}), \end{aligned}$$

where F is the hypergeometric function of Gauss and $\eta>0$, satisfy the differential equation

$$(3)\quad d^2y/d\eta^2+\coth\eta\, dy/d\eta+(1/4-n^2)y=0.$$

This differential equation arises from Legendre's equation (satisfied by the Legendre functions $P_n(\cosh\eta)$ and $Q_n(\cosh\eta)$) by replacing n by $n-1/2$. The functions

$$(4)\quad (\cosh\eta-\cos\theta)^{1/2}\cos n\theta\begin{cases}P_{n-1/2}(\cosh\eta)\\ Q_{n-1/2}(\cosh\eta)\end{cases}$$

are harmonic. They are called the *ring* (*toroidal*) *functions.* As to the convergence of (1) we note that $Q_{n-1/2}(\cosh\eta)$ and $P_{n-1/2}(\cosh\eta)$ are of the order $e^{-n\eta}$ and $e^{n\eta}$, respectively, as $n\rightarrow\infty$.

The capacity of the anchor ring is

$$(5)\quad C=\frac{4a}{\pi}\sum_{n=0}^{\infty}{}'\frac{Q_{n-1/2}(\cosh\eta_0)}{P_{n-1/2}(\cosh\eta_0)}.$$

Taking the two first terms of this sum, we have the following approximation of the capacity [6, p. 643]:

$$(6)\quad 2aK/K'+4a(K-E)/E',$$

where K and E denote, as usual, the complete elliptic integrals of the first and second kind, and the modulus is

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[5] The constant factor in $Q_{n-1/2}$ has to be replaced by 1 for $n=0$.

(7) $$k = e^{-\eta_0} = R^{-1}(D - (D^2 - R^2)^{1/2}).$$

Here D is the distance of the center of the revolving disk, which generates the ring, from the z-axis and R is the radius of the disk.

Other problems connected with anchor rings have been treated by Niven [19].

15. **Example 3. Spindle.** In dipolar coordinates a spindle is defined by the condition $\theta = \theta_0$, $0 < \theta_0 < \pi$. In the exterior space we have $0 < \theta < \theta_0$. The potential function of the spindle is (C. Neumann [18, p. 224])

(1) $$u(p) = 2^{-1/2}(\cosh \eta - \cos \theta)^{1/2} \int_{-\infty}^{\infty} \frac{L_q(\cos \theta_0) L_q(-\cos \theta)}{L_q(-\cos \theta_0)} \frac{\cos q\eta}{\cosh q\pi} dq.$$

The function

(2) $$\begin{aligned} L_q(\cos \theta) &= \pi^{-1} \cosh q\pi \int_{-\infty}^{\infty} (2 \cosh \eta' - 2 \cos \theta)^{-1/2} \cos q\eta' d\eta' \\ &= F(1/2 + iq, 1/2 - iq; 1; \cos^2 (\theta/2)) \end{aligned}$$

satisfies the differential equation

(3) $$d^2y/d\theta^2 + \cot \theta \, dy/d\theta - (q^2 + 1/4)y = 0.$$

This equation arises from Legendre's differential equation (satisfied by $P_n(\cos \theta)$ and $Q_n(\cos \theta)$) by substituting $n = -1/2 + iq$. The functions

(4) $$(\cosh \eta - \cos \theta)^{1/2} L_q(- \cos \theta) \cos q\eta$$

are harmonic. They are called the *conal functions.*

As to the convergence of the integral (1) we note that

(5) $$L_q(- \cos \theta)/L_q(- \cos \theta_0)$$

is a positive quantity less than 1 and that

(6) $$\int_{-\infty}^{\infty} \frac{L_q(\cos \theta_0)}{\cosh q\pi} dq$$

is a convergent integral.

For the capacity of the spindle we obtain

(7) $$\begin{aligned} C &= a \int_{-\infty}^{\infty} \frac{L_q(\cos \theta_0)}{L_q(- \cos \theta_0)} \frac{dq}{\cosh q\pi} \\ &= a \int_{-\infty}^{\infty} \frac{F(1/2 + iq, 1/2 - iq; 1; \cos^2 (\theta_0/2))}{F(1/2 + iq, 1/2 - iq; 1; \sin^2 (\theta_0/2))} \frac{dq}{\cosh q\pi}. \end{aligned}$$

For $\theta_0=\pi/2$ (sphere) we have $C=a$, since

$$\int_{-\infty}^{\infty}\frac{dq}{\cosh q\pi}=1. \tag{8}$$

Another form of the potential u, involving an infinite series instead of an integral, is due to H. M. Macdonald [10].

16. **Example 4. Two spheres.** We consider two separated spheres (W. Thomson [35, pp. 86–97]; C. Neumann [14]; Maxwell [11, pp. 266–273]). In this case it is convenient to use again dipolar co-ordinates. But instead of the conal functions we employ now the harmonics

$$(\cosh\eta-\cos\theta)^{1/2}P_n(\cos\theta)e^{\pm(n+1/2)\eta}, \tag{1}$$

where P_n denotes Legendre's polynomials.

We can represent the two spheres in the form $\eta=\alpha<0$ and $\eta=\beta>0$. In the exterior space $\alpha<\eta<\beta$. Using the harmonics mentioned, we have for the potential function

$$\begin{aligned}u(p) &= (2\cosh\eta-2\cos\theta)^{1/2}\\ &\cdot\sum_{n=0}^{\infty}\left(e^{N\alpha}\frac{e^{N(\beta-\eta)}-e^{N(\eta-\beta)}}{e^{N\delta}-e^{-N\delta}}+e^{-N\beta}\frac{e^{N(\eta-\alpha)}-e^{N(\alpha-\eta)}}{e^{N\delta}-e^{-N\delta}}\right)P_n(\cos\theta),\\ &N=n+1/2,\qquad \delta=\beta-\alpha,\end{aligned} \tag{2}$$

and for the capacity

$$C=2a\sum_{n=0}^{\infty}\left(e^{N\alpha}\frac{e^{N\beta}-e^{-N\beta}}{e^{N\delta}-e^{-N\delta}}+e^{-N\beta}\frac{e^{-N\alpha}-e^{N\alpha}}{e^{N\delta}-e^{-N\delta}}\right). \tag{3}$$

An interesting limiting case arises when the spheres are tangent to each other (Maxwell [11, pp. 273–276]). Denoting their radii by r and r', we have in this case

$$C=\frac{rr'}{r+r'}\left(-\frac{\Gamma'}{\Gamma}\left(\frac{r}{r+r'}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{r'}{r+r'}\right)-2\gamma\right), \tag{4}$$

where Γ denotes Euler's gamma function and γ Euler's constant. For $r=r'$ we have $C=2r\log 2=1.39\,r$.

Let us consider the generalized condenser in the sense of §6 bounded by the first sphere s_1, the second sphere s_2, and the sphere of infinitely large radius s_0. Then the potential functions $u_1(p)$ and $u_2(p)$ of §6 can be expressed in the form

$$
(5)\qquad
\begin{aligned}
u_1(p) &= (2\cosh\eta - 2\cos\theta)^{1/2}\sum_{n=0}^{\infty} e^{N\alpha}\,\frac{e^{N(\beta-\eta)} - e^{N(\eta-\beta)}}{e^{N\delta} - e^{-N\delta}}\,P_n(\cos\theta),\\
u_2(p) &= (2\cosh\eta - 2\cos\theta)^{1/2}\sum_{n=0}^{\infty} e^{-N\beta}\,\frac{e^{N(\eta-\alpha)} - e^{N(\alpha-\eta)}}{e^{N\delta} - e^{-N\delta}}\,P_n(\cos\theta).
\end{aligned}
$$

The capacity coefficients C_{11}, $C_{12}=C_{21}$, C_{22} can also be computed (Maxwell [11, p. 271]).

17. **Example 5. Lens.** (a) In this case we return again to the peripolar (toroidal) coordinates defining the boundary of the lens by the conditions $\theta=\theta_1$ and $\theta=\theta_2$. We can always assume that the "upper" surface of the lens is in the upper half-space, $0<\theta_1<\pi$. The "lower" surface may be in the upper or lower half-space, $\theta_1<\theta_2<2\pi$. In the space *exterior* to the lens we define θ according to the condition

$$
(1)\qquad \theta_2 - 2\pi < \theta < \theta_1,
$$

the value $\theta=0$ corresponding to the point at infinity.

We employ the harmonic functions

$$
(2)\qquad (\cosh\eta - \cos\theta)^{1/2} L_q(-\cosh\eta)\begin{cases}\cosh q\theta,\\ \sinh q\theta,\end{cases}
$$

where $L_q(-\cosh\eta)$ is a function arising from the conal function $L_q(-\cos\theta)$ by a proper analytic continuation. We mention the following representation:

$$
(3)\qquad
\begin{aligned}
&L_q(-\cosh\eta)\\
&\quad = \pi^{-1}\cosh q\pi\int_{-\infty}^{\infty}(2\cosh\eta' + 2\cosh\eta)^{-1/2}\cos q\eta'\,d\eta', \quad q \text{ and } \eta \text{ real.}
\end{aligned}
$$

It can be shown that for $q\rightarrow\pm\infty$, η fixed,

$$
(4)\qquad L_q(-\cosh\eta) = O(e^{\epsilon|q|}),
$$

where ϵ is an arbitrary positive number independent of q.

Now we have for the potential function (Mehler [12, p. 141])

$$
(5)\qquad
\begin{aligned}
&u(p) = (2\cosh\eta - 2\cos\theta)^{1/2}\\
&\quad\cdot\int_{-\infty}^{\infty}\frac{\sinh q(\theta_1-\theta)\cosh q(\pi-\theta_2) + \cosh q(\pi-\theta_1)\sinh q(2\pi-\theta_2+\theta)}{2\sinh q(2\pi+\theta_1-\theta_2)\cosh q\pi}\\
&\quad\cdot L_q(-\cosh\eta)\,dq.
\end{aligned}
$$

For the capacity we find

$$
\begin{aligned}
(6)\qquad C &= a\int_{-\infty}^{\infty}\frac{\sinh q\theta_1\cosh q(\pi-\theta_2)+\cosh q(\pi-\theta_1)\sinh q(2\pi-\theta_2)}{\sinh q(2\pi+\theta_1-\theta_2)\cosh q\pi}\,dq\\
&= a\int_{-\infty}^{\infty}\frac{\sinh q(\pi+\theta_1-\theta_2)+\cosh q(2\pi-\theta_1-\theta_2)\sinh q\pi}{\sinh q(2\pi+\theta_1-\theta_2)\cosh q\pi}\,dq.
\end{aligned}
$$

As to the convergence of (5) and (6) we observe that for $q\to+\infty$ the order of magnitude of the factor of L_q on the range (1) is less than

$$
(7)\qquad e^{-q(\pi-|\pi-\theta_2|)}+e^{-q\theta_1},
$$

and this expression approaches 0 exponentially. Now we use (4).

(b) The following special cases of the preceding general result are of particular interest.

(α) Spherical bowl (calotte), $\theta_1=\theta_2$, $0<\theta_1<\pi$ (W. Thomson [35, p. 186]; C. Neumann [17]); the capacity is

$$
\begin{aligned}
(8)\qquad C &= a\int_{-\infty}^{\infty}\left(\frac{\cosh q(\pi-\theta_1)}{\cosh q\pi}\right)^2 dq\\
&= \frac{a}{\pi}+\left(1-\frac{\theta_1}{\pi}\right)\frac{a}{\sin\theta_1}=\frac{a+a'}{\pi},
\end{aligned}
$$

where a is the radius of the "rim" and a' the "spherical radius" of the bowl.

(β) Circular disk, limiting case of (α), $\theta_1=\theta_2\to\pi$; $C=2a/\pi$.[6]

(γ) Half spherical shell (thin), special case of (α), $\theta_1=\theta_2=\pi/2$; $C=a(1/\pi+1/2)$.

(δ) Half spherical solid, $\theta_1=\pi/2$, $\theta_2=\pi$; $C=2a(1-3^{-1/2})$.

(ϵ) Symmetrical lens, $\theta_1+\theta_2=2\pi$; in this case

$$
(9)\qquad C = a\int_{-\infty}^{\infty}\frac{\cosh q(\pi-\theta_1)}{\cosh q\theta_1\cosh q\pi}\,dq.
$$

(ζ) Sphere, $\theta_2-\theta_1=\pi$;

$$
(10)\qquad C = a\int_{-\infty}^{\infty}\frac{\cosh q(\pi-2\theta_1)}{\cosh q\pi}\,dq=\frac{a}{\sin\theta_1}.
$$

(η) Two spheres tangent to each other, a suitable limiting case, $\theta_1\to 0$, $\theta_2\to 2\pi$, $a\to 0$; cf. §16.

(θ) The two spheres forming the lens are orthogonal in such a way that the exterior (dielectric) angle is $\pi/2$, $\theta_2-\theta_1=3\pi/2$ (Maxwell [11, p. 261]);

---

[6] According to a remark of W. Thomson [35, p. 180] Cavendish found by experiment $a/C=1.571$.

$$C = r + r' - a, \tag{11}$$

where r and r' are the radii of the spheres.

The capacity integral (6) can be computed in closed form in all cases when the dielectric angle $2\pi+\theta_1-\theta_2$ is a rational multiple of π. As to the potential function $u(p)$, it can be computed in terms of elementary functions only for certain special multiples; in other cases it leads to elliptic and more complicated functions. (See Macdonald [9].)

In certain cases the problem of the lens can be also treated by using the famous method of electric images due to W. Thomson [35, pp. 144–191]; see also Maxwell [11].

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STANFORD UNIVERSITY

REPORT OF THE MEETING OF THE COUNCIL OF THE SOCIETY IN NEW YORK CITY ON FEBRUARY 24, 1945

Since permission to hold the regular February meeting of the Society was denied by the War Committee on Conventions of the Office of Defense Transportation, it was necessary to cancel the scientific program. However, the scheduled meeting of the Council was held in the Men's Faculty Club at Columbia University at 1:00 P.M. on Saturday, February 24, 1945. The following members of the Council were present: Garrett Birkhoff, S. S. Cairns, H. B. Curry, Arnold Dresden, Nelson Dunford, B. P. Gill, M. R. Hestenes, Einar Hille, T. R. Hollcroft, Harold Hotelling, Marston Morse, O. E. Neugebauer, J. F. Ritt, P. A. Smith, I. S. Sokolnikoff, Oswald Veblen, Hermann Weyl, and the Secretary.

At the request of the authors, four of the papers submitted for presentation at the February New York meeting have been reserved for presentation at future meetings of the Society. All other papers submitted were read by title at the meeting of the Council; titles and cross references to the abstracts of these papers appear at the end of this report.

The Secretary announced the election of the following thirty-two persons to membership in the Society:

Professor Isaac L. Battin, Drew University;
Mr. Raymond Buckingham Block, Wright Field, Dayton, Ohio;
Mr. George F. Bradfield, Northwestern University;
Dr. Lee Byrne, Elgin Academy, Elgin, Ill.;
Mr. George Clyde Campbell, Major, U.S.A.;
Mr. William Walter Durding, Lieutenant Colonel, A.U.S.;
Mr. Bernard Epstein, Washington, D. C.;
Mr. Edward J. Evensen, Technician Fourth Grade, U.S.A.;
Mr. Ivan Arthur Flodin, A. S. & R. Co., Whiting, Ind.;
Mr. George Grossman, Manhattan High School of Aviation Trades, New York, N. Y.;
Mr. Juan P. Guglialmelli, San Martin, Buenos Aires, Argentina;
Mr. Robert Jastrow, Columbia University;
Dr. Robert Clark Jones, Polaroid Corporation, Cambridge, Mass.;
Dr. Jerome C. R. Li, Queens College, Flushing, N. Y.;
Mrs. Edith H. Luchins, Brooklyn College;
Mr. James Wallace Marshall, Wayne University;
Dr. Ella Marth, St. Louis, Mo.;
Dr. Margaret Evelyn Mauch, Michigan State College;
Mr. Edward Anthony Maziarz, St. Joseph's College, Collegeville, Ind.;
Mrs. Helen Kelsall Nickerson, Wheaton College, Norton, Mass.;
Mr. Robert Harrison Shaw, Ensign, U.S.N.R.;
Miss Jean Jadviga Siebielec, Lieutenant (junior grade), U.S.N.R.;

Dr. Charles V. L. Smith, Lieutenant Commander, U.S.N.R., United States Naval Academy;
Mr. Abraham Barnett Soble, University of Cincinnati;
Sister Mary Philip Steele, Rosary College, Rivert Forest, Ill.;
Mr. Harry I. Treiber, Princeton University;
Mr. Eugene Francis Trombley, Bendix Research Laboratories, Detroit, Mich.;
Mr. Bradford Tye, Bethany College, Bethany, W. Va.;
Mr. Jose Antonio Valladares, International Telephone and Telegraph Corporation, New York, N. Y.;
Miss Jean Brosius Walton, Swarthmore College;
Dr. Sidney Weinbaum, Curtiss-Wright Corporation, Buffalo, N. Y.;
Mr. Harry Weingarten, Seward Park High School, New York, N. Y.

It was reported that the following had been elected to membership on nomination of institutional members as indicated:

Brown University: Messrs. Abraham Franck and Leonard Greenstone;
Bryn Mawr College: Misses Jean Ann Schaffer and Enriqueta González-Baz;
California Institute of Technology: Mr. Donald Harry Potts;
University of California: Messrs. Way Ming Chen and Frank Jones Massey, Jr.;
University of California at Los Angeles: Mr. William Sharp Gustin, Mrs. Margaret B. Lehman;
University of Chicago: Miss Betty Ella Alexander;
University of Cincinnati: Mr. William R. Harlow;
College of the City of New York: Messrs. Gerald Freilich and Samuel Goldberg;
Columbia University: Mr. John Lee Bogdanoff, Mrs. Denyse Mosseri Dreyfus, Messrs. Hermann Karl Meyer and John Raleigh;
Cornell University: Miss Mary Ann Lee, Mr. Abraham Morton Mark, Miss Irma Ruth Moses, Mr. Irving Reiner;
Duke University: Mr. Eckford Larry Cohen;
University of Illinois: Miss Grace E. Bates, Mr. James Lane Boswell, Misses Marianne R. Freundlich, Annette Sinclair and Christine Sykes Williams;
Indiana University: Mrs. Loraze Brackett Taylor;
Johns Hopkins University: Mrs. June Horn Clearman, Messrs. Calvin Richard Putnam and Irving Sussman;
Massachusetts Institute of Technology: Messrs. Frank Edgar Bothwell and Stephen Harry Crandall;
Michigan State College: Miss (Mary) Isobel Blyth;
University of Michigan: Reverend Richard Sigmund Cebula, Sister Mary Celine Fasenmyer;
University of Minnesota: Messrs. Monroe David Donsker and Walter Fleming;
New York University: Mr. Peter D. Lax;
Northwestern University: Miss Alice Jean Froseth;
Ohio State University: Miss Miriam C. Ayer;
University of Rochester: Mr. George Milton Wing;
Smith College: Miss Felice Hilda Davidson;
Syracuse University: Mr. Shepard Bartnoff;
University of Toronto: Messrs. James Allister Jenkins and Robert Steinberg;
University of Virginia: Messrs. Edwin E. Floyd and W. Roy Utz;
University of Washington: Mr. Tom Mike Apostol;

University of Wisconsin: Miss Violet Grace Hachmeister, Mr. Richard Ralph Heacock, Miss Gloria Olive, Messrs. Whitney Gillete Scobert and Theodore Wysocki;
Yale University: Messrs. Joseph Ross Lee, Jr., and Frank Herbert McGar, Jr.

The following appointments by the President were reported: as a member of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor Morgan Ward (committee now consists of Professors A. C. Schaeffer, Chairman, D. H. Lehmer, and Morgan Ward); as representative of the Society on the Supervisory Board of the American Year Book for the period 1945–1947, Professor G. T. Whyburn; as a Committee on Nomination of Officers and Members of the Council for 1946, Professors R. E. Langer (Chairman), R. P. Agnew, G. C. Evans, G. T. Whyburn, and D. V. Widder; as Chairman of the Committee on Printing Contracts, Professor R. M. Foster (committee now consists of Professor R. M. Foster, Chairman, Dean M. H. Ingraham, and Professor J. D. Tamarkin); as a Committee on Arrangements for the 1945 Summer Meeting in Montreal, Canada, Professors R. E. O'Connor (Chairman), A. H. Gillson, T. R. Hollcroft, B. W. Jones, and W. L. G. Williams; as a committee to consider a suitable memorial to the late Professor G. D. Birkhoff, Professors R. E. Langer (Chairman), G. A. Bliss, Marston Morse, J. D. Tamarkin, and D. V. Widder; as auditors of the Society's accounts for 1945, Professors T. F. Cope and A. E. Meder, Jr.; as a member of the War Policy Committee (replacing Professor W. D. Cairns), Professor C. C. MacDuffee; as a member of the Committee on the Role of the Society in Mathematical Publication (replacing Professor Oscar Zariski), Professor F. D. Murnaghan (committee now consists of Professors R. L. Wilder, Chairman, Garrett Birkhoff, G. C. Evans, J. R. Kline, ex officio, R. E. Langer, F. D. Murnaghan, and J. D. Tamarkin); as Committee on the Society Visiting Lectureship, Professor Solomon Lefschetz (Chairman), Dean R. G. D. Richardson, and Professor Norbert Wiener.

The Secretary reported that the following had been chosen as the voting representatives of editorial committees on the Council for the year 1945: Bulletin—Professor P. A. Smith; Transactions—Professor A. A. Albert; Colloquium Publications—Professor J. F. Ritt; American Journal of Mathematics—Professor Hermann Weyl; Mathematical Reviews—Professor O. E. Neugebauer; Mathematical Surveys—Professor J. D. Tamarkin.

It was announced that Professor J. C. Slater had accepted the invitation to deliver the Josiah Willard Gibbs Lecture at the 1945 Annual Meeting and that Professor Hassler Whitney had accepted the invitation to deliver the Colloquium Lectures at the 1946 Summer Meeting.

Certain invitations to deliver addresses were announced: Professor Samuel Eilenberg for the 1945 Summer Meeting; Professors Jerzy Neyman and George Polya for the April, 1945, meeting at Stanford University.

It was reported that Professor J. D. Tamarkin had been selected as Chairman of the Mathematical Surveys Editorial Committee.

It was reported that Dr. R. P. Boas had accepted the position as Executive Editor of Mathematical Reviews, effective July 1, 1945. Dr. Boas will replace Professor Will Feller who has been the Executive Editor of Mathematical Reviews since the foundation of that journal in 1940.

On recommendation of the Colloquium Editorial Committee, the Council accepted for publication in the Colloquium Series the manuscript of Professor André Weil entitled *Foundations of algebraic geometry.*

The Council voted to cancel the scientific meetings of the Society which have been scheduled for April 27–28, 1945, in Chicago and April 27–28, 1945, in New York. The meeting scheduled for April 28, 1945, in Stanford University will be held since this meeting is not subject to the recent regulations of the Office of Defense Transportation. The problem of the Summer Meeting, now scheduled for Montreal, Canada, on June 24–25, 1945, and the other meetings which have been planned for 1945 are to be discussed at a future meeting of the Council.

Titles and cross references to the papers read at the Council meeting follow below.

1. E. W. Barankin: *Bounds for the characteristic roots of a matrix.* (Abstract 51-3-51-*t*.)

2. Felix Bernstein: *The integral equations of the theta function.* (Abstract 51-3-61-*t*.)

3. Garrett Birkhoff: *Lattice-ordered Lie groups.* (Abstract 51-3-52-*t*.)

4. H. W. Eves: *Analytical and graphical rectification of a tilted photograph.* (Abstract 51-1-26-*t*.)

5. A. L. Foster: *Boolean-like rings, a generalization of Boolean rings. The logical algebra of general commutative rings.* (Abstract 51-3-53-*t*.)

6. H. L. Garabedian: *Stieltjes integral transformations.* Preliminary report. (Abstract 51-3-62-*t*.)

7. G. N. Garrison: *Note on invariant complexes of a quasigroup.* (Abstract 51-3-54-*t*.)

8. E. J. Gumbel: *Expected values of a statistical variate.* (Abstract 51-3-70-*t*.)

9. A. P. Hillman: *On derivatives of differential polynomials.* (Abstract 51-1-7-*t*.)

10. Dunham Jackson: *The boundedness of certain sets of orthonormal polynomials in one, two, and three variables.* (Abstract 51-3-63-*t*.)

11. Mark Kac: *Random walk in the presence of absorbing barriers.* (Abstract 51-1-44-*t*.)

12. Irving Kaplansky: *The commutativity of generalized Boolean rings.* (Abstract 51-1-9-*t*.)

13. Edward Kasner and John DeCicco: *Some properties of minimal surfaces.* (Abstract 51-3-69-*t*.)

14. R. L. Moore: *Concerning tangents to continua in the plane.* (Abstract 51-5-102-*t*.)

15. F. J. Murray: *Gears, cams and differentials.* (Abstract 51-1-29-*t*.)

16. F. J. Murray: *On solving a set of n linear equations.*(Abstract 51-1-30-*t*.)

17. Harry Pollard: *A theory of the Laplace integral.* Preliminary report. (Abstract 51-3-65-*t*.)

18. H. E. Salzer: *Inverse interpolation for eight-, nine-, ten-, and eleven-point direct interpolation.* (Abstract 51-1-34-*t*.)

19. F. E. Satterthwaite: *Error control in matrix calculation.* II. (Abstract 61-5-82-*t*.)

20. M. F. Smiley: *A remark on metric boolean rings.* (Abstract 51-3-57-*t*.)

21. Ernest Snapper: *Partial differentiation and elementary divisors.* (Abstract 51-3-58-*t*.)

22. R. H. Sorgenfrey: *A condition that a continuum be irreducible between two points.* (Abstract 51-3-71-*t*.)

23. W. J. Sternberg: *On a special set of algebraic nonlinear equations.* (Abstract 51-3-59-*t*.)

24. C. A. Truesdell: *The membrane theory of shells of revolution.* (Abstract 51-3-67-*t*.)

25. P. A. White: *On new types of regular convergence.* (Abstract 51-3-72-*t*.)

26. G. T. Whyburn: *Extensions of plane continua mappings.* (Abstract 51-5-103-*t*.)

27. G. T. Whyburn: *On uniqueness of the inverse of a transformation.* (Abstract 51-5-104-*t*.)

J. R. KLINE,
Secretary

BOOK REVIEWS

Infinite series. By J. M. Hyslop. New York, Interscience, 1942. 120 pp. $1.75.

This textbook gives a concise presentation of the classic theory of convergence of real sequences and series. It is intended for students having had an elementary course in calculus. The brevity of the text precludes extensive motivation of the definitions and theorems, but numerous examples worked out in the text provide illumination. Problems, with answers given, appear at the end of each chapter.

Chapter 1 (17 pages) treats limits of functions by the ϵ-δ methods, and introduces the σ-O notation. Chapter 2 (7 pages) gives Taylor formulas, with remainders, for several elementary functions. Chapter 3 (11 pages) gives definitions, examples, and fundamental properties of convergence, including the Cauchy-sequence criterion. Chapter 4 (22 pages) gives the integral, comparison, ratio, Cauchy nth root (Cauchy condensation, sic), Kummer, and Raabe tests for convergence of series of nonnegative terms. For positive integers n, the Stirling formula $n!=(2n\pi)^{1/2}n^ne^{-n}[1+O(1/n)]$ is derived. Chapter 5 (10 pages) treats absolute and conditional convergence, giving the Abel and Dirichlet tests and Riemann's theorem on rearrangements. Chapter 6 (18 pages), on series of functions, treats uniform convergence, termwise integration and differentiation, and power series. Chapter 7 (7 pages) treats convergence of "the product series." The product series, defined without motivation, is that of Cauchy; one may feel that the author should have called the series by its familiar name so that the possibility of other interesting product series (Dirichlet, for example) would not be summarily dismissed. Chapter 8 (12 pages) treats infinite products, including those of the sine, cosine, and factorial (gamma) functions.

Chapter 9 (11 pages) treats double series. Summation "by triangles," "by squares," "by rows" and "by columns" are defined; to correct the formulation of the second of these definitions (p. 107, formula 2), replace N by k inside the parentheses and insert $\sum_{k=1}^{N}$ before the first parenthesis. The definition of convergence and divergence of a double series $\sum a_{mn}$ runs as follows: "Naturally, we wish our definition of the sum of a double series to conform as closely as possible to the definition of the sum of a single series. This analogy may be preserved by starting at the top left-hand corner of the array and taking successive groups of terms, where each group consists of only a finite number of terms of the series and contains all the elements of the preceding group, and then examining the limit of the pth group as p tends to infinity. These successive groups correspond in fact to successive partial sums in the case of single series. Generally speaking, the limit of the pth group will depend on the system by means of which the groups are formed. When, however, the limit is finite and independent of the system of grouping, we say that the double series is CONVERGENT and that the limit in question is the sum of the series. In all other cases, the series is said to be DIVERGENT." This unorthodox definition of "sum" of a double series seems to be inadequate. The author must have intended that each term a_{mn} of the series would appear in the union of the groups; without such an understanding, no double series would be convergent unless all of its term were 0. Unless one adds a further requirement limiting the characters of the groups, one can arrange the terms of the double series into a simple series in any way one chooses and then determine the groups so that the pth group will consist of the first p terms of the simple series; this implies that $\sum u_{mn}$ cannot be convergent unless it is absolutely convergent. Since the chapter gives the classic theorems pertaining to Pringsheim convergence, giving the customary relations between convergence and absolute convergence, it would seem that "convergence" should be replaced by Pringsheim convergence in this chapter.

The text is carefully written, with due regard to such details as the distinction between positive and nonnegative. It is, as a whole, precise and accurate. Some lapses are to be found. There are a few places where $<$ should be replaced by $\leqq$, and a few where $=$ should be replaced by $\leqq$. There are a few places where statements involving "any" are ambiguous; when these statements are removed from their contexts, one cannot tell whether the "any" means "each" or "at least one." Sometimes the intended meanings of statements are destroyed by unfortunate wordings. In the process of defining the least upper bound K of a function $f(x)$, the author is guilty of saying "there is at least one value of x for which $f(x) > K - \epsilon$, where ϵ is any positive number" when he should (and doubtless does) mean that "to each $\epsilon > 0$ corresponds at least one value of x for which $f(x) > K - \epsilon$." A student who reads the author's statement and concludes that $f(x) \geqq K$ is not to be condemned; under the author's definition the set of numbers x for which $0 < x < 1$ does not have a least upper bound. Several theorems in which an equation involving θ is said to hold "where $0 < \theta < 1$" could be made more precise and intelligible by saying that there is at least one θ such that $0 < \theta < 1$ and the equation holds.

One who has finished a first course in calculus should find this book to be an intelligible introduction to basic methods in modern mathematical analysis as well as to the classic theory of convergence of series.

RALPH P. AGNEW

Miscellaneous physical tables. Planck's radiation functions and electronic functions. New York, Work Projects Administration, 1941. 7+61 pp. $1.50.

In the first part of this volume of the well known W.P.A. series, values are given for the spectral radiancy of a black body, the rate of photon emission, and the indefinite integrals (from 0 to λ in wavelength) of the functions. These are tabulated at constant intervals of $\lambda \cdot T$ (wavelength multiplied by Kelvin temperature), the size of the intervals changing occasionally. Auxiliary tables give values of some of the preceding quantities as tabular functions of wavelength alone at a few definite temperatures.

The second part tabulates relativistically correct values of the following properties of an electron, as functions of its velocity (entered with constant intervals in fractions of c, the velocity of light): the energy, in units of the electron rest energy m_0c^2; the momentum, in units of m_0c; the energy in electron-kilovolts; and the $H \cdot \rho$-values (ρ is the radius of a circle in which the electron would move if projected with the tabulated velocity normal to a magnetic field of H oersteds).

Each part of the volume was inspired by suggestions of well known scientists, as stated in the introductory discussions. The first part has been published previously in the February 1940 number of the Journal of the Optical Society of America. The introduction contains brief descriptions of the functions, examples illustrating use of and interpolation in the tables, a summary of the methods used in assembling or computing the entries, and a bibliography. In the major tables of the first part, first differences and second central differences are given for greater ease and accuracy in interpolation. A complication not met in purely mathematical tables exists here due to the fact that all the functions tabulated depend on measured physical constants: the velocity of light, Planck's constant, and electronic properties. The tables were based on the present best values of these constants, but are carried to extra figures. Directions are given for making corrections in case these are necessitated by newer values of the basic physical constants.

The tables are in the usual format of the W.P.A. series. They appear simple to use, and are of obvious assistance in certain fields of pure and applied science. A single minor criticism may be offered. In the second part, the electron velocity might perhaps have been entered in cgs units as well as in fractions of the velocity of light.

A. O. WILLIAMS, JR.

Galois theory. By Emil Artin. With a chapter on applications by A. N. Milgram. (Notre Dame Mathematical Lectures No. 2.) University of Notre Dame, 1942. 70 pp. $1.25.

This is the first appearance in print of a complete presentation of Artin's brilliant improvement on the classical Galois theory of fields and equations. It provides a fundamentally better development of this most important algebraic theory, and algebraists will be very pleased that it has appeared.

The field theory is presented in the forty-four pages of Chapter II and is preceded by a twelve page first chapter on the linear algebra of systems of linear equations with coefficients in a quasi-field. The fields are then assumed to be commutative. The first five sections present the concepts and elementary properties of extension fields, relative degrees and reducibility of polynomials. They also contain proofs of the existence of simple algebraic extensions and of splitting fields and of the uniqueness of the latter.

The principal non-classical part of the approach occurs in Section F where the concept of fixed field is given and in Section G where the background of the fundamental theorem of the Galois theory is presented and the theorem is proved. It should be noted here that in the fundamental theorem the hypothesis that the irreducible factors of $p(x)$ are separable is required and was probably inadvertently omitted.

There are additional sections on finite fields, roots of unity, Noether equations, Kummer fields, simple extensions, the existence of a normal basis, and the Galois group of the composite of two fields. These results are applied in the final chapter by A. N. Milgram to the proof of the theorem that the general equation of degree $n>4$ is not solvable by radicals. There is also a section in this last chapter on a condition that an equation of prime degree be solvable by radicals, and a final section on ruler and compass construction.

The book as a whole is a model of elegant, compact presentation.

A. A. ALBERT

Received by the Editor of Book Reviews January, 1943; the delay in publication is due to the subsequent loss of the manuscript in the mail.

NOTES

The Chauvenet Prize for 1941–1943 has been awarded by the Mathematical Association of America to Professor R. H. Cameron of the Massachusetts Institute of Technology.

Professor A. E. Johns of McMaster University has been elected president of the Royal Astronomical Society of Canada. Professor J. W. Campbell of the University of Alberta has been elected vice president.

Dr. Harry Pollard has received from the American Telephone and Telegraph Company one of the five Frank B. Jewett fellowships for research in the physical sciences.

Dr. N. A. Hall has been appointed mechanical engineer in charge of thermodynamic analysis for the Research Division of the United Aircraft Corporation.

Professor W. H. Hill of Kansas State Teachers College of Pittsburg, Kansas, has retired.

Assistant Professor J. E. Ikenberry of Franklin and Marshall College has been appointed to a professorship at Madison College, Harrisonburg, Virginia.

Assistant Professor E. R. Lorch of Columbia University has been promoted to an associate professorship.

Associate Professor G. E. Moore of the University of Illinois has been appointed assistant dean of liberal arts and sciences.

Mr. S. T. Parker of the University of Louisville has been promoted to an assistant professorship.

Associate Professor George Polya of Stanford University has been appointed to a visiting professorship at the University of California at Berkeley.

Dr. R. D. Specht of the University of Florida has been appointed to an assistant professorship at the University of Wisconsin.

Assistant Professor B. C. Wong of the University of California has retired.

The following appointments to instructorships are announced: Polytechnic Institute of Brooklyn: Mr. M. J. Forray; University of

Kentucky: Mrs. Betty D. Crawley; Wayne University: Mr. Arvid Jacobson.

Professor C. H. Rowe of the University of Dublin died in November 1944. He was a member of the Society for thirteen years.

Dean J. L. Gibson of the University of Utah died February 10, 1945. He had been a member of the Society since 1916.

Mr. M. M. S. Moriarty, formerly a teacher in Holyoke, Massachusetts, High School, died April 30, 1944. He was a member of the Society for thirteen years.

Associate Professor A. L. Underhill of the University of Minnesota died January 18, 1945. He had been a member of the Society since 1907.

Professor N. R. Wilson of the University of Manitoba died December 27, 1944. He had been a member of the Society since 1900.

The following thirty-nine doctorates, with mathematics or mathematical physics as a major subject, were conferred during 1944 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which degree was conferred, minor subject (other than mathematics) and the title of the dissertation are given in each case if available.

W. H. Brothers, Jr., Michigan, November, *On the solution of boundary value problems in hyperbolic differential equations.*

H. D. Brunk, Rice, February, *Some generalizations for Dirichlet series of Hadamard's theorem, with applications.*

Enrique Bustamante-Llaca, Princeton, October, *Transfinite type theory.*

C. L. Clark, Virginia, June, *Arc reversing transformations.*

V. F. Cowling, Rice, October, *On functions defined by a Taylor series.*

W. J. R. Crosby, Toronto, June, *Generic simple algebras.*

J. J. Dennis, Northwestern, August, *Some points in the theory of positive definite J-fractions.*

W. J. Dixon, Princeton, April, *Further contributions to the theory of serial correlation.*

C. L. Dolph, Princeton, April, *Non-linear integral equations of the Hammerstein type.*

Wade Ellis, Michigan, June, *On relations satisfied by linear operators on a three dimensional linear vector space.*

A. B. Farnell, California, February, *Characteristic roots of matric polynomials and a study of infinite matrices.*

Herbert Federer, California, February, *Surface area.*

Mariano García, Jr., Virginia, June, *Orbit-components and component orbits.*

Ruth E. Goodman, Pennsylvania, February, *On the Bloch-Landau constant for schlicht functions.*

W. H. Gottschalk, Virginia, February, *An investigation of continuous mappings with almost periodic properties.*

A. A. Grau, Michigan, November, *Ternary operations and Boolean algebra.*

Charles Hatfield, Jr., Cornell, June, *On the average number of roots of certain random functions.*

Solomon Hurwitz, Columbia, August, *On a class of functions suggested by the zeta of Riemann.*

A. W. Jones, Columbia, June, *Isomorphism between lattices of certain finite groups.*

Ida Roettinger Kaplan, Michigan, November, *An operational approach to the solution of boundary value problems by generalized Fourier series.*

Meyer Karlin, Columbia, March, *Characterization of the system of extremals of a variation problem of higher order in the plane.*

Jennie P. Kormes, New York, February, *The solution of the differential equation* $(\partial^2/\partial t^2-\Delta)(a^2\partial^2/\partial t^2-\Delta)u=f(x, y, z, t)$ *by Hadamard's method.*

D. P. Ling, Columbia, June, *The im Grossen behavior of the geodesics on surfaces of revolution.*

J. N. B. Livingood, Pennsylvania, July, *A partition function with prime modulus* $P>3$.

Ella C. Marth, Saint Louis, May, minor in education, *Further properties of Garvin's F-series.*

Margaret P. Martin, Minnesota, August, minor in biostatistics, *Some new systems of orthogonal polynomials on algebraic curves.*

Sister Mary Ingonda von Mezynski, S.Sp.S., Marquette, June, *Projective description of plane sextic curves derived from conics as base curves.*

Knox Millsaps, California Institute of Technology, October, minor in physics, *Contributions to the theories of topological groups.*

Ethel C. Muggli, Washington, August, minor in education, *Some formal aspects of the Riquier theory of systems of partial differential equations.*

C. E. Noble, Iowa, August, *The isogonal transformations associated with the plane triangle.*

M. J. Norris, Harvard, June, *Topological spaces having the same regular open sets.*

Mary K. Peabody, Yale, June, *Differential operators of infinite order.*

A. M. Peiser, Cornell, February, *Some applications of Fourier analysis and calculus of probability to the study of real roots of algebraic equations.*

P. C. Rosenbloom, Stanford, April, *On sequences of polynomials, especially sections of power series.*

Edward Rosenthall, California Institute of Technology, October, minor in physics, *Some Diophantine problems.*

S. A. Schaaf, California, June, *Cooling of non-homogeneous media with cylindrical symmetry.*

L. I. Schoenfeld, Pennsylvania, July, *A transformation formula in the theory of partition.*

P. M. Stehle, Princeton, June, major in mathematical physics, *Star streaming.*

C. A. Truesdell, Princeton, January, *The membrane theory of shells of revolution.*

Sister M. Claudia Zeller, Michigan, March, *The development of trigonometry from Regiomontanus to Pitiscus.*

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

73. A. P. Hillman: *On identities for differential polynomials.*

The author generalizes his previous theorems (see Bull. Amer. Math. Soc. vol. 49 (1943) pp. 711–712) as follows. Let $F \neq 0$, $C_1, \cdots, C_s$ be differential polynomials, and let $C_1P_1 + \cdots + C_sP_s = 0$, where the P_i are distinct power products of degree d_i and order p_i in F and its derivatives. Also let $p_j > p_k$ whenever $d_j < d_k$. Then each C_i holds F. There is an analogue for partial differential polynomials. (Received March 29, 1945.)

74. A. P. Hillman: *Theorems obtained from the Newton polygon process for differential polynomials.*

Two of Ritt's theorems (see §8, Amer. J. Math. vol. 60 (1938) pp. 1–43) are generalized to include the following. Let A and B be forms (ordinary or partial differential polynomials), let A have constant coefficients, and let B hold A. Then the form B^* composed of the terms of B for which $ad+bw$ is least holds the corresponding form A^*. Here d stands for degree, w stands for weight, and a and b are any real numbers. (Received March 29, 1945.)

75. Bjarni Jonsson: *On unique factorization problem for torsionfree abelian groups.*

The problem has been raised whether the unique factorization theorem holds for all groups which are (finite) direct products of indecomposable groups (cf. A. Kurosch, Math. Ann. vol. 106). The solution is negative, even for the class $\mathfrak{H}$ of torsionfree abelian groups of finite rank. In fact, let a, b, c, d be rationally independent real numbers. Let A, B, C, D be additive groups of real numbers, defined as follows. A and B consist of all numbers $5^n ia$ and $5^n ib + 7^n jc + 11^n kd + (l/3)(b-c) + (m/2)(b+d)$ respectively; i, j, k, l, m, n being integers. C and D consist of those members of $A \times B$ which are of the form $t(3a-b)+uc$ and $t(2a-b)+ud$ respectively; t and u being rational numbers. Then $A \times B = C \times D$, all four groups being indecomposable and non-isomorphic. However, a unique factorization theorem in a weaker form applies to $\mathfrak{H}$. The two groups A, B are called equivalent, $A \equiv B$, if A is isomorphic with a subgroup of B and conversely. A group A, of positive rank, is strongly indecomposable if $A \equiv B \times C$ never holds unless $A \equiv B$ or $A \equiv C$. Then every group in $\mathfrak{H}$ is equivalent to a direct product of strongly indecomposable groups; and if it is equivalent to two such products, then the number of factors in both products is the same, and the factors are (apart from order) respectively equivalent. (Received March 24, 1945.)

76. G. K. Kalisch: *A spectral theory for symmetric operators over generalized Hilbert spaces.*

In a previous paper the author defined generalized Hilbert spaces S over fields F with non-Archimedean valuations. In the present paper he considers symmetric bounded linear operators R over S, such that $(Ra, b)=(a, Rb)$ for all a and b in S. Besides a spectral theorem for totally continuous operators which is exactly the same as in the classical case, the principal result is a spectral theorem valid for symmetric operators R all of whose characteristic manifolds are proper in the terminology of the above-mentioned paper. Characteristic manifolds are manifolds $M(\lambda, \eta)$ for λ and η in F which reduce R, and on which $|Rx-\lambda x|<|\eta|$, and $(R-\lambda I)^n/\eta^n\rightarrow 0$. Any such operator can be decomposed into $R=U+V$. U has a pure point spectrum (countable) and a corresponding canonical form as in the classical case, and V has a pure continuous spectrum and can be represented as $V=\int\lambda dE_\lambda$ which is an abbreviation for a limiting process akin to the Riemann-Stieltjes integral, based on a decomposition of the identity, E_λ, which in turn is based on the decomposition of S by the (orthogonal) characteristic manifolds with properly chosen λ's and η's. Also the nature of the continuous spectrum is discussed, and some consequences of the spectral theorem are drawn, such as $f(R)=\int f(\lambda)dE_\lambda$ for polynomials $f(\lambda)$ and certain limits thereof. (Received February 3, 1945.)

77. Irving Kaplansky: *A contribution to von Neumann's theory of games.*

Let $\|a_{ij}\|$ be the matrix of a zero-sum two-person game G, as defined by von Neumann (*Theory of games and economic behaviour*, Princeton, 1944). The author gives a method of finding the value of v' of G in a finite number of steps. If G is "completely mixed" in the sense that all strategies must actually be used, then $\|a_{ij}\|$ is square and $v'=|a_{ij}|/\sum A_{ij}$, where A_{ij} is the cofactor of a_{ij}. If G is not completely mixed (for which a criterion is given), one passes to matrices of lower order. (Received April 2, 1945.)

78. Irving Kaplansky: *A note on groups without isomorphic subgroups.*

A theorem of R. A. Beaumont asserts that an abelian group of finite rank with elements of finite order has no isomorphic subgroups. It is shown that the theorem extends to vector spaces over a ring R provided that R is a principal ideal ring all of whose proper residue class rings are finite. (Received April 2, 1945.)

79. Irving Kaplansky: *Maximal fields with valuations.* II.

Schilling showed that a maximal field with discrete value group of finite rank is a power series field (Ann. of Math. vol. 38 (1937) pp. 551–576). In this paper the result is carried over to (suitably defined) discrete groups of infinite rank. Together with other known results, this virtually completes the determination of the structure of maximal fields, a problem proposed by Krull. (Received April 2, 1945.)

80. N. H. McCoy: *Subdirectly irreducible commutative rings.*

A special case of a theorem of Garrett Birkhoff (Bull. Amer. Math. Soc. vol. 50 (1944) pp. 764–768) states that every ring is isomorphic to a subdirect sum of subdirectly irreducible rings. Birkhoff also showed that a subdirectly irreducible commutative ring without nilpotent elements is a field. The purpose of the present paper is to make some further characterizations of subdirectly irreducible commutative rings. Let R be a commutative ring, not all elements of R being divisors of zero, and denote by D the set of all elements of R which are divisors of zero. Then R is subdirectly irreducible if, and only if, it has the following four properties: (i) the set of all elements x of R such that $Dx=0$ is a principal ideal $J=(j)\neq 0$, (ii) the set of all elements y of R such that $Jy=0$ is precisely D, (iii) R/D is a field, (iv) if d_1 is any element of D, not in J, there exists an element d_2 of D, not in J, such that $d_1d_2=j$. Some additional related results are also obtained. (Received February 20, 1945.)

81. Kathryn A. Morgan: *Representation of a positive binary form by a positive quaternary form.*

The conditions for representing a positive binary form as a sum of squares of linear forms were discussed by Mordell (Quart. J. Math. Oxford Ser. vol. 1 (1930) pp. 276–288) and Chao Ko (Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 81–98). This paper presents a method for finding the representation of a positive primitive binary form by a positive quaternary form and especially the number of representations of a primitive binary form as the sum of four squares. (Received March 3, 1945.)

82. F. E. Satterthwaite: *Error control in matrix calculation.* II.

The arithmetic calculation of the inverse of a matrix or of the solution of a set of simultaneous equations is often complicated by a serious magnification of rounding errors. The proposed method is as follows: (1) Each equation (or line) of the Doolittle solution is expressed approximately as an exact linear function of the original equations. (2) The discrepancy between the approximation and the ideal is adjusted by the same type of process as is used to adjust the original equations in a standard Doolittle method. (3) If the approximation is close enough, the coefficients in the adjustment will be small enough so that they can not cause any significant carrying forward of errors. A second approximation is sometimes necessary to satisfy this condition. The advantages of this method are: (1) It works for any matrix or set of equations. (2) It does not require an advance approximate solution. (3) Any number of decimal places may be carried with complete assurance that errors are never greater than one or two in the last decimal place. (4) Each step is self-checking. (5) The method is ideally suited for use with modern high speed calculating machines. (6) The routine can be easily taught to the average clerk. (Received February 5, 1945.)

83. J. E. Wilkins: *A generalization of the Euler ϕ-function.*

One defines the function $\phi_n(k)$ so that exactly $\phi_n(k)$ of the k arithmetic progressions $mk+l$ $(m=0, 1, \cdots; l=0, 1, \cdots, k-1)$ contain infinitely many numbers not divisible by an nth power greater than 1. It is shown that $\phi_n(k)$ is that multiplicative function for which $\phi_n(p^r)=p^r$ if $r<n$ and p is prime and $\phi_n(p^r)=p^r-p^{r-n}$ if $r\geqq n$ and p is prime. Thus $\phi_1(k)$ is the Euler ϕ-function. It is also shown that the function $\phi_n(k)/k$ is uniformly almost periodic in the sense of Bohr for $n\geqq 2$, and its asymptotic mean value is $1/\zeta(2n)$. For $n=2$, these results are due to Haviland, whose discussion, however, is not free from errors. (Received March 20, 1945.)

ANALYSIS

84. R. P. Agnew: *Spans of translations of peak functions.*

A peak function $F(x)$ is defined, in terms of positive constants a and b, by

$F(x)=a(b-|x|)$ when $|x|\leqq b$ and $F(x)=0$ when $|x|>b$. It is shown that, for each $p>1$, the closure in the Lebesgue space L_p of the linear manifold determined by the translations of a peak function is the whole space L_p. Explicit formulas for approximations are given. Uniform approximation and truncated step functions are treated. (Received March 9, 1945.)

85. R. P. Boas: *Fundamental sets of analytic functions.*

A set $\{u_n(z)\}$ of functions analytic in $|z|<1$ is called fundamental if each function analytic in $|z|<1$ can be uniformly approximated in each circle $|z|\leqq r<1$ by a linear combination of the $u_n(z)$. If $F(z)$ is entire, conditions on $F(z)$ and on the sequence $\{a_n\}$ which make the set $\{F(a_nz)\}$ fundamental have been given by Gelfond (Rec. Math. (Mat. Sbornik) N.S. vol. 4 (1938) pp. 149–156) and Markouchevitch (C.R. (Doklady) Acad. Sci. URSS vol. 43 (1944) pp. 3–6). In this paper it is shown that $\{F(a_nz)\}$ is fundamental if $F^{(n)}(0)\neq 0$ $(n=0, 1, 2, \cdots)$, and if when $F(z)$ is of order ρ and type τ, $\{a_n\}$ satisfies one of the following: (1) $\liminf n/|a_n|^\rho>\rho\tau$, (2) $\limsup n/|a_n|^\rho>e\rho\tau$. (1) improves a result of Gelfond; (2) is a result of Markouchevitch. It is further shown that (1) and (2) are best possible for all ρ. (Received March 15, 1945.)

86. Glenn James: *Certain general polynomial expansions.*

This paper develops an expansion of differentiable functions by means of a general operator, $\prod_{i=1}^{n}(I+D/(m-i+1))$. Special interpretations of D and m result in Taylor's theorem for one, and for several variables. The coefficients in this general expansion are, in general, functions of the number of terms. The resulting infinite series is shown to converge in instances where the corresponding Taylor's series diverges. (Received March 6, 1945.)

87. D. H. Lehmer: *On the Graeffe process for power series.*

The Graeffe root-squaring method, originally devised for the solution of algebraic equations, has been extended to cope with the problem of determining the absolute values of the zeros of entire functions (Polya: Zeitschrift fur Mathematik Physik vol. 63 (1915) pp. 275–290, Ostrowski: Acta Math. vol. 72 (1940) pp. 99–257). The problem of the determination of the arguments of the complex zeros is considered in the present paper. A method is given which not only determines approximately each complex root but also affords a method of verification different from the forthright one of substituting into the proposed equation. A discussion is given of certain practical difficulties peculiar to power series. The method is illustrated in the case of the difficult Bessel function equation of F. Wolf: $J_0(z)J_2(z)=J_1^2(z)$. (Received March 5, 1945.)

88. Morris Marden: *A note on the zeros of the sections of a partial fraction.*

The following theorem was proved by Bôcher and Grace in the case $m_1=m_2=m_3=1$ and by Linfield in the general case: The zeros of the logarithmic derivative of the rational function $f(z)=(z-z_1)^{m_1}(z-z_2)^{m_2}(z-z_3)^{m_3}$ lie at the foci of the conic which touches the line-segments (z_1, z_2), (z_2, z_3) and (z_3, z_1) in points dividing these segments in the ratios $m_2:m_1$, $m_3:m_2$ and $m_1:m_3$ respectively. In the present note this theorem is given an elementary proof based upon the optical properties of conics and upon a relation between the zeros of the partial fraction $F(z)=\sum_1^p m_j(z-z_j)^{-1}$ and its sections

$F_1(z)=\sum_1^k m_j(z-z_j)^{-1}$ and $F_2(z)=\sum_{k-1}^p m_j(z-z_j)^{-1}$ when $p=3$ and $k=2$. The latter relation is then generalized to the case of arbitrary p and k and connected with the theorem due to Siebeck, Van den Berg, Heawood, Juhel-Renjoy and Linfield that the zeros of the logarithmic derivative of the rational function $f(z)=(z-z_1)^{m_1}(z-z_2)^{m_2} \cdots (z-z_p)^{m_p}$ lie at the foci of the curve of class $p-1$ which touches each of the $p(p-1)/2$ line-segments (z_j, z_k) in a point dividing this segment in the ratio $m_k:m_j$. (Received March 16, 1945.)

89. R. M. Robinson: *Univalent majorants.*

$g(z)\prec G(z)$ in a circle about the origin if both functions are regular there and $G(z)$ is univalent, if $g(0)=G(0)$, and if the set of values assumed by $g(z)$ is included in the set of values assumed by $G(z)$. It is shown that if $g(z)\prec G(z)$ for $|z|<1$ then $zg'(z)\prec zG'(z)$ for $|z|\leqq 3-2^{3/2}$. For any given $G(z)$ there is a largest r such that $g(z)\prec G(z)$ for $|z|<1$ implies $zg'(z)\prec zG'(z)$ for $|z|\leqq r$; this r assumes its smallest value $3-2^{3/2}$ for $G(z)=z/(1-z)^2$ and its largest value $2^{1/2}-1$ for $G(z)=z$. The passage from $g(z)\prec G(z)$ to $(zg(z))'\prec(zG(z))'$ is also considered, as well as the two inverse operations. The possibility of deriving one of the relations $f(z)/z\prec F(z)/z$, $f'(z)\prec F'(z)$, $zf'(z)/f(z)\prec zF'(z)/F(z)$ from another is studied, particular attention being paid to the case in which the given majorant is $(1+z)/(1-z)$, so that the majorized function has a positive real part. Typical results are: If $f(z)=z+\cdots$ is regular for $|z|<1$ and $\Re f'(z)>0$, then $f(z)/z\prec(2/z)\log(1-z)^{-1}-1$ for $|z|<1$. If $f(z)=z+\cdots$ is regular for $|z|<1$ and $\Re[zf'(z)/f(z)]>0$ (star mapping), then $f'(z)\prec(1+z)/(1-z)^3$ at least for $|z|\leqq(5-17^{1/2})/2$ (which extends a theorem of A. Marx). (Received March 7, 1945.)

90. A. R. Schweitzer: *Functional relations valid in the domains of abstract groups and Grassmann's space analysis.* III.

The author extends a type of generalized algebra of logic to $(n+1)$-ary composition $(n=1, 2, 3, \cdots)$ by defining a special case of the following postulates considered in a previous paper (abstract 51-1-20): 1. $f(u_1, u_2, \cdots, u_{n+1})=f(u, t_1, t_2, \cdots, t_n)$ where $u_i=f(x_i, t_1, t_2, \cdots, t_n)$ and $u=f(x_1, x_2, \cdots, x_{n+1})$. 2. $f(v_0, v_1, v_2, \cdots, v_n)=\phi(v, t_1, t_2, \cdots, t_n)$ where $v_0=\phi(x, t_1, t_2, \cdots, t_n)$, $v_i=\phi(t_i, t_1, t_2, \cdots, t_n)$ and $v=f(x, t_1, t_2, \cdots, t_n)$. 3. $f(x, x, \cdots, x)=x$. 4. Duals, in f and ϕ, of the prceeding postulates. 5. Closure of the set of elements x_i under the compositions f and ϕ. The special case noted is obtained by assuming (in postulate 2): 6. If $n>1$, then $v_1=v_2=\cdots=v_n$. 7. $f(v_0, v_1, v_2, \cdots, v_n)=\phi(t_1, t_1, t_2, \cdots, t_n)$. 8. Duals, in f and ϕ, of postulates 6 and 7. Then for $n=1$ under postulates 1–8, $f\{\phi(x, t), t\}=\phi\{f(x, t), t\}=t$. Also if $x_2=x_3=\cdots=x_{n+1}$ in $f(x_1, x_2, \cdots, x_{n+1})$ and $\phi(x_1, x_2, \cdots, x_{n+1})$ then postulates 1–5 become abstractly identical with those obtained from the latter for $n=1$. (Received March 23, 1945.)

91. Otto Szász: *On some summability methods with triangular matrix.*

In a recent paper (Ann. of Math. vol. 43) the author associated with a given series or sequence transform a class of triangular matrix transforms and discussed the regularity conditions of such transforms which correspond to some standard summability methods such as Abel summability. He now establishes for some of these methods the regularity conditions relative to Cesàro summability of a given order. As a result triangular transforms are found which are more powerful than the Cesàro scale. Such is the case, for example, for the series to sequence transform: $A_n = \sum_0^n u_\nu x_n^\nu$, where $x_n \uparrow 1$ as $n \to \infty$ so slowly that $n^r x_n^n \to 0$ for all integers r. (Received March 17, 1945.)

92. F. T. Wang: *Strong summability of Fourier series.*

Let $S_n(x)$ be the partial sum of the Fourier series $f(t) \sim a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$ at $t = x$, and let $\phi(t) = (1/2)\{f(x+t) + f(x-t) - 2S\}$. The following result gives the solution of a problem proposed by Hardy and Littlewood (Fund. Math. vol. 25 (1935)): *If* $\int_0^t |\phi(u)| \{1 + \log^+ |\phi(u)|\} du = o(t)$ as $t \to 0$, then $\sum_{\nu=0}^{n} |S_\nu(x) - S|^2 = o(n)$ as $n \to \infty$. (Received March 12, 1945.)

93. F. T. Wang: *Tauberian theorem of oscillating series.*

Let $\sigma_n^{(r)}$ be the rth Cesàro mean of the series $\sum_{n=0}^{\infty} a_n$; and $\sigma_n^{(r)} - s = o(n^{(\delta-1)})$ as $n \to \infty$ for $r > 0$, $0 < \delta < 1$; and $a_n > -Kn^{-\delta}$; then the series $\sum_{n=0}^{\infty} a_n$ converges to S. An example shows that the order in the above inequality is the best possible in its kind. (Received March 12, 1945.)

94. H. J. Zimmerberg: *A class of definite boundary value problems.*

This paper is concerned with an extension of the results of Reid (Trans. Amer. Math. vol. 52 (1942) pp. 381–425) to differential systems consisting of the vector differential equation $y' = A(x)y + \lambda B(x)y$, and the two-point boundary conditions $(M_0 + \lambda M_1)y(a) + (N_0 + \lambda N_1)y(b) = 0$, in which the elements of the coefficient matrices of the system are allowed to be complex-valued. It is shown that under suitable assumptions of definiteness such systems possess fundamental properties similar to those previously established for real-valued, definitely self-adjoint problems by Bobonis (doctoral dissertation, Chicago, 1939; *Contributions to the calculus of variations*, 1938–1941, pp. 99–138). In particular, this study yields new results for the definitely self-adjoint systems considered by Bobonis. It is also shown that certain important types of boundary value problems associated with the second variation of an isoperimetric problem of Bolza in the calculus of variations which are not definitely self-conjugate adjoint do belong to this new class of problems. (Received March 19, 1945.)

GEOMETRY

95. L. K. Hua: *Geometries of matrices.* I. *Generalizations of von Staudt's theorem.*

A geometry is studied whose points are defined as the symmetric matrices Z of degree n; a class of points at infinity is to be added. The group of transformations of this geometry consists of all mappings $Z = (AZ^* + B)(CZ^* + D)^{-1}$ where the matrices A, B of degree n form the upper half of a symplectic matrix S of degree $2n$ while C, D form the lower half of S. The question of equivalence of systems of points with regard to this group is investigated. A generalization of von Staudt's theorem is obtained. Finally, some other geometries of a similar nature are discussed. (Received February 10, 1945.)

96. L. K. Hua: *Geometries of matrices.* I_1. *Arithmetical construction.*

The paper forms an illustration and a supplement to the first part. The geometry of two-rowed matrices is studied in more detail. It is shown that one of the conditions appearing in the generalization of von Staudt's theorem is redundant. (Received February 10, 1945.)

97. H. F. DeBaggis: *A simplified projective theory of order and parallelism in the hyperbolic plane.*

The point B is said to lie between A and C if every line through B intersects at least one of each pair of intersectining lines through A and C (F. P. Jenks, Reports of a Mathematical Colloquium vol. 1, p. 46). On the basis of this definition the entire theory of linear and planar order in the hyperbolic plane can be derived from Jenks' Postulates I, V–VIII and the following assumptions: II′. Each line contains at least two points; III′. There are three collinear and three noncollinear points; IV′. Through a point outside of two intersecting lines l_1 and l_2 there is a line intersecting exactly one l_i. In particular, the "postulate" of Pasch can be proved. Postulate IV′, which is one-half of Jenks' Postulate IV, is satisfied in the hyperbolic plane enlarged by "ends" (pairs of parallel lines) whereas IV is not. The theory of hyperbolic parallelism can be developed by proving that lines and points including ends satisfy I, II′, III′, IV′, V–VIII. (Received February 21, 1945.)

98. Karl Menger: *A projective definition of hyperbolic perpendicularity without reference to parallelism.*

Abbott's projective definition of hyperbolic perpendicularity (Reports of a Mathematical Colloquium vol. 4, p. 22) is based on parallelism. R. Baer asked whether perpendicularity could be defined in hyperbolic planes not containing parallel lines. If l is a finite line, Q a finite or ideal point not on l, then for a finite point X let $\operatorname{sgn}(X; Q, l)$ be equal to 1 if $(X+Q)\cdot l$ is a finite point, and equal to 0 otherwise. The finite line l' is perpendicular to the finite line l if and only if l' contains an ideal point P' such that $\operatorname{sgn}(X; P', l)=1$ for each finite point X. The ideal point is a pair of nonintersecting finite lines. Equality of such points, as well as their joins with finite points, are defined by means of Pappus constructions. (Received February 21, 1945.)

99. Karl Menger: *A projective definition of hyperbolic congruency independent of parallelism.*

Let Q and Q' be two distinct finite points on distinct lines l and l', respectively. There is exactly one finite line $k(Q, Q'; l, l')=h$ such that (1) $t(X)=\operatorname{sgn}(X; Q, l')+\operatorname{sgn}(X; Q', l)$ is 0 or 2 for each finite X on h, and (2) there exist three finite points on h in the order ABC for which $t(A)=t(C)\neq t(B)$ if l and l' are not parallel or (2′) there exist two finite points for which $t(A)\neq t(B)$ if l and l' are parallel. Two noncollinear pairs of finite points Q, R and Q', R' are congruent if and only if $h(Q, Q'; l, l')=h(R, R'; l, l')$ where $l=Q+R$ and $l'=Q'+R'$. Two collinear pairs of points are congruent if they are congruent with the same noncollinear pair. (Received February 21, 1945.)

100. Peter Scherk: *On differentiable arcs and curves.* V. Preliminary report.

The terminology used here is that of the preceding paper of this series [Ann. of Math. vol. 46 (1945) pp. 68–82]. Each point s of a K^{n+1} is mapped on the (possibly empty) set $t_m^n(s)$ of those points of the K^{n+1} which are projected from the osculating m-space of s into singular points $(0\leqq m<n-1)$. The existence, continuity and direction of the components of these mappings are studied outside of their fixed points. As a corollary the finiteness of the numbers N_{ik}^n is obtained $(0\leqq i,\ 0\leqq k,\ i+k\leqq n-2$; cf. ibid. pp. 78–81). (Received March 12, 1945.)

STATISTICS AND PROBABILITY

101. Irving Kaplansky: *The asymptotic distribution of runs of consecutive elements.*

In a random permutation of $1, \cdots, n$ let r be the number of instances in which i is next to $i+1$. Wolfowitz (Ann. Math. Statist. vol. 15 (1944) pp. 97–98) proved that asymptotically r has the Poisson distribution with mean 2. In this paper an asymptotic series is derived for the distribution, beginning with $2^r e^{-2}(r!)^{-1} \cdot [1-(r^2-3r)/n \cdots]$. (Received April 2, 1945.)

TOPOLOGY

102. R. L. Moore: *Concerning tangents to continua in the plane.*

Among other things it is shown that if in the plane a compact dendron has a tangent at each of its points then the set of all its end points is a countable inner limiting set and the closure of the set of all its junction points is totally disconnected. (Received February 3, 1945.)

103. G. T. Whyburn: *Extensions of plane continua mappings.*

In this paper a study is made of conditions under which a given non-alternating transformation of locally connected continuum on a plane or sphere can be extended monotonically to the whole plane or sphere. The problem is completely solved for mappings into dendrites. Also conditions are found under which the extended transformation will be interior, assuming that the given mapping is interior. (Received February 23, 1945.)

104. G. T. Whyburn: *On uniqueness of the inverse of a transformation.*

Given a continuous mapping $f(A)=B$ on a metric (compact) continuum A, it is shown (i) that if the images of cut points of A are dense in B and, for each $y \in B$, any two points of $f^{-1}(y)$ are conjugate, then the set G of points in B with unique inverses are uncountably everywhere dense in B; (ii) the same conclusion holds if f is monotone and the images of the local separating points of A are dense in B, provided no point locally separates any $f^{-1}(y)$ locally in A. Also for non-alternating interior mappings $f(A)=B$, it is shown that any nondegenerate A-set, cut point, or end point in A is necessarily an inverse set. (Received February 23, 1945.)

NEW PUBLICATIONS

Artin, E. See Ford, L. R.

Copeland, A. H. See Ford, L. R.

Ford, L. R., Copeland, A. H., and Artin, E. Nomography. Probability. Complex functions. (Notre Dame Mathematical Lectures, no. 4.) Notre Dame, 1944. 70 pp. $1.25.

Hadamard, J. An essay on the psychology of invention in the mathematical field. Princeton University Press, 1945. 13+143 pp. $2.00.

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Taibo-Fernández, A. Geometría descriptiva y sus aplicaciones. Vol. 1. Punto, recta y plano. 309 pp. Vol. 2. Curvas y superficies. 448 pp. Madrid, 1943.

A NOTE ON KLOOSTERMAN SUMS

ALBERT LEON WHITEMAN

1. Introduction. In recent years the Kloosterman sum

$$A_k(n) = \sum_{h \bmod k}{}' \exp(2\pi i n(h+\bar{h})/k)$$

has played an increasingly important role in the analytic theory of numbers. The dash $'$ beside the summation symbol indicates that the letter of summation runs only through a reduced residue system with respect to the modulus. The number $\bar{h}$ is defined as any solution of the congruence $h\bar{h} \equiv 1 \pmod{k}$, and n denotes an arbitrary integer. It was shown by Salié[1] almost fifteen years ago that $A_k(n)$ may be evaluated explicitly when k is a power of a prime. Salié's result is given by the following theorem.

THEOREM. *Let $k = p^\alpha$, $\alpha \geqq 2$, $(n, k) = 1$, where p denotes an odd prime. Then,*

(i) *if α is even,*

$$A_k(n) = 2k^{1/2}\cos(4\pi n/k);$$

(ii) *if α is odd,*

$$A_k(n) = \begin{cases} 2(n \mid k)k^{1/2}\cos(4\pi n/k) & \text{for } p \equiv 1 \pmod{4}, \\ -2(n \mid k)k^{1/2}\sin(4\pi n/k) & \text{for } p \equiv 3 \pmod{4}. \end{cases}$$

The symbol $(n \mid k)$ denotes, as is usual, the Legendre symbol.

Salié's proof of his theorem is based upon induction. In the present note a direct proof is given. The method consists in introducing a transformation which expresses the Kloosterman sum in terms of Gauss sums and certain types of Ramanujan sums.

2. Two lemmas. A Gauss sum may be defined by

$$G_{h,k} = \sum_{m=0}^{k-1} \exp(2\pi i h m^2/k).$$

We shall find it convenient to write G instead of $G_{1,k}$. The following lemma[2] is classical.

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[1] Hans Salié, *Über die Kloostermanschen Summen $S(u, v; q)$*, Math. Zeit. vol. 34 (1931) pp. 91–109.

[2] See, for example, Edmund Landau, *Vorlesungen uber Zahlentheorie*, vol. 1, p. 153.

LEMMA 1. *If k is an odd integer and $(h, k)=1$, then*

$$G_{h,k} = (h\mid k)G \tag{1}$$

and

$$G = i^{((k-1)/2)^2}k^{1/2}. \tag{2}$$

We shall also need the following lemma.

LEMMA 2. *Let p denote an odd prime; let n and α denote positive integers. Then*

$$\sum_{h \bmod p^\alpha}{}' \exp(2\pi i n h/p^\alpha) = \begin{cases} p^\alpha - p^{\alpha-1} \text{ if } p^\alpha \mid n, \\ -p^{\alpha-1} \text{ if } p^\alpha \nmid n \text{ but } p^{\alpha-1} \mid n, \\ 0 \textit{ if } p^{\alpha-1} \nmid n (\alpha > 1). \end{cases} \tag{3}$$

Furthermore, if α is odd, and if we put $n_1 = n/p^{\alpha-1}$ when $p^{\alpha-1}\mid n$, we have

$$\sum_{h \bmod p^\alpha}{}' (h\mid p^\alpha) \exp(2\pi i n h/p^\alpha) = \begin{cases} 0 \textit{ if } p^\alpha \mid n, \\ i^{((p-1)/2)^2}(n_1 \mid p)p^{\alpha-1/2} \\ \textit{if } p^{\alpha-1} \mid n \textit{ but } p^\alpha \nmid n, \\ 0 \textit{ if } p^{\alpha-1} \nmid n (\alpha > 1). \end{cases} \tag{4}$$

The first part of this lemma follows at once from a well known transformation formula[3] for Ramanujan sums or may easily be proved directly. The second part of the lemma may be established in the following way:

If $p^\alpha \mid n$, then

$$\sum_{h \bmod p^\alpha}{}' (h\mid p^\alpha) \exp(2\pi i n h/p^\alpha) = \sum_{h \bmod p^\alpha}{}' (h \mid p^\alpha) = 0.$$

If $p^\alpha \nmid n$ but $p^{\alpha-1}\mid n$, then by (1)

$$\begin{aligned} \sum_{h \bmod p^\alpha}{}' (h\mid p^\alpha) \exp(2\pi i n h/p^\alpha) &= \sum_{h \bmod p^\alpha}{}' (h\mid p) \exp(2\pi i n_1 h/p) \\ &= (n_1 \mid p)p^{\alpha-1}\sum_{h=1}^{p-1} (h\mid p) \exp(2\pi i h/p). \end{aligned}$$

But it is easy to show that[4]

$$G_{1,p} = \sum_{h=1}^{p-1} (h\mid p) \exp(2\pi i h/p). \tag{5}$$

Hence, by (2), the lemma is established in this case. Finally, if $p^{\alpha-1}\nmid n$,

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[3] See, for example, Landau, loc. cit., vol. 1, bottom of p. 280.

[4] See, for example, Landau, loc. cit., vol. 1, p. 155.

$$\begin{aligned}\sum_{h \bmod p^\alpha}{}' (h \mid p^\alpha) \exp(2\pi i n h/p^\alpha) &= \sum_{h \bmod p^\alpha}{}' (h + p \mid p^\alpha) \exp(2\pi i n(h + p)/p^\alpha) \\ &= \exp(2\pi i n/p^{\alpha-1}) \sum_{h \bmod p^\alpha}{}' (h \mid p^\alpha) \exp(2\pi i n h/p^\alpha) = 0,\end{aligned}$$

where we have noted that $\exp(2\pi i n/p^{\alpha-1}) \neq 1$ since $p^{\alpha-1} \nmid n$. This completes the proof of Lemma 2.

3. **Proof of Salié's theorem.** Let us first observe that (2) may be written in the form $1 = (-1 \mid k)G^2/k$. Using (1) we may now transform the Kloosterman sum $A_k(n)$ in the following manner.

$$\begin{aligned}A_k(n) &= (-1 \mid k)G^2/k \sum_{h \bmod k}{}' \exp(2\pi i(-n^2 h - \bar{h})/k) \\ &= (-1 \mid k)G/k \sum_{h \bmod k}{}' \exp(2\pi i(-n^2 h - \bar{h})/k) \\ &\qquad\qquad \cdot \sum_{m=0}^{k-1} (h \mid k) \exp(2\pi i h m^2/k) \\ &= (-1 \mid k)G/k \sum_{h \bmod k}{}' \sum_{m=0}^{k-1} (h \mid k) \exp(2\pi i h(m^2 - n^2 - \bar{h}^2)/k) \\ &= (-1 \mid k)G/k \sum_{h \bmod k}{}' \sum_{m=0}^{k-1} (h \mid k) \exp(2\pi i h(m^2 - n^2 + 2m\bar{h})/k)\end{aligned}$$

since $m + \bar{h}$ runs through a complete residue system with respect to the modulus k whenever m does. Interchanging signs of summation we get

$$\begin{aligned}A_k(n) &= (-1 \mid k)G/k \sum_{m=0}^{k-1} \exp(4\pi i m/k) \\ &\qquad \cdot \sum_{h \bmod k}{}' (h \mid k) \exp(2\pi i(m^2 - n^2)h/k.\end{aligned} \tag{6}$$

At this point we divide the discussion into two cases according as α is even or odd. For α even, we have

$$A_k(n) = G/p^\alpha \sum_{m=0}^{p^\alpha-1} \exp(4\pi i m/p^\alpha) \sum_{h \bmod p^\alpha}{}' \exp(2\pi i(m^2 - n^2)h/p^\alpha).$$

Referring to (3) we see that the last sum equals zero except when $p^{\alpha-1} \mid m^2 - n^2$. Now the solutions[5] of the congruence $m^2 \equiv n^2 \pmod{p^\alpha}$

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[5] See, for example, G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, pp. 95–96.

are all given by $m \equiv \pm n \pmod{p^\alpha}$, and the solutions of the congruence $m^2 \equiv n^2 \pmod{p^{\alpha-1}}$, $m \bmod p^\alpha$, where $m^2 \not\equiv n^2 \pmod{p^\alpha}$, are $m \equiv \pm n + qp^{\alpha-1} \pmod{p^\alpha}$, $1 \leqq q \leqq p-1$. Hence, applying the first part of Lemma 2, we obtain

$$\begin{aligned} A_k(n) &= G/p^\alpha \Big\{ (p^\alpha - p^{\alpha-1}) \exp (4\pi i n/p^\alpha) \\ &+ (p^\alpha - p^{\alpha-1}) \exp (-4\pi i n/p^\alpha) - p^{\alpha-1} \sum_{q=1}^{p-1} \exp (4\pi i(\pm n + qp^{\alpha-1})/p^\alpha) \Big\} \\ &= 2G \cos (4\pi n/k), \end{aligned}$$

which completes the proof of the theorem in the case in which α is even.

We next consider the case which arises when α is odd. For this purpose we return to (6) and obtain

$$\begin{aligned} A_k(n) = (-1 \mid p^\alpha) G_{1,p^\alpha}/p^\alpha \sum_{m=0}^{p^\alpha - 1} \exp (4\pi i m/p^\alpha) \sideset{}{'}\sum_{h \bmod p^\alpha} (h \mid p^\alpha) \\ \cdot \exp (2\pi i (m^2 - n^2) h/p^\alpha). \end{aligned}$$

From (4) we see that the last sum is zero except when $p^{\alpha-1} \mid m^2 - n^2$ but $p^\alpha \nmid m^2 - n^2$. Furthermore, let us observe that the number n_1, defined in Lemma 2, is here of the form $\pm 2nq + q^2 p^{\alpha-1}$. Hence, proceeding as we did in the case in which α is even, we get

$$\begin{aligned} A_k(n) &= (-1 \mid p^\alpha) G/p^\alpha \sum_{q=1}^{p-1} \exp (4\pi i(\pm n + qp^{\alpha-1})/p^\alpha) \\ &\qquad \cdot \left\{ i^{((p-1)/2)^2} (\pm 2nq \mid p) p^{\alpha-1/2} \right\} \\ &= (-1 \mid p^\alpha) G_{1,p^\alpha}/p^\alpha \Big\{ (n \mid p^\alpha) G_{1,p} p^{\alpha-1} \\ &\qquad \cdot \exp (4\pi i n/p^\alpha) \sum_{q=1}^{p-1} (2q \mid p) \exp (4\pi i q/p) \\ &\qquad + (-n \mid p^\alpha) G_{1,p} p^{\alpha-1} \exp (-4\pi i n/p^\alpha) \sum_{q=1}^{p-1} (2q \mid p) \exp (4\pi i q/p) \Big\} \\ &= (n \mid p^\alpha) G_{1,p^\alpha}/p^\alpha \left\{ (-1 \mid p^\alpha) G_{1,p}^2 p^{\alpha-1} \exp (4\pi i n/p^\alpha) \right. \\ &\qquad \left. + G_{1,p}^2 p^{\alpha-1} \exp (-4\pi i n/p^\alpha) \right\}. \end{aligned}$$

This completes the proof of the theorem in this case in view of Lemma 1.

4. **Concluding remarks.** The reader may have wondered why the case $\alpha = 1$ is excluded in Salié's theorem. The reason is that Salié's

method breaks down in this case as, indeed, does ours. For the sake of completeness we shall now show that when $\alpha=1$ our method leads merely to a transformation formula.

For $k=p$, the last sum in (6) becomes a Gauss sum in view of (5). Thus we have by (1) and (2)

$$\begin{aligned}
A_p(n) &= (-1\,|\,p)G/p\sum_{m=0}^{p-1}\exp\,(4\pi i m/p)\sum_{h=1}^{p-1}(h\,|\,p)\exp\,(2\pi i(m^2-n^2)h/p)\\
&= (-1\,|\,p)G^2/p\sum_{m=0}^{p-1}(m^2-n^2\,|\,p)\exp\,(4\pi i m/p)\\
&= \sum_{m=0}^{p-1}(m^2-4n^2\,|\,p)\exp\,(2\pi i m/p).
\end{aligned}$$

Hence, we obtain the transformation formula

$$\sum_{h=1}^{p-1}\exp\,(2\pi i n(h+\bar{h})/p) = \sum_{m=0}^{p-1}(m^2-4n^2\,|\,p)\exp\,(2\pi i m/p),$$

which may, of course, be established directly without much difficulty.

Various sums related to the Kloosterman sum $A_k(n)$ have been evaluated by Salié[6] and Lehmer.[7] The author has verified that the method of this paper may be employed to obtain new derivations of these results.

WASHINGTON, D. C.

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[6] Loc. cit.

[7] D. H. Lehmer, *On the series for the partition function*, Trans. Amer. Math. Soc. vol. 43 (1938) pp. 271–295.

A REMARK ON METRIC BOOLEAN RINGS

MALCOLM F. SMILEY

The purpose of this note is to prove that if, on a ring $B \equiv [a, b, c, \cdots]$ with unity element 1, a real valued function $\mu(a)$ is defined satisfying

$$(1) \qquad \mu(a) > 0 \quad \text{for every} \quad a \neq 0,$$

$$(2) \qquad \mu(a + b) + 2\mu(ab) = \mu(a) + \mu(b)$$

for every $a, b \in B$, then B is a metric Boolean ring[1] [2, pp. 41 and 96]. This result is analogous to one of Glivenko's [3] which states that every metric lattice is modular [2, p. 42]. We discuss also the following modification of (1):

$$(3) \qquad \mu(a) \geqq 0 \quad \text{for every} \quad a \in B.$$

The conditions (2) and (3) also lead, via identification, to a metric Boolean ring.

THEOREM 1. *Let B be a ring with unity element* 1 *on which is defined a real valued function $\mu(a)$ satisfying* (1) *and* (2). *Then B is a metric Boolean ring.*

The following lemma lists the steps in our proof of Theorem 1.

LEMMA 1. *For every $a, b \in B$, we have* (i) $\mu(a) = 0$ *if and only if* $a = 0$, (ii) $\mu(ab) = \mu(ba)$, (iii) $\mu(1 + a) = \mu(1) - \mu(a)$, (iv) $\mu(a^2b) = \mu(ab^2)$, (v) $\mu(a^2) = \mu(a)$, (vi) $a + a = 0$, (vii) $a^2 = a$.

PROOF. (i) Set $b = 0$ in (2) and use (1). (ii) This is clear by (2). (iii) Set $b = 1$ in (2). (iv) From (2) and (iii) we have

$$\mu(a + b + 1) + 2\mu(ab + a) = \mu(a) + \mu(1) - \mu(b).$$

Using (2) again gives

$$\mu(a + b + 1) + 2\mu(ab) + 2\mu(a) - 4\mu(a^2b) = \mu(a) + \mu(1) - \mu(b).$$

Rearranging, we find that

$$4\mu(a^2b) = \mu(a + b + 1) + 2\mu(ab) + \mu(a) + \mu(b) - \mu(1).$$

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Opinions and assertions contained in this paper are those of the author and are not to be construed as official or as reflecting the views of the United States Navy Department or of the Naval Service at large.

[1] Numbers enclosed in brackets denote references given at the end of the paper.

Interchanging a and b in this equation yields (iv). (v) Set $b=1$ in (iv). (vi) Use (2) with $a=b$, (v), and then (i). (vii) By (2), (vi), and (iii), we have

$$\mu(a^2+a) = \mu(a(a+1)) = (1/2)[\mu(a)+\mu(a+1)-\mu(1)] = 0.$$

Now (i) and (vi) give (vii).

The condition (vii) is Stone's definition of a Boolean ring [4]. That B is a metric Boolean ring is then immediate by (1) and (2).

REMARK. The referee has pointed out that the method of proof of Lemma 1 actually yields the following theorem.

THEOREM. *Let B be a ring with unity element and let R be a ring with unity element* 1 *in which $2=1+1$ is not a divisor of zero. If a function $\mu(a)$ is defined on B with values in R, which satisfies* (2) *and*

$$(1') \qquad \textit{if} \quad a \neq 0, \quad \textit{then} \quad \mu(a) \neq 0;$$

then B is a Boolean ring.

We turn now to the conditions (2) and (3).

THEOREM 2. *Let B be a ring with unity element* 1 *on which is defined a real valued function $\mu(a)$ satisfying* (2) *and* (3). *Then the set B_0 of all $a \in B$ for which $\mu(a)=0$ is an ideal of B and the difference ring $B-B_0$ is a metric Boolean ring.*

PROOF. First, if a, $b \in B_0$, then, by (2), we have

$$\mu(a+b)+2\mu(ab) = \mu(a)+\mu(b) = 0,$$

and consequently, by (3), $\mu(a+b)=\mu(ab)=0$. Thus B_0 is a subring of B. To show that B_0 is an ideal, we first note that (ii) and (iii) of Lemma 1 are still valid. Now, using (2) and (iii) of Lemma 1, we obtain for $a \in B_0$, $c \in B$,

$$\begin{aligned} 2\mu(a(1+c)) &= \mu(a)+\mu(1+c)-\mu(1+a+c), \\ &= \mu(1)-\mu(c)-\mu(1)+\mu(a+c), \\ &= -\mu(c)+\mu(a)+\mu(c)-2\mu(ac), \\ &= -2\mu(ac). \end{aligned}$$

Hence, by (3), $\mu(ac)=0$, and (ii) of Lemma 1 yields also $\mu(ca)=0$. We conclude that B_0 is an ideal of B. Note also that if $a \in B_0$, $c \in B$, we have

$$\mu(a+c) = \mu(a)+\mu(c)-2\mu(ac) = \mu(c).$$

Thus the function $\mu(a)$ is constant on each element of $B-B_0$. The conclusion of Theorem 2 is now clear by Theorem 1.

Note added in proof. The assumption of a unity element for B made in Theorems 1 and 2 may be avoided as follows. Expand $\mu(a+b+c)$ as in the proof of (iv) of Lemma 1 and interchange a and b to obtain (4): $\mu(abac)=\mu(babc)$. Setting $b=c=-a$ then gives $\mu(a^4)=\mu(-a^4)$. Putting $b=a$ in (2) yields, via (3), $\mu(a)\geqq\mu(a^2)$, and hence also $\mu(-a)\geqq\mu(a^2)$. With $b=-a$ in (2), we obtain $\mu(a)+\mu(-a)=2\mu(-a^2)$. Thus we have $2\mu(-a^4)=\mu(a^2)+\mu(-a^2)\geqq 2\mu(a^4)=2\mu(-a^4)$, and consequently $2\mu(a^4)=2\mu(a^2)=2\mu(-a^2)=\mu(a)+\mu(-a)$, from which (5): $\mu(a^2)=\mu(a)=\mu(-a)$ follows. From (4) and (5) we obtain (6): $\mu(aba^2)=\mu(baba)=\mu(ba)$ and also $\mu(a^6)=\mu(a^8)=\mu(a^3)=\mu(a)$. Now (2) gives $\mu(a+a^2)=0$. To show that B_0 of Theorem 2 is an ideal we have, for $a\in B_0$, $b\in B$, $\mu(a^2+ab)=\mu(a^2)+\mu(ab)-2\mu(a^3b)=-\mu(ab)$, by (5), (6), and (ii).

On the other hand, the assumption of the unity element for B in the referee's theorem of our Remark *is* essential. For we may take R to be the ring of integers modulo an odd integer m, the elements and addition of B to be those of R, and define $ab\equiv 0$, $\mu(a)\equiv a$ for every $a, b\in B$. All the hypotheses of the theorem of our Remark, *except that B have a unity element*, are valid, but B is no Boolean ring. The standard process of introducing a unity element [1, p. 23] thus cannot preserve (1′) and (2).

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UNITED STATES NAVAL ACADEMY, POSTGRADUATE SCHOOL

GROUPS WITH ISOMORPHIC PROPER SUBGROUPS

ROSS A. BEAUMONT

1. **Introduction.** Since isomorphic groups have orders which are the same cardinal number, no finite group has an isomorphic proper subgroup. Moreover if there exists an isomorphism f_1 of a group G such that G^{f_1} is a proper subgroup of G, then f_1 induces an isomorphism f_2 of G^{f_1} upon a proper subgroup $(G^{f_1})^{f_2}=G^{f_1f_2}$ of G^{f_1}. An infinite chain of isomorphic subgroups is generated such that

$$G > G^{f_1} > G^{f_1f_2} > \cdots > G^{f_1f_2\cdots f_n} > \cdots > 1.$$

Thus any group G with an isomorphic proper subgroup does not satisfy the descending chain condition. Therefore a survey of groups with isomorphic proper subgroups is concerned only with infinite groups which do not satisfy the descending chain condition. Among these groups, it is obvious that infinite cyclic groups have isomorphic proper subgroups.

In a recent paper[1] Baer has given a criterion for groups which do not have isomorphic proper subgroups. This criterion has not been used in the present note since the structure of the groups considered allows simpler proofs. Because of its interest in this problem, however, it is repeated here.

CRITERION. *The group G does not have an isomorphic proper subgroup if there exists a well ordered ascending chain of subgroups $N(v)$ of G with the following properties.*

(i) $N(v)^f \leqq N(v)$ *for every isomorphism f of G into itself.*

(ii) $N(0)=1$ *and* $N(t)=G$ *for some ordinal t.*

(iii) $N(v+1)/N(v)$ *does not have an isomorphic proper subgroup.*

(iv) *If v is a limit ordinal, then every element of $N(v)$ is contained in some $N(u)$ for $u<v$.*

We shall call a group G completely reducible if it is the direct product of groups of rank 1. The theorems of this note give a complete survey of those completely reducible groups which have isomorphic proper subgroups.

DEFINITION 1. *A group G is an I-group if there exists a proper sub-*

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[1] R. Baer, *Groups without proper isomorphic quotient groups*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 267–277.

group H of G such that G and H are isomorphic.

LEMMA 1. *If a group G has a direct factor S which is an I-group, then G is an I-group.*

PROOF. $G=S\times T$ for some subgroup T of G and there exists a proper subgroup S' of S such that S and S' are isomorphic. The product group $H=S'T=S'\times T$ is a proper subgroup of G and is isomorphic to G since the corresponding direct factors S and S', T and T, of G and H are isomorphic. Hence G is an I-group.

LEMMA 2. *An infinite direct power group, that is, a group G which is the direct product of infinitely many groups which are isomorphic to each other, is an I-group.*

PROOF. There is a (1-1) correspondence between the direct factors of G and the direct factors of any proper subgroup H which is the direct product of almost all of the direct factors of G, and the corresponding direct factors are isomorphic.

2. **Abelian groups all of whose elements are of finite order.** The following result proves that the discussion of abelian groups all of whose elements are of finite order may be limited to primary abelian groups without loss in generality.

LEMMA 3. *If A is an abelian group such that all the elements of A are of finite order, then A is an I-group if, and only if, one of the primary components $A(p)$ of A is an I-group.*

PROOF. $A=\prod_q A(q)$ where the $A(q)$ are primary abelian groups. If $A(p)$ is an I-group for some p, it follows from Lemma 1 that A is an I-group. Conversely, if A is an I-group, then there exists a proper subgroup A' of A such that A and A' are isomorphic. Since the orders of elements are preserved by isomorphism, the isomorphism of A on A' induces an isomorphism of $A(q)$ on $A'(q)$ for all q, where $A'(q)$ is the primary q-component of A'. Now $A'(q)\leqq A(q)$ for all q since $A'<A$, and it follows from $A'=\prod_q A'(q)$ that for some p, $A'(p)<A(p)$, for otherwise $A=A'$. Thus $A(p)$ is an I-group.

DEFINITION 2. *A group G is said to be of finite rank $r(G)$ if there exists an integer $r(G)$ with the property that $r(G)$ is the smallest number such that every finite subset of elements of G which does not consist of the unity element alone is contained in a subgroup of G which is generated by $r(G)$ elements. The rank of the unit group $\{1\}$ is defined to be 0. If no such integer exists, then G is said to be of infinite rank and we write $r(G)=\infty$.*

It is known[2] that the only primary abelian groups of rank 1 are cyclic groups and groups of type p^∞. G is a group of type p^∞ if G is generated by an infinite sequence $a(i)$ of elements where $a(0)$ is of order p and $a(i+1)^p=a(i)$ for every i.

THEOREM 1. *If A is a primary abelian group of finite rank r, then A is not an I-group.*

PROOF. It follows from the hypothesis and known results[3] that A is the direct product of groups of rank 1 except in the trivial case where $r=0$ and $A=\{1\}$. Assume that A is an I-group. Denote by $A(p^k)$ the subgroup of A consisting of all those elements of A whose orders divide p^k. The definition of a group of type p^∞ implies that $A(p^k)$ is a finite group. The isomorphism of A on a proper subgroup A' induces an isomorphism of $A(p^k)$ on a subgroup $A'(p^k)$ of A'. Since orders are preserved and since $A'<A$, $A'(p^k)\leqq A(p^k)$. Since no finite group is an I-group, $A'(p^k)=A(p^k)$. If a is an element of A, a is contained in $A(p^j)$ for some integer j. By the above $A(p^j)=A'(p^j)$ $<A'$ so that a is contained in A'. Thus $A\leqq A'$ which contradicts $A'<A$ and the assumption that A is an I-group.

REMARK. Note that the subgroups $A(p^0), A(p^1), \cdots, A(p^k), \cdots$ form an ascending chain of subgroups of A which have the properties of Baer's criterion.

THEOREM 2. *If A is a primary abelian group of infinite rank and if A is the direct product of groups of rank 1, then A is an I-group.*

PROOF. *Case* I. *The orders of the elements of A are bounded.*

A is the direct product of infinitely many cyclic groups of order dividing p^k for some integer k. In this direct factorization there must be infinitely many cyclic groups of order p^i for some i where $k\geqq i>0$. The direct product of these groups is an infinite direct power group, and hence by Lemmas 2 and 1, A is an I-group.

Case II. *The orders of the elements of A are not bounded.*

If there are infinitely many direct factors which are groups of type p^∞, then the direct product of these groups is an infinite direct power group, and A is an I-group as in Case I.

If there are only a finite number of direct factors of type p^∞, then since A is of infinite rank, there are infinitely many cyclic groups in the direct factorization. If there are infinitely many cyclic direct factors of order p^i for some integer i, then the proof that A is an I-group is that of Case I. The alternative is that there be only a finite number of cyclic direct factors of A of order p^i for all i, so that in the direct factorization of A there are cyclic groups of arbitrarily high order. (Otherwise, A would be of finite rank.) Then

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[2] H. Prüfer, *Untersuchungen über die Zerlegbarkeit der abzahlbaren primaren Abelschen Gruppen*, Math. Zeit. vol. 17 (1923) pp. 35–61.

[3] H. Prüfer, ibid.

$$A = \prod_{i=1}^{n} G_i(\infty) \times \prod_{i=1}^{n_1} G_i(j_1) \times \prod_{i=1}^{n_2} G_i(j_2) \times \cdots \times \prod_{i=1}^{n_k} G_i(j_k) \times \cdots$$

where $G_1(\infty) = \{1\}$; $G_i(\infty)$, $i \neq 1$, is of type p^∞; $G_i(j_k)$ is cyclic of order $q_k = p^{j_k}$, and $j_1 < j_2 < \cdots < j_k < \cdots$. Let

$$A' = \prod_{i=1}^{n} G_i(\infty) \times \prod_{i=1}^{n_1-1} G_i(j_1) \times (G_{n_2}(j_2))^{q_2/q_1} \times \prod_{i=1}^{n_2-1} G_i(j_2)$$
$$\times (G_{n_3}(j_3))^{q_3/q_2} \times \cdots$$

where if $n_k - 1 = 0$,

$$\prod_{i=1}^{n_k-1} G_i(j_k)$$

is replaced by $\{1\}$.

By construction, A' is a proper subgroup of A and there is a (1-1) correspondence between the direct factors of A and those of A'. Since these corresponding direct factors are isomorphic, A and A' are isomorphic and A is an I-group.

3. Abelian groups all of whose elements are of infinite order. In this section, the abelian groups considered will be written additively.

THEOREM 3. *If G is an abelian group such that all the elements of G except the identity are of infinite order, and if G does not admit the field of rational numbers, R, as a field of operators, then G is an I-group.*

PROOF. If $\beta G = G$ for every integer $\beta \neq 0$, then for every integer α and every g in G, $\alpha g = \beta g'$ for some g' in G. Thus the equation $\alpha g = \beta x$ has a solution g' in G for every pair of integers α and $\beta \neq 0$ and for every g in G. This is just the statement[4] that G admits R, which contradicts the hypothesis. Hence there exists an integer $\beta \neq 0$ such that βG is a proper subgroup of G. Moreover, the homomorphism $g \rightarrow \beta g$ is an isomorphism of G on βG, for if $\beta g = \beta f$, $\beta(g - f) = 0$; and since $\beta \neq 0$, this implies that $g - f$ has finite order. By the hypothesis on G, $g - f = 0$ and $g = f$.

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[4] If G admits R, the equation has a unique solution, but the existence of a solution implies its uniqueness. For a discussion of the properties of rational numbers as operators of the abelian group, see R. Baer, *Abelian groups without elements of finite order*, Duke Math. J. vol. 3 (1937) p. 70.

COROLLARY. *Every proper subgroup, $G\neq 0$, of the additive group of all rational numbers, R, is an I-group.*

PROOF. Assume that G admits R as a field of operators. Let s/t be an element of G where s and t are nonzero integers. Then $r(s/t)=g$ in G for every r in R, and hence $(t/s)g$ is in G. But $(t/s)g=(t/s)[r(s/t)]=r$, so that r is in G for every r in R. This contradicts $G<R$ and with Theorem 3 completes the proof of the corollary.

THEOREM 4. *If G admits R as a field of operators, then G is an I-group if, and only if, G is infinite-dimensional over R.*

PROOF. If G is infinite-dimensional over the field R, G possesses an infinite basis,[5] B, with respect to R. Let G' be the subgroup of G with basis B', where B' contains almost all of the elements of B. The (1-1) correspondence between B and B' may be extended to an isomorphism of G upon its proper subgroup G' in the usual way.

Conversely, if G is an I-group which admits R, then G contains an isomorphic proper subgroup G' which admits R. For if g' is in G' and s and $t\neq 0$ are integers, there exists an element h of G such that $sg=th$, where g is the correspondent of g' under the isomorphism of G upon G'. We have

$$sg' = (sg)' = (th)' = th'$$

where h' is in G', and this is the statement that G' admits R. It follows from the above equation that the isomorphism of G upon G' is an operator-isomorphism. If the dimension of G over R is finite, G' would have the same finite dimension over R, and this would imply $G=G'$ which contradicts the hypothesis. Hence G is of infinite dimension over R.

COROLLARY 1. *If G is the direct sum of ω groups each isomorphic to R, then G is an I-group if, and only if, ω is infinite.*

PROOF. G admits R as a field of operators and G is infinite-dimensional over R if, and only if, ω is infinite.

COROLLARY 2. *The additive group of all rational numbers, R, is not an I-group.*

PROOF. R satisfies the hypothesis of Corollary 1 with $\omega=1$.

It is interesting to note that the hypothesis of Theorem 4, that an abelian group G admits the field R of rational numbers as an operator domain, implies that every element of G except the identity is of infinite order. For if $ng=0$ with g in G and n a nonzero integer, $(1/n)(ng)=g=0$.

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[5] H. Zassenhaus, *Lehrbuch der Gruppenthorie*, p. 66, Satz 15.

The results of this section may be generalized in the following way.[6] Let G be an abelian group admitting an integral domain D of operators. Let F be the quotient field of D.

DEFINITION 3. *G is an I-group with respect to D, or more simply, a D-group, if G contains an operator-isomorphic proper subgroup.*

The following theorems whose proofs are simplified versions of the proofs of Theorem 3 and Theorem 4 may be stated.

THEOREM 5. *If G is an abelian group admitting D as a domain of operators, such that $\alpha g=0$ for α in D and g in G implies that either $\alpha=0$ or $g=0$, and if G does not admit F as a field of operators, then G is a D-group.*

THEOREM 6. *If G admits F as a field of operators, then G is a D-group if, and only if, G is infinite-dimensional over D.*

4. **Summary.** If H is a group of rank 1, then H is an unmixed group, that is, either every element of H is of finite order or every element of H except the identity is of infinite order. If H does not contain elements of infinite order, the primary components of H are either finite cyclic groups or groups of type p^{∞}. If H does contain elements of infinite order, H is essentially a subgroup of the additive group of all rational numbers,[7] R. Hence if a group G is completely reducible it is the direct sum of groups of the three types mentioned above. Let $F(G)$ be the direct sum of those direct summands of G which do not contain elements of infinite order. Then $F(G)$ is the subgroup of all of the elements of finite order in G, and $G=F(G)+I(G)$ where $I(G)$ is the direct sum of rational groups.

THEOREM 7. *If G is a completely reducible group, then G is an I-group if, and only if, either $F(G)$ or $I(G)$ is an I-group.*

PROOF. If either $F(G)$ or $I(G)$ is an I-group, then G is an I-group by Lemma 1. If G contains an isomorphic proper subgroup G', then $G'=F'(G)+I'(G)$ where $F'(G)$ is the map of $F(G)$ and $I'(G)$ is the map of $I(G)$ under the isomorphism. But $F'(G)\leqq F(G)$ since orders are preserved. If $F'(G)<F(G)$, the theorem is proved; therefore, let us assume that $F'(G)=F(G)$. $I(G)$ is isomorphic to the differ-

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[6] The author wishes to thank the referee for suggesting these generalizations with their proofs, which in turn suggested a revision of this section.

[7] See the reference of footnote 4.

ence group $G-F(G)$ which contains properly the difference group $G'-F'(G)$ since $F'(G)=F(G)$. Now $I'(G)$ is isomorphic to $G'-F'(G)$ and to $I(G)$. Hence $G-F(G)$ is an I-group and it follows that $I(G)$ which is isomorphic to $G-F(G)$ is also an I-group.

It follows from Theorem 7 that the theorems and corollaries of §§2 and 3 survey completely all completely reducible groups, G, which are I-groups.

UNIVERSITY OF WASHINGTON

AN EXISTENCE THEOREM FOR LATIN SQUARES

MARSHALL HALL

1. **Introduction.** A latin square may be interpreted as a representation of a 3-web or as the multiplication table of a quasi-group. Hence the following theorem has application both in the theory of projective planes and in the theory of quasi-groups. It is derived from a very interesting result of P. Hall.

2. **The existence theorem.** Is there any combinatorial restriction which prevents us from constructing a latin square by adding a row at a time? The following theorem shows that such a procedure is permissible.

THEOREM. *Given a rectangle of $n-r$ rows and n columns such that each of the numbers* $1, 2, \cdots, n$ *occurs once in every row and no number occurs twice in any column, then there exist r rows which may be added to the given rectangle to form a latin square.*

PROOF. Let C_i, $i=1, 2, \cdots, n$ be the subset of the numbers $1, 2, \cdots, n$ which do not occur in the ith column of the given rectangle. Then each C_i contains r numbers and each number occurs r times in all the C's. For there are $n-r$ numbers in the ith column and each number has appeared in $n-r$ columns. It will be shown that the subsets satisfy the requirements of P. Hall's theorem:[1]

In order that a complete system of distinct representatives of subsets

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[1] P. Hall, *On representatives of subsets*, J. London Math. Soc. vol. 10 (1935) pp. 26–30.

$T_1, \cdots, T_m$ of a set S shall exist, it is sufficient that for each $k=1, \cdots, m$ any selection of k of the subsets shall contain between them at least k elements of S.

The necessity of these requirements is evident. Let us apply this theorem to the subsets C_i. Any selection of k C's will contain kr numbers and at least k of these must be distinct since each number is contained in only r C's. The distinct representatives $c_1, \cdots, c_n$ of the subsets $C_1, \cdots, C_n$ may be added as a row to the given rectangle. For $c_1, \cdots, c_n$ must contain each of the numbers $1, \cdots, n$ once and no c_i has appeared in the ith column of the given $n-r$ rows. Repeatedly applying this process, we continue adding rows to the rectangle until it becomes a complete latin square.

UNITED STATES NAVY

PROOF THAT THE MERSENNE NUMBER M_{167} IS COMPOSITE

CHARLES B. BARKER

The Mersenne numbers are of the form

$$M_p = 2^p - 1,$$

where p is a prime. It is known, except in a few instances, whether M_p is prime or composite for all p not greater than 257. The unknown cases are those for which $p = 167$, 193, 199, 227, and 229.

The author of this paper has recently completed the proof that M_{167} is composite. This proof is based upon the well known theorem of Luças, which subsequently was amplified by Lehmer.[1] The most recent contribution is that of H. S. Uhler,[2] who proved that M_{157} is composite.

The method employed by the author was direct computation upon an eight-bank electric calculating machine. Each residue was checked by computing it two ways, that is, by calculating r_i from both

$$(r_{i-1})^2 - 2, \quad \text{and} \quad (M_{167} - r_{i-1})^2 - 2.$$

Obviously one cannot list the whole series of residues, so only the last one will be given here. This final residue was found to be

$$163\ 32098278\ 81677538\ 71550317\ 93792426\ 84838281\ 73373557.$$

Since this residue is not zero, it follows that M_{167} is composite.

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[1] D. H. Lehmer, *On Lucas's test for the primality of Mersenne's numbers*, J. London Math. Soc. vol. 10 (1935) pp. 162–165.

[2] H. S. Uhler, *First proof that the Mersenne number M_{157} is composite*, Proc. Nat. Acad. Sci. U.S.A. vol. 30 (1944) pp. 314–316.

ALGEBRAIC DETERMINATION OF THE SECOND FUNDAMENTAL FORM OF A SURFACE BY ITS MEAN CURVATURE

T. Y. THOMAS

1. Introduction. The main purpose of the following paper is to show that in general the mean curvature of a surface yields an *algebraic* determination of its second fundamental form. We do this by deriving the explicit equations giving this determination. The continuity and differentiability properties of the various functions entering into the discussion will be assumed without special mention since these requirements are obvious from the methods and equations employed.

We denote the mean curvature by H and the Gaussian curvature by K. The symbols $g_{\alpha\beta}$ and $b_{\alpha\beta}$ will be used to denote the symmetric components of the first and second fundamental forms of the surface (two-dimensional surface in Euclidean three space). Between these quantities we have the relations

$$(1.1)\qquad H = g^{\alpha\beta} b_{\alpha\beta}/2 \quad \text{and} \quad |g_{\alpha\beta}| K = |b_{\alpha\beta}| = b_{11}b_{22} - b_{12}^2,$$

where $|g_{\alpha\beta}|$ and $|b_{\alpha\beta}|$ stand for determinants. The first of these relations can be regarded as defining the mean curvature. The second is known as the Gauss equation. We may mention here also the Codazzi equations which play an important role in the following, that is, the equations $b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta}$, where the "comma" denotes covariant differentiation based on the first fundamental form of the surface.

We shall find that the combination $H^2 - K$ enters into most of the following equations. This quantity satisfies the condition $H^2 - K \geqq 0$. For, if we choose a coordinate system such that at a point P we have $g_{\alpha\beta} = \delta_{\alpha\beta}$, then at this point $2H = b_{11} + b_{22}$ and hence

$$\begin{aligned} 4(H^2 - K) &= (b_{11}^2 + 2b_{11}b_{22} + b_{22}^2) - 4(b_{11}b_{22} - b_{12}^2) \\ &= (b_{11}^2 - 2b_{11}b_{22} + b_{22}^2) + 4b_{12}^2 \\ &= (b_{11} - b_{22})^2 + 4b_{12}^2 \geqq 0. \end{aligned}$$

Suppose that $H^2 - K = 0$ in a region R of the surface. Then at a point P of R and relative to a coordinate system for which $g_{\alpha\beta} = \delta_{\alpha\beta}$ at P we have

$$(1.2)\qquad b_{11} + b_{22} = 2H \quad \text{and} \quad b_{11}b_{22} - b_{12}^2 = H^2.$$

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Squaring the first of these equations and combining with the second leads to the result $(b_{11}-b_{22})^2=0$. Hence $b_{11}=b_{22}$ at P. The first equation (1.2) then gives $b_{11}=H$ and $b_{22}=H$. From this and the second equation (1.2) we now have $b_{12}=0$. From these relations and the fact that $g_{\alpha\beta}=\delta_{\alpha\beta}$ we can now unite $b_{\alpha\beta}=Hg_{\alpha\beta}$. These latter relations are independent of the coordinate system employed and hold at all points of the region R in which the condition $H^2-K=0$ is satisfied. By differentiating both members of the relations $b_{\alpha\beta}=Hg_{\alpha\beta}$ covariantly and then making use of the Codazzi equations we now easily deduce that $H=\text{const}$. From $H^2-K=0$ it then follows that $K=\text{const}$. *Hence the condition $H^2-K=0$ in a region R of a surface implies that the mean curvature H and the Gaussian curvature K are constant in R.* We shall make use of this result in the following discussion.

2. **Determination of the covariant derivative of the b tensor.** Differentiate each of the equations (1.1) covariantly to obtain

$$\begin{aligned} H_{,1} &= [g^{11}b_{11,1}+2g^{12}b_{12,1}+g^{22}b_{22,1}]/2 \\ H_{,2} &= [g^{11}b_{11,2}+2g^{12}b_{12,2}+g^{22}b_{22,2}]/2, \\ |g_{\alpha\beta}|\,K_{,1} &= b_{11,1}b_{22}+b_{11}b_{22,1}-2b_{12}b_{12,1}, \\ |g_{\alpha\beta}|\,K_{,2} &= b_{11,2}b_{22}+b_{11}b_{22,2}-2b_{12}b_{12,2}. \end{aligned}$$

Now make the substitutions

$$g^{11}=g_{22}/|g_{\alpha\beta}|,\qquad g^{12}=-g_{12}/|g_{\alpha\beta}|,\qquad g^{22}=g_{11}/|g_{\alpha\beta}|$$

in the first two of the above equations and then write the resulting equations in the form

$$\begin{aligned} b_{11,1} &= (1/g_{22})[2|g_{\alpha\beta}|\,H_{,1}+2g_{12}b_{12,1}-g_{11}b_{22,1}], \\ b_{22,2} &= (1/g_{11})[2|g_{\alpha\beta}|\,H_{,2}+2g_{12}b_{12,2}-g_{22}b_{11,2}]. \end{aligned}\tag{2.1}$$

Use these latter equations to eliminate the quantities $b_{11,1}$ and $b_{22,2}$ from the right members of the above equations for $K_{,1}$ and $K_{,2}$. Then making use of the Codazzi equations we find that

$$\begin{aligned} 2(g_{12}b_{22}-g_{22}b_{12})b_{11,2}+(g_{22}b_{11}-g_{11}b_{22})b_{22,1}& \\ =|g_{\alpha\beta}|\,g_{22}K_{,1}-2|g_{\alpha\beta}|\,b_{22}H_{,1},& \end{aligned}$$

$$\begin{aligned} (g_{11}b_{22}-g_{22}b_{11})b_{11,2}+2(g_{12}b_{11}-g_{11}b_{12})b_{22,1}& \\ =|g_{\alpha\beta}|\,g_{11}K_{,2}-2|g_{\alpha\beta}|\,b_{11}H_{,2}.& \end{aligned}$$

We consider these equations as equations to determine the quantities $b_{11,2}$ and $b_{22,1}$. The determinant of this system is

$$\Delta = \begin{vmatrix} 2(g_{12}b_{22} - g_{22}b_{12}) & (g_{22}b_{11} - g_{11}b_{22}) \\ (b_{11}g_{22} - g_{22}b_{11}) & 2(g_{12}b_{11} - g_{11}b_{12}) \end{vmatrix}.$$

When this determinant is expanded and the terms collected we find that $\Delta = 4[H^2 - K]|g_{\alpha\beta}|^2$. The solution by Cramer's rule gives

$$\begin{aligned} b_{11,2} = (|g_{\alpha\beta}|/\Delta)[&2g_{12}g_{22}b_{11}K_{,1} - 2g_{11}g_{22}b_{12}K_{,1} - 4g_{12}b_{11}b_{22}H_{,1} \\ &+ 4g_{11}b_{12}b_{22}H_{,1} - g_{11}g_{22}b_{11}K_{,2} + g_{11}^2 b_{22}K_{,2} \\ &+ 2g_{22}b_{11}^2 H_{,2} - 2g_{11}b_{11}b_{22}H_{,2}], \end{aligned} \tag{2.2}$$

$$\begin{aligned} b_{22,1} = (|g_{\alpha\beta}|/\Delta)[&g_{11}g_{12}b_{22}K_{,2} - 2g_{12}b_{11}b_{22}H_{,2} - 2g_{11}g_{22}b_{12}K_{,2} \\ &+ 4g_{22}b_{11}b_{12}H_{,2} - g_{11}g_{22}b_{22}K_{,1} + 2g_{11}b_{22}^2 H_{,1} \\ &+ g_{22}^2 b_{11}K_{,1} - 2g_{22}b_{11}b_{22}H_{,1}]. \end{aligned} \tag{2.3}$$

Since in the right members of (2.1) we have $b_{12,1} = b_{11,2}$ and $b_{12,2} = b_{22,1}$, the equations (2.1), (2.2) and (2.3) give the complete determination of the quantities $b_{\alpha\beta,\gamma}$. If we consider these equations at an arbitrary point P but relative to a coordinate system such that $g_{\alpha\beta} = \delta_{\alpha\beta}$ at P they will be found to simplify considerably. Thus we have

$$\begin{aligned} b_{11,1} &= 2H_{,1} - b_{22,1}, \qquad b_{22,2} = 2H_{,2} - b_{11,2}, \\ b_{11,2} &= \frac{(H^2 - K)^{-1}}{4}[-2b_{12}K_{,1} + 4b_{12}b_{22}H_{,1} - b_{11}K_{,2} + b_{22}K_{,2} \\ &\qquad + 2b_{11}^2 H_{,2} - 2b_{11}b_{22}H_{,2}], \\ b_{22,1} &= \frac{(H^2 - K)^{-1}}{4}[-2b_{12}K_{,2} + 4b_{11}b_{12}H_{,2} - b_{22}K_{,1} \\ &\qquad + 2b_{22}^2 H_{,1} + b_{11}K_{,1} - 2b_{11}b_{22}H_{,1}] \end{aligned} \tag{2.4}$$

at a point P where $g_{\alpha\beta} = \delta_{\alpha\beta}$. The quantities $b_{11,2}$ and $b_{22,1}$ in the right members of the first two equations (2.4) are considered to have the determination given by the last two of these equations.

3. Integrability conditions. We now deduce the integrability conditions derivable from the system (2.4). In conformity with the way in which the third and fourth equations of this system have been written we assume that $H^2 - K > 0$. Later we shall show that the final integrability relations are independent of this condition.

Since the covariant derivative of the fundamental tensor (components $g_{\alpha\beta}$) vanishes, it is permissible to differentiate the equations of the system (2.4) covariantly, the result of such differentiation giving relations which are valid at a point P for which $g_{\alpha\beta} = \delta_{\alpha\beta}$. Thus let us differentiate the first equation (2.4) covariantly with respect to x^2 and then subtract the quantity $b_{11,2,1}$ from both members of the resulting equation. This gives

$$b_{11,1,2} - b_{11,2,1} = 2H_{,1,2} - b_{22,1,2} - b_{11,2,1}. \tag{3.1}$$

The quantities $b_{11,2,1}$ and $b_{22,1,2}$ in the right member of (3.1) can be determined by covariant differentiation of the third and fourth equations of (2.4) respectively. For the difference appearing in the left member of (3.1) we have by Bianchi's identity

$$\begin{aligned} b_{11,1,2} - b_{11,2,1} &= - b_{\sigma 1}\overset{\sigma}{B}_{112} - b_{1\sigma}\overset{\sigma}{B}_{112} = - 2b_{1\sigma}\overset{\sigma}{B}_{112} \\ &= - 2b_{1\sigma}g^{\sigma\tau}B_{\tau 112} = - 2b_{1\sigma}g^{\sigma\tau}(g_{\tau 2}g_{11} - g_{\tau 1}g_{12})K \\ &= - 2b_{12}g_{11}K + 2b_{11}g_{12}K = - 2b_{12}K, \end{aligned}$$

where the B's are the components of the curvature tensor and use has been made of the fact that $g_{\alpha\beta} = \delta_{\alpha\beta}$ in obtaining the final expression. When these substitutions are made in (3.1) we obtain a rather complicated equation which is capable of simplification. In connection with this process of simplification let us define the quantities

$$b^{\alpha\beta} = B^{\alpha\beta}/|g_{\mu\nu}|,$$

where $B^{\alpha\beta}$ denotes the cofactor of the element $b_{\alpha\beta}$ in the determinant $|b_{\alpha\beta}|$. As so defined the quantities $b^{\alpha\beta}$ are the components of an absolute symmetric contravariant tensor. Then the above equation reduces to one which can be written in the form

$$\begin{aligned} &[2(H^2 - K)b^{\alpha\beta}H_{\alpha\beta} - 4Hb^{\alpha\beta}H_\alpha H_\beta + 2b^{\alpha\beta}K_\alpha H_\beta + 2Hg^{\alpha\beta}K_\alpha H_\beta \\ &\qquad - g^{\alpha\beta}K_\alpha K_\beta - (H^2 - K)g^{\alpha\beta}K_{\alpha\beta} - 4K(H^2 - K)^2]b_{12} = 0, \end{aligned} \tag{3.2}$$

in which for simplicity we have omitted the "commas" denoting covariant differentiation. Again we have

$$b_{12,1,2} - b_{12,2,1} = b_{11,2,2} - b_{22,1,1}$$

in view of the Codazzi equations. By Bianchi's identity the left member of this equation becomes $(b_{11} - b_{22})K$. The quantities $b_{11,2,2}$ and $b_{22,1,1}$ in the right member can be calculated by covariant differentiation of the third and fourth equations (2.4). When these substitutions are made we obtain an equation which, after considerable reduction, becomes

$$[\quad](b_{11} - b_{22}) = 0, \tag{3.3}$$

where the bracket stands for the bracket expression in (3.2).

If $H^2-K>0$ it follows from §1 that either $b_{11}-b_{22}\neq 0$ or $b_{12}\neq 0$. Hence it follows from (3.2) and (3.3) that

$$\begin{aligned}(3.4)\quad &2(H^2-K)b^{\alpha\beta}H_{\alpha\beta}-4Hb^{\alpha\beta}H_\alpha H_\beta+2b^{\alpha\beta}K_\alpha H_\beta+2Hg^{\alpha\beta}K_\alpha H_\beta\\ &\qquad -g^{\alpha\beta}K_\alpha K_\beta-(H^2-K)g^{\alpha\beta}K_{\alpha\beta}-4K(H^2-K)^2=0\end{aligned}$$

at a point P where $H^2-K>0$. From continuity, (3.4) is also valid at a point P where $H^2-K=0$ provided that this point P is a limit of points at which $H^2-K>0$. Finally suppose that $H^2-K=0$ at a point P which is not a limit of points at which $H^2-K>0$. Then this point P will be contained in a region R of the surface within which $H^2-K=0$. But then the left member of (3.4) is seen directly to vanish at P since $H=$const. and $K=$const. in R by the italicized result in §1. *Hence the equation* (3.4) *is satisfied at all points of the surface* (*without regard to whether, the quantity* H_2-K *vanishes or not*).

4. **Closed surfaces of constant mean curvature.** On the basis of the equation (3.4) we can obtain a simple direct proof of the known result that a closed surface (open and compact Riemann space) of constant mean curvature H and non-negative Gaussian curvature K is of constant Gaussian curvature. In fact if $H=$const., (3.4) becomes

$$(H^2-K)g^{\alpha\beta}K_{\alpha\beta}+g^{\alpha\beta}K_\alpha K_\beta+4K(H^2-K)^2=0.$$

Integrating the left member of this equation over the surface we find immediately that

$$\int g^{\alpha\beta}K_\alpha K_\beta dS+2\int K(H^2-K)^2dS=0.$$

If $K\geqq 0$ it follows that each integral must vanish separately. The vanishing of the first integral implies that $K_\alpha=0$ or $K=$const. over the surface.

5. **Determination of the quantities $b_{\alpha\beta}$ in general.** It is desirable for the requirements of the following calculations to put equation (3.4) into a more contracted form. This can be accomplished by writing

$$(5.1)\qquad b^{\alpha\beta}Q_{\alpha\beta}=g^{\alpha\beta}P_{\alpha\beta},$$

where

$$\begin{aligned}Q_{\alpha\beta}&=2(H^2-K)H_{\alpha\beta}-4HH_\alpha H_\beta+K_\alpha H_\beta+K_\beta H_\alpha,\\ P_{\alpha\beta}&=(H^2-K)K_{\alpha\beta}+K_\alpha K_\beta-HK_\alpha H_\beta-HK_\beta H_\alpha+2K(H^2-K)^2g_{\alpha\beta}.\end{aligned}$$

As so defined the P's and Q's are the components of symmetric tensors.

Now consider the equation (5.1) and also the first of the equations (1.1). Write these in the following expanded form

$$\begin{cases} Q_{22}b_{11} - 2Q_{12}b_{12} + Q_{11}b_{22} = \left| g_{\mu\nu} \right| g^{\alpha\beta}P_{\alpha\beta}, \\ g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22} = 2\left| g_{\mu\nu} \right| H. \end{cases} \tag{5.2}$$

We use these equations in the determination of the $b_{\alpha\beta}$. The matrix of the coefficient of the b's in these equations has the same rank as

$$\left\| \begin{matrix} Q_{22} & Q_{12} & Q_{11} \\ g_{22} & g_{12} & g_{11} \end{matrix} \right\|. \tag{5.3}$$

In general the rank of this matrix will be two. In the calculation which follows we assume this general case and suppose furthermore for definiteness that the last of the second order determinants of the matrix does not vanish. The results obtained will be seen to be independent of the nonvanishing of this particular determinant and will depend in fact only on the assumption that the above matrix has rank two.

Considerable simplification will be obtained by carrying out the calculations at a point P relative to a coordinate system with respect to which $g_{\alpha\beta} = \delta_{\alpha\beta}$ at P. The results found under this condition can immediately be put into a general invariant form. The above nonvanishing determinant in the matrix (5.3) now has the value $Q_{12} \neq 0$ and under this condition the equations (5.2) can be solved for b_{12} and b_{22} in terms of b_{11}. We have

$$b_{12} = [(Q_{22} - Q_{11})b_{11} + (2HQ_{11} - \sum P_{\alpha\alpha})](Q_{12})^{-1}/2, \tag{5.4}$$

$$b_{22} = 2H - b_{11}. \tag{5.5}$$

Now turn to the second of the equations (1.1) and into this equation substitute the above values of b_{12} and b_{22}. The result is a quadratic equation in b_{11} the solution of which is

$$b_{11} = \frac{4HQ_{12}^{2} - (Q_{22} - Q_{11})(2HQ_{11} - \sum P_{\alpha\alpha}) \pm A^{1/2}}{(Q_{22} - Q_{11})^2 + 4Q_{12}^2}, \tag{5.6}$$

where

$$\begin{aligned} A = {} & [(Q_{22} - Q_{11})(2HQ_{11} - \sum P_{\alpha\alpha}) - 4HQ_{12}^{2}]^2 \\ & - [(Q_{22} - Q_{11})^2 + 4Q_{12}^2][(2HQ_{11} - \sum P_{\alpha\alpha})^2 + 4KQ_{12}^2]. \end{aligned}$$

By multiplying out the factors occurring in the numerator of the expression for b_{11} and combining in an appropriate manner we find that

$$\begin{aligned} & 4HQ_{12}^{2} - (Q_{22} - Q_{11})(2HQ_{11} - \sum P_{\alpha\alpha}) \\ & \quad = [(\sum P_{\alpha\alpha})(\sum Q_{\beta\beta}) - 4H\left| Q_{\alpha\beta} \right|] + 2Q_{11}[H\sum Q_{\alpha\alpha} - \sum P_{\alpha\alpha}], \end{aligned}$$

and

$$A = 4Q_{12}^2\left[2H\left(\sum P_{\alpha\alpha}\right)\left(\sum Q_{\beta\beta}\right) - K\left(\sum Q_{\alpha\alpha}\right)^2 - \left(\sum P_{\alpha\alpha}\right)^2 - 4(H^2 - K)\,|\,Q_{\alpha\beta}\,|\right].$$

Substitution of these values into (5.6) gives

$$\begin{aligned}(5.7)\quad b_{11} = {} & \frac{\left[\sum P_{\alpha\alpha}\sum Q_{\beta\beta} - 4H\,|\,Q_{\alpha\beta}\,|\right]}{\left[(Q_{22}-Q_{11})^2+4Q_{12}^2\right]} + \frac{2\left[H\sum Q_{\alpha\alpha} - \sum P_{\alpha\alpha}\right]Q_{11}}{\left[(Q_{22}-Q_{11})^2+4Q_{12}^2\right]} \\ & \pm \frac{2Q_{12}\left[2H\sum P_{\alpha\alpha}\sum Q_{\beta\beta} - K\left(\sum Q_{\alpha\alpha}\right)^2 - \left(\sum P_{\alpha\alpha}\right)^2 - 4(H^2-K)\,|\,Q_{\alpha\beta}\,|\right]^{1/2}}{\left[(Q_{22}-Q_{11})^2+4Q_{12}^2\right]}.\end{aligned}$$

Now consider the ratio

$$\frac{2\left[\sum P_{\alpha\alpha}\sum Q_{\beta\beta} - 4H\,|\,Q_{\alpha\beta}\,|\right] + 2\left[H\sum Q_{\alpha\alpha} - \sum P_{\alpha\alpha}\right]\sum Q_{\beta\beta}}{\left[(Q_{22} - Q_{11})^2 + 4Q_{12}^2\right]}.$$

If we expand and rearrange the terms of the numerator and denominator this ratio becomes

$$\frac{2H\left(\sum Q_{\alpha\alpha}\right)^2 - 8H\,|\,Q_{\alpha\beta}\,|}{\left(\sum Q_{\alpha\alpha}\right)^2 - 4\,|\,Q_{\alpha\beta}\,|} = 2H.$$

Since $b_{11}+b_{22}=2H$ we can now immediately deduce the expression for b_{22}. We have in fact

$$\begin{aligned}(5.8)\quad b_{22} = {} & \frac{\left[\sum P_{\alpha\alpha}\sum Q_{\beta\beta} - 4H\,|\,Q_{\alpha\beta}\,|\right]}{\left[(Q_{22}-Q_{11})^2+4Q_{12}^2\right]} + \frac{2\left[H\sum Q_{\alpha\alpha} - \sum P_{\alpha\alpha}\right]Q_{22}}{\left[(Q_{22}-Q_{11})^2+4Q_{12}^2\right]} \\ & \mp \frac{2Q_{12}\left[2H\sum P_{\alpha\alpha}\sum Q_{\beta\beta} - K\left(\sum Q_{\alpha\alpha}\right)^2 - \left(\sum P_{\alpha\alpha}\right)^2 - 4(H^2-K)\,|\,Q_{\alpha\beta}\,|\right]^{1/2}}{\left[(Q_{22}-Q_{11})^2+4Q_{12}^2\right]}.\end{aligned}$$

To obtain the corresponding expression for b_{12} we substitute the expression for b_{11} given by (5.7) into the right member of (5.4). After cancellation of various terms the resulting equation reduces to

$$\begin{aligned}(5.9)\quad b_{12} = {} & \frac{2\left(H\sum Q_{\alpha\alpha} - \sum P_{\alpha\alpha}\right)Q_{12}}{\left[(Q_{22}-Q_{11})^2+4Q_{12}^2\right]} \\ & \pm \frac{(Q_{22}-Q_{11})\left[2H\sum P_{\alpha\alpha}\sum Q_{\beta\beta} - K\left(\sum Q_{\alpha\alpha}\right)^2 - \left(\sum P_{\alpha\alpha}\right)^2 - 4(H^2-K)\,|\,Q_{\alpha\beta}\,|\right]^{1/2}}{\left[(Q_{22}-Q_{11})^2+4Q_{12}^2\right]}.\end{aligned}$$

The problem now is to combine equations (5.7), (5.8) and (5.9) into a tensor system of recognizable invariantive character. A step in this direction has already been accomplished by the way in which certain terms in the right members of these equations have been written. For example $\sum P_{\alpha\alpha}$ becomes $g^{\alpha\beta}P_{\alpha\beta}$ when referred to arbitrary coordinates. For this purpose we define the following scalars:

$$\begin{aligned} W &= [(Q_{22} - Q_{11})^2 + 4Q_{12}^2] = (g^{\alpha\beta}Q_{\alpha\beta})^2 - 4|Q_{\alpha\beta}|/|g_{\alpha\beta}|, \\ R &= \sum P_{\alpha\alpha}\sum Q_{\beta\beta} - 4H|Q_{\alpha\beta}| = g^{\alpha\beta}P_{\alpha\beta}g^{\mu\nu}Q_{\mu\nu} - 4H|Q_{\alpha\beta}|/|g_{\alpha\beta}|, \\ U &= H\sum Q_{\alpha\alpha} - \sum P_{\alpha\alpha} = Hg^{\alpha\beta}Q_{\alpha\beta} - g^{\alpha\beta}P_{\alpha\beta}, \end{aligned}$$

in which the middle members in these equations are the expressions for these scalars when $g_{\alpha\beta} = \delta_{\alpha\beta}$. In terms of these scalars the expression underneath the radical in the above equations can easily be shown to be given by $(H^2 - K)W - U^2$. Next define the symmetric tensor V with components

$$V_{\alpha\beta} = (Q_{\alpha\mu}\epsilon_{\beta\nu} + Q_{\beta\mu}\epsilon_{\alpha\nu})g^{\mu\nu},$$

in which the ϵ's are the components of the skew-symmetric tensor defined by $\epsilon_{12} = -\epsilon_{21} = (|g_{\alpha\beta}|)^{1/2}$ and $\epsilon_{11} = \epsilon_{22} = 0$. We note in particular that $V_{11} = 2Q_{12}$, $V_{22} = -2Q_{12}$ and $V_{12} = Q_{22} - Q_{11}$ when $g_{\alpha\beta} = \delta_{\alpha\beta}$. On the basis of this observation and the above definitions of the scalars W, R, and U we now find that equations (5.7), (5.8) and (5.9) can be combined into the system

$$(5.10) \quad b_{\alpha\beta} = RW^{-1}g_{\alpha\beta} + 2UW^{-1}Q_{\alpha\beta} \pm \{[(H^2 - K)W - U^2]^{1/2}/W\}V_{\alpha\beta}.$$

The determination (5.10) *of the quantities $b_{\alpha\beta}$ is valid in any region of the surface in which the condition $W > 0$ is satisfied.* If $W = 0$ at a point P we see from the above expression for this scalar that $Q_{11} = Q_{22}$ and $Q_{12} = 0$ provided that $g_{\alpha\beta} = \delta_{\alpha\beta}$ at P. But then we can write $Q_{\alpha\beta} = \lambda g_{\alpha\beta}$, that is, the quantities $Q_{\alpha\beta}$ are proportional to the quantities $g_{\alpha\beta}$ and hence the matrix (5.3) has rank less than two at P. Conversely if this matrix has rank less than two, the Q's and the g's are proportional and the scalar $W = 0$.

In (5.10) the mean curvature H must be such a function that the conditions

$$(5.11) \qquad W > 0, \qquad (H^2 - K)W - U^2 \geqq 0$$

are satisfied. *Conversely if H is any function satisfying the conditions* (5.11) *it is evident from the derivation of* (5.10) *that the quantities $b_{\alpha\beta}$ given by* (5.10) *satisfy the algebraic conditions* (1.1).

6. **Relations between the invariants.** The scalars W, R, and U and the tensor V satisfy the following relations

$$2W = g^{\alpha\beta}g^{\mu\nu}V_{\alpha\mu}V_{\beta\nu},$$

$$HW = R + g^{\alpha\beta}Q_{\alpha\beta}U.$$

The first of these can be expressed by saying that the scalar W is one-half the square of the tensor V. The derivation of these relations can easily be carried out on the basis of the definitions of the invariants W, R, U, and V in §5.

7. **The exceptional case** $W=0$. Suppose $W=0$ over a closed surface. Then $Q_{\alpha\beta}=\lambda(x)g_{\alpha\beta}$, and substituting into this equation the value of $Q_{\alpha\beta}$ from §5 we have

$$2(H^2 - K)H_{\alpha\beta} - 4HH_\alpha H_\beta + K_\alpha H_\beta + K_\beta H_\alpha = \lambda g_{\alpha\beta}. \tag{7.1}$$

Now $b^{\alpha\beta}Q_{\alpha\beta}=b^{\alpha\beta}g_{\alpha\beta}\lambda=2H\lambda$. Hence (5.1) becomes

$$\begin{aligned}(H^2 - K)g^{\alpha\beta}K_{\alpha\beta} + g^{\alpha\beta}K_\alpha K_\beta - 2Hg^{\alpha\beta}K_\alpha H_\beta& \\ + 4K(H^2 - K)^2 &= 2H\lambda.\end{aligned} \tag{7.2}$$

Next put $H^2-K=\theta(\geqq 0)$ or $K=H^2-\theta$, from which we deduce

$$K_\alpha = 2HH_\alpha - \theta_\alpha,$$

$$K_{\alpha\beta} = 2H_\alpha H_\beta + 2HH_{\alpha\beta} - \theta_{\alpha\beta}.$$

When these substitutions are made in (7.2) and the quantities $H_{\alpha\beta}$ in the resulting equations are eliminated by means of (7.1), we have an equation which reduces to

$$2\theta g^{\alpha\beta}H_\alpha H_\beta - \theta g^{\alpha\beta}\theta_{\alpha\beta} + g^{\alpha\beta}\theta_\alpha\theta_\beta + 4K\theta^2 = 0. \tag{7.3}$$

Now integrate the left member of (7.3) over the closed surface. We thus find that

$$\int \theta g^{\alpha\beta}H_\alpha H_\beta dS + \int g^{\alpha\beta}\theta_\alpha\theta_\beta dS + 2\int K\theta^2 dS = 0.$$

If $K>0$ at every point, each of the integrands is non-negative and hence must vanish over the surface. Hence from the third integral we must have $\theta=0$ and from §1 this means that the mean curvature and Gaussian curvature are constant. *Hence if the scalar W vanishes over a closed surface of positive Gaussian curvature, the mean curvature H and the Gaussian curvature K are constant over the surface.*

It follows from the above result that for the case of a closed analytic surface which is of positive but not of constant Gaussian curvature the scalar W cannot vanish over any region R of the surface. Hence W can vanish only at exceptional points and elsewhere on the surface the determination of the second fundamental form given by (5.10) will apply.

8. The differential equations satisfied by the mean curvature. We have observed (§5) that the quantities given by (5.10) satisfy (1.1) provided that H is a function satisfying (5.11). However in order for the function H to be the mean curvature of a surface and have the above quantities $b_{\alpha\beta}$ as the components of its second fundamental form the Codazzi equations must also be satisfied, that is, we must have $b_{\alpha\beta,\gamma}=b_{\alpha\gamma,\beta}$. It may be observed that for the case of the two-dimensional surface under consideration these latter equations can be written in the contracted form

$$b_{\alpha\beta.\gamma}\epsilon^{\beta\gamma} = 0. \tag{8.1}$$

Understanding that the $b_{\alpha\beta}$ are given by (5.10) *in terms of the mean curvature H and its derivatives, we see that the two equations* (8.1) *are the differential equations satisfied by the mean curvature H.* Conversely if the scalar function H satisfies the conditions (5.11) and the differential equations (8.1) over a given abstract two-dimensional Riemann space, then the quantities $b_{\alpha\beta}$ defined by (5.10) together with the function H will be the components of a possible second fundamental form and the mean curvature respectively for the given Riemann space considered as a surface in three-dimensional Euclidean space.

INDIANA UNIVERSITY

A NOTE ON HYPERGEODESICS AND CANONICAL LINES

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In this note we introduce two families of hypergeodesics on a non-ruled surface in ordinary projective space. Consideration of the properties of these hypergeodesics leads to certain geometrical constructions which yield canonical lines of the first kind from a given canonical line of the second kind.

We shall assume that the differential equations of a non-ruled surface S are written in the Fubini canonical form[1]

$$(1)\quad x_{uu} = px + \theta_u x_u + \beta x_v, \qquad x_{vv} = qx + \gamma x_u + \theta_v x_v \quad (\theta = \log \beta\gamma).$$

We select an ordinary point P_x of the surface S as one vertex of the usual local tetrahedron of reference. When a curve C_λ through the point P_x is regarded as being imbedded in the one-parameter family of curves represented on S by the equation

$$(2)\qquad dv - \lambda(u, v)du = 0,$$

the osculating plane at the point P_x of the curve C_λ has the local equation

$$(3)\qquad 2\lambda(\lambda x_2 - x_3) + (\lambda' + \beta - \theta_u\lambda + \theta_v\lambda^2 - \gamma\lambda^3)x_4 = 0,$$

in which we have placed $\lambda' = \lambda_u + \lambda\lambda_v$.

It will be recalled that two lines $l_1(a, b)$, $l_2(a, b)$ are reciprocal lines[2] at a point P_x of a surface if the line $l_1(a, b)$ joins the point P_x and the point y defined by placing

$$y = -ax_u - bx_v + x_{uv}$$

and the line $l_2(a, b)$ joins the points ρ, σ defined by

$$\rho = x_u - bx, \qquad \sigma = x_v - ax,$$

where a, b are functions of u, v. As the point P_x varies over the surface S, the lines $l_1(a, b)$, $l_2(a, b)$ generate two reciprocal congruences Γ_1, Γ_2, respectively.

The two reciprocal lines $l_1(a, b)$, $l_2(a, b)$ are canonical lines $l_1(k)$, $l_2(k)$ of the first and second kind respectively in case

$$a = -k\psi, \qquad b = -k\phi,$$

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[1] E. P. Lane, *Projective differential geometry of curves and surfaces*, Chicago, 1932, p. 69.

[2] E. P. Lane, *loc. cit.*, pp. 82–85.

where k is a constant and ϕ, ψ are defined by

$$\phi = (\log \beta\gamma^2)_u, \qquad \psi = (\log \beta^2\gamma)_v.$$

Canonical lines of the first kind lie in the canonical plane whose local equation is

$$\phi x_2 - \psi x_3 = 0. \tag{4}$$

The equation of the plane $\pi(a, b)$ which is the harmonic conjugate of the tangent plane of the surface at the point P_x with respect to the two focal planes of a general line $l_2(a, b)$ of the congruence Γ_2 is given by

$$x_1 + bx_2 + ax_3 + [2^{-1}(a_u + b_v) + ab]x_4 = 0. \tag{5}$$

For a general canonical line $l_2(k)$, the local coordinates $\xi_1, \cdots, \xi_4$ of the plane $\pi(k)$ are given by

$$\xi_1 = 1, \quad \xi_2 = -k\phi, \quad \xi_3 = -k\psi, \quad \xi_4 = -3k\theta_{uv}/2 + k^2\phi\psi. \tag{6}$$

The equations, in plane coordinates, of the two asymptotic osculating quadrics Q_u, Q_v of a curve C_λ are respectively

$$\begin{aligned} &2\lambda^3(\xi_2\xi_3 - \xi_1\xi_4) - 2\beta\xi_1(\beta\xi_1 - \lambda\xi_2 + \lambda^2\xi_3) - C\xi_1^2 = 0, \\ &2(\xi_2\xi_3 - \xi_1\xi_4) - 2\gamma\lambda\xi_1(\gamma\lambda\xi_1^2 + \xi_2 - \lambda\xi_3) - D\xi_1^2 = 0, \end{aligned} \tag{7}$$

where we have placed

$$\begin{aligned} C &= \beta[\lambda' - \beta + (\phi - \theta_u)\lambda - (2\psi - \theta_v)\lambda^2] - (\beta\gamma + \theta_{uv})\lambda^3, \\ D &= \gamma[-\lambda' - \gamma\lambda^3 + (\psi - \theta_v)\lambda^2 - (2\phi - \theta_u)\lambda] - (\beta\gamma + \theta_{uv}). \end{aligned}$$

Let us now suppose that, at each point P_x of the curve C_λ, the asymptotic osculating quadric Q_u of C_λ is tangent to the plane $\pi(k)$. Then we find that the function λ satisfies the differential equation

$$\lambda' = A_1 + B_1\lambda + C_1\lambda^2 + D_1\lambda^3, \tag{8}$$

where the coefficients A_1, B_1, C_1, D_1 are given by

$$\begin{aligned} A_1 &= -\beta, & B_1 &= \theta_u - (1 + 2k)\phi, \\ C_1 &= 2(1 + k)\psi - \theta_v, & D_1 &= (1/\beta)[\beta\gamma + (1 + 3k)\theta_{uv}], \end{aligned} \tag{9}$$

in which, for the present, we assume $k \neq -1/3$, so that the canonical line $l_2(k)$ is not the second axis a_2 of Čech.

Similarly, the asymptotic osculating quadric Q_v of C_λ is tangent to the plane $\pi(k)$ if, and only if,

$$\lambda' = A_2 + B_2\lambda + C_2\lambda^2 + D_2\lambda^3, \tag{10}$$

where the coefficients A_2, B_2, C_2, D_2 are given by

$$
(11)\quad \begin{aligned} A_2 &= -(1/\gamma)[\beta\gamma + (1+3k)\theta_{uv}], & B_2 &= \theta_u - 2(1+k)\phi, \\ C_2 &= (1+2k)\psi - \theta_v; & D_2 &= \gamma, \end{aligned}
$$

in which $k \neq -1/3$. Thus we reach the following conclusion:

At each point P_x of a curve C_λ, each of the asymptotic osculating quadrics Q_u and Q_v of C_λ is tangent to the plane $\pi(k)$ which is the harmonic conjugate of the tangent plane of the surface at the point P_x with respect to the two focal planes of any canonical line $l_2(k)$, except the second axis a_2 of Čech, if, and only if, C_λ is an integral curve (hypergeodesic) of the respective differential equations (8), (10).

By means of equation (3), together with equation (8), we find that the osculating plane at a point P_x of a hypergeodesic of the family (8) has the local equation

$$
(12)\quad \begin{aligned} &2\beta(\lambda x_2 - x_3) \\ &\qquad + [-(1+2k)\beta\phi + 2(1+k)\beta\psi\lambda + (1+3k)\theta_{uv}\lambda^2]x_4 = 0. \end{aligned}
$$

The envelope of the osculating planes at the point P_x of all the hypergeodesics of the family (8) is found from equation (12) to be the nondegenerate quadric cone whose equation is

$$
(13)\quad \beta[x_2 + (1+k)\psi x_4]^2 + (1+3k)\theta_{uv}[2x_3 + (1+2k)\phi x_4]x_4 = 0.
$$

Similarly, the envelope of the osculating planes at the point P_x of the hypergeodesics (10) is the quadric cone

$$
(14)\quad \gamma[x_3 + (1+k)\phi x_4]^2 + (1+3k)\theta_{uv}[2x_2 + (1+2k)\psi x_4]x_4 = 0.
$$

The vertex of each of the cones (13), (14) is, of course, the point P_x. Furthermore, the cone (13) is tangent to the tangent plane of the surface at the point P_x along the asymptotic v-tangent at P_x, while the cone (14) has the asymptotic u-tangent of S at P_x for its line of contact with the tangent plane. The polar plane of any point on the u-tangent with respect to the cone (13) intersects this cone, besides in the v-tangent at P_x, also in a generator which is the line $l_1(a, b)$ with a and b defined by the formulas

$$
(15)\quad a = (1+k)\psi, \qquad b = 2^{-1}(1+2k)\phi.
$$

We may also regard this line as being determined by the planes whose equations are

$$
(16)\quad x_2 + (1+k)\psi x_4 = 0, \qquad x_3 + 2^{-1}(1+2k)\phi x_4 = 0.
$$

Similarly, the polar plane of any point on the v-tangent with respect to the cone (14) intersects this cone in the u-tangent at P_x and in the line $l_1(a, b)$ for which

$$a = 2^{-1}(1 + 2k)\psi, \qquad b = (1 + k)\phi. \tag{17}$$

This line may also be regarded as determined by the planes

$$x_2 + 2^{-1}(1 + 2k)\psi x_4 = 0, \qquad x_3 + (1 + k)\phi x_4 = 0. \tag{18}$$

The locus of the line (16) for all canonical lines $l_2(k)$ is found by eliminating k from equations (16) to be the plane

$$\phi x_2 - \psi x_3 + 2^{-1}\phi\psi x_4 = 0. \tag{19}$$

Similarly, the locus of the line (18) is the plane

$$\phi x_2 - \psi x_3 - 2^{-1}\phi\psi x_4 = 0. \tag{20}$$

Thus we find that *the tangent plane of the surface at the point P_x and the canonical plane* (4) *separate the planes defined by equations* (19), (20) *harmonically.*

We now describe simple geometrical constructions which yield canonical lines of the first kind from a given canonical line $l_2(k)$ of the second kind, except the second axis a_2 of Čech. In the first place, the plane determined by the two lines (16), (18) has the equation

$$\phi x_2 + \psi x_3 + 2^{-1}(3 + 4k)\phi\psi x_4 = 0, \tag{21}$$

and is found to intersect the canonical plane (4) in the canonical line $l_1(k_1)$ for which

$$k_1 = -4^{-1}(3 + 4k). \tag{22}$$

The polar plane of any point on the u-tangent at P_x with respect to the cone (13) intersects the polar plane of any point on the v-tangent at P_x with respect to the cone (14) in the canonical line $l_1(k_2)$ for which

$$k_2 = -(1 + k). \tag{23}$$

Furthermore, the plane which is tangent to the cone (13) along the line (16) intersects the plane which is tangent to the cone (14) along the line (18) in the canonical line $l_1(k_3)$ for which

$$k_3 = -2^{-1}(1 + 2k). \tag{24}$$

We remark that if the line $l_2(k)$ is the second edge e_2 of Green, for which $k = -1/4$, then the three canonical lines $l_1(k_1)$, $l_1(k_2)$, and $l_1(k_3)$ obtained by the preceding constructions are respectively the first directrix d_1 of Wilczynski, the first canonical line for which $k=-3/4$, and the first edge e_1 of Green.

It may be seen from the formulas (22), (23), and (24) that a given canonical line $l_2(k)$ will yield in turn each of the canonical lines $l_1(k_1)$, $l_1(k_2)$, and $l_1(k_3)$ for its reciprocal if, and only if, the given canonical line $l_2(k)$ is the second canonical line for which $k=-3/8$, the second directrix d_2 of Wilczynski, and the second edge e_2 of Green.

Finally, if the given canonical line $l_2(k)$ is the second axis a_2 of Čech for which $k=-1/3$, *the asymptotic osculating quadric* Q_u *of* C_λ *is tangent to the plane* $\pi(-1/3)$ *if, and only if,*

$$\lambda' = -\beta + (\theta_u - \phi/3)\lambda + (4\psi/3 - \theta_v)\lambda^2 + \gamma\lambda^3, \tag{25}$$

so that the curve C_λ is a union curve of the congruence Γ_1 of lines $l_1(a, b)$ for which

$$a = 2\psi/3, \qquad b = \phi/6.$$

Similarly, the asymptotic osculating quadric Q_v of C_λ is tangent to the plane $\pi(-1/3)$ if, and only if,

$$\lambda' = -\beta + (\theta_u - 4\phi/3)\lambda + (\psi/3 - \theta_v)\lambda^2 + \gamma\lambda^3, \tag{26}$$

in which case C_λ is a union curve of the congruence Γ_1 of lines $l_1(a, b)$ for which

$$a = \psi/6, \qquad b = 2\phi/3.$$

The plane determined by the two lines thus defined at the point P_x intersects the canonical plane in the canonical line $l_1(k)$ for which $k=-5/12$, namely, the first axis of Bompiani.

We conclude with the statement that if at the point P_x both of the asymptotic osculating quadrics Q_u and Q_v of C_λ are tangent to the plane $\pi(-1/3)$, then $\phi+\psi\lambda=0$, so that the curve C_λ is tangent at P_x to the second canonical tangent t_2.

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NOTE ON CONVEX CURVES ON THE HYPERBOLIC PLANE

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1. **Introduction.** In a previous note [5](1) we have obtained some properties referring to convex curves on the sphere. Following an analogous way our purpose is now to obtain the same properties for convex curves on a surface of constant negative curvature $K=-1$, or, what is equivalent, for convex curves on the hyperbolic plane.

In §§6 and 7 we consider the curves of constant breadth, for which we obtain the formula (7.3) which relates the length L and area F with the breadth α.

For the curves which are not of constant breadth the formula (4.5), which contains (7.3) as a particular case, holds. But (4.5) is true only if we suppose that the curve has in all its points geodesic curvature κ_g greater than one.

2. **Definitions.** A closed curve C on a surface of constant negative curvature $K=-1$ is said to be convex when it cannot be cut by any geodesic in more than two points, except that a complete arc of geodesic may belong to the curve. Any closed convex curve C has a finite length L and bounds a finite area F. In the following, unless otherwise specified, we shall suppose that C is composed of a finite number of arcs each with continuous geodesic curvature κ_g.

Let ω_i be the exterior angles which these arcs form at the vertices of C. Then we have the Gauss-Bonnet formula [3, p. 191],

$$\int_C \kappa_g ds + \sum \omega_i = 2\pi + F. \tag{2.1}$$

If a point O on C is taken as origin, any point A of C can be determined by the length of the arc $OA=s$ or by the angle τ defined by

$$\tau = \int_0^s \kappa_g ds + \sum_s \omega_i \tag{2.2}$$

where $\sum_s \omega_i$ is extended over all the vertices of C contained in the arc OA.

Any geodesic with only one common point or with a complete arc in common with C is called a "geodesic of support" of C. In each

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[1] Numbers in brackets refer to the references cited at the end of the paper.

point of C for which there exists a tangent geodesic the geodesic of support coincides with this. Any geodesic which passes through a vertex without crossing C is also a geodesic of support. To any geodesic of support of C corresponds a value of the angle τ (2.2).

Let g be a geodesic of support of C and let A be its point of support. Let g' be the orthogonal geodesic to g at the point A and g_1 another geodesic of support of C which is also orthogonal to g'. The curve C will be contained between g and g_1. If A' is the point in which g_1 cuts g', we shall call *breadth* α of C corresponding to the point A the length of the arc AA' of geodesic g'. The breadth α is a function of the angle τ or the arc s corresponding to the point A.

3. Closed convex curves with $\kappa_g > 1$ at any point. Let us suppose that C has $\kappa_g > 1$ at any point. We shall call *pseudospherical osculating circle* of C at the point A the limit of the geodesic circle determined by the points A, A_1, A_2 of C when $A_1, A_2 \to A$.

If we suppose $\kappa_g > 1$, the radius R of the pseudospherical osculating circle (which we shall call "radius of pseudospherical curvature") has a finite value and is related to the geodesic curvature κ_g by

$$\kappa_g = \coth R. \tag{3.1}$$

This equality is obtained by applying the Gauss-Bonnet formula to a geodesic circle and using the following formulas for its length and area:

$$L = 2\pi \sinh R, \qquad F = 2\pi(\cosh R - 1). \tag{3.2}$$

The center O of pseudospherical curvature is the limiting position of the point in which the orthogonal geodesic to C at $A(s)$ cuts the orthogonal geodesic at $A(s+\Delta s)$ when $\Delta s \to 0$.

The condition $\kappa_g > 1$, which is equivalent to $R < \infty$, will then be necessary and sufficient for two sufficiently close orthogonal geodesics to C to intersect each other.

Before proceeding it is necessary that we prove the following lemma.

LEMMA. *Let g_1 and g_2 be two geodesics which are orthogonal to the geodesic g. Let MN be an arc of the curve C with $\kappa_g > 1$ at each point and which is tangent to g_1 and g_2 at the ends M, N respectively. We affirm that: on the arc MN there is only one point with the property that the orthogonal geodesic to C which passes through it is also orthogonal to g.*

We suppose the arc MN is composed of a finite number of arcs with continuous geodesic curvature, and a geodesic will be considered orthogonal to MN at a corner if it is orthogonal to a geodesic of support through the corner.

PROOF. The angle which the geodesics of support of the arc MN form with the orthogonal geodesic to g through their contact points increases from 0 to π. Hence there is a point at which this angle equals $\pi/2$. It remains to be proved that there is no other point with this property.

Let us consider the curvilinear coordinate system formed by the orthogonal geodesics to g as curves $v=\text{const.}$ and their orthogonal trajectories as curves $u=\text{const.}$, $u=0$ being the geodesic g. Then the element of length is given by [3, p. 282]

$$ds^2 = du^2 + \cosh^2 u\,dv^2. \tag{3.3}$$

If $u=u(s)$, $v=v(s)$ are the equations of the curve C, calling ϕ the angle which C forms at each point with the corresponding geodesic $v=\text{const.}$, we have $\tan\phi=\cosh u v'/u'$, hence

$$d\tan\phi/ds = [(u'v'' - u''v')\cosh u + u'^2v'\sinh u]u'^{-2}. \tag{3.4}$$

The geodesic curvature of the curve $u=u(s)$, $v=v(s)$ is given by [3, p. 187]

$$\kappa_g = (u'v'' - u''v')\cosh u + 2u'^2v'\sinh u + v'^3\cosh^2 u\sinh u. \tag{3.5}$$

From this and from (3.3) we deduce

$$(u'v'' - u''v')\cosh u + u'^2v'\sinh u = \kappa_g - v'\sinh u. \tag{3.6}$$

From $u'^2+v'^2\cosh^2 u=1$ and $\cosh^2 u-\sinh^2 u=1$ we have $v'^2\sinh^2 u =1-u'^2-v'^2$, that is, $v'^2\sinh^2 u\leqq 1$. Hence, from (3.4) and (3.6) under the assumption that $\kappa_g>1$, we get (if $u'\neq 0$)

$$d\tan\phi/ds = (\kappa_g - v'\sinh u)u'^{-2} > 0.$$

The angle ϕ is then always increasing from M to N, that is, from 0 to π. Consequently at only one point will $\phi=\pi/2$, which proves our lemma.

4. **Principal formula.** Let g be a geodesic which cuts the closed convex curve C and let O be a fixed point on the surface with $K=-1$ that contains C. Let w be the distance from O to g and θ the angle which the orthogonal geodesic from O to g makes with a fixed direction at O. Then it is known [4, p. 687] that the measure of a set of geodesics is the integral of the expression $dg=\cosh w\,d\theta\,dw$ extended to the set.

Consider the set of "oriented" geodesics which cut C; it is known [4, p. 691] that

$$\int_{C,g\neq 0} dg = \int \cosh w d\theta dw = 2L, \tag{4.1}$$

where L is the length of C.

According to the former lemma if we suppose that $\kappa_g > 1$ in each point of C, on each side of g there will be only one point A in which the orthogonal geodesic to C will also be orthogonal to g. Consequently the geodesic g can be determined by the point A (that is, by the corresponding value of s or τ) and the distance a from A to g (Fig. 1). We wish to express the density $dg = \cosh w d\theta dw$ in terms of τ and a. It is known that the differential expression $dg = \cosh w d\theta dw$ does not depend on the point O or the direction origin of the angles θ. Consequently we can consider for a moment that the point O is the pseudospheric center of curvature of C at A, that is, the point in which the orthogonal geodesic to C at $A(\tau)$ is intersected by the orthogonal geodesic at $A' = A(\tau + d\tau)$. Hence $OA = R$, $w = R - a$. Let H be the point in which the geodesic of support of C at A intersects the geodesic of support at A'. From the Gauss-Bonnet theorem we deduce that the area of the geodesic quadrilateral $OAHA'$ has the value

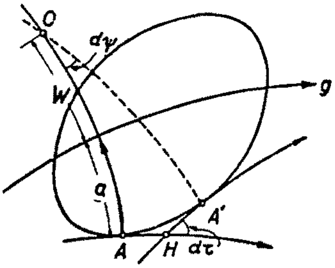

FIG. 1

$$(\pi/2 + d\tau + \pi/2 + \pi - d\psi) - 2\pi = d\tau - d\psi, \tag{4.2}$$

$d\tau$ being the angle which the geodesic of support at A forms with the geodesic of support at A' and $d\psi$ the angle AOA'. But save for infinitesimals of second order the same area equals the area of the sector of geodesic circle AOA' which has the value $(\cosh R - 1)d\psi$. Consequently

$$d\tau = \cosh R d\psi. \tag{4.3}$$

Since R is independent of w, from $w = R - a$, $d\theta = d\psi$, and (4.3) we deduce

$$|\partial(a, \tau)/\partial(\theta, w)| = \cosh R.$$

Hence $dg = \cosh w d\theta dw = (\cosh (R - a)/\cosh R) da d\tau$ or, since $d\tau = \kappa_g ds$, we have

$$dg = \cosh a da d\tau - \sinh a da ds. \tag{4.4}$$

Let us substitute this expression (4.4) in (4.1). For each value of s (or τ) the arc a can vary from 0 to the breadth α of C corresponding to the point s (or τ). Therefore

$$\int_{C,\,g\neq 0} dg = \int_C d\tau \int_0^\alpha \cosh a da - \int_C ds \int_0^\alpha \sinh a da$$

or, in accordance with (4.1),

$$L = \int_C \sinh \alpha d\tau - \int_C \cosh \alpha ds. \tag{4.5}$$

This is our principal formula from which we wish to obtain some consequences. The formula (4.5) is analogous to that obtained for convex spherical curves in a previous paper [5] and holds for any convex curve with $\kappa_g > 1$ at each point on a surface of constant negative curvature $K = -1$.

5. **Consequences.** (a) Let Δ be the minimum breadth of C, that is to say, the minimum value of α, and δ the maximum value of α, that is, the diameter of C. From (4.5), (2.1), and (2.2) we deduce

$$\sinh \Delta/(1 + \cosh \delta) \leqq L/(2\pi + F) \leqq \sinh \delta/(1 + \cosh \Delta). \tag{5.1}$$

Therefore: *on a surface of constant negative curvature $K = -1$, for any convex curve C with $\kappa_g > 1$ the inequalities* (5.1) *are verified.*

(b) If C has a continuous geodesic curvature, from (4.5) we deduce

$$\int_C \sinh \alpha \kappa_g ds - \int_C (1 + \cosh \alpha) ds = 0$$

or

$$\int_C \cosh \frac{\alpha}{2} \left(\kappa_g \sinh \frac{\alpha}{2} - \cosh \frac{\alpha}{2} \right) ds = 0. \tag{5.2}$$

From this and according to (3.1), we have: *In any closed convex curve C on a surface of constant negative curvature $K = -1$, with continuous goedesic curvature $\kappa_g > 1$, there are at least two points for which* *the radius R of pseudospherical curvature equals $\alpha/2$, where α is the breadth of K corresponding to the point considered.*

6. **Convex curves of constant breadth.** In §2 we have defined the breadth α of a closed convex curve C on the surface with $K=-1$, corresponding to a point A of C. When α is constant, C is called a *curve of constant breadth.*

The geodesic circles of finite radius R are the first examples of curves of constant breadth $\alpha=2R$. Another class of curves of constant breadth is the generalization on the surfaces of constant negative curvature of the Reuleaux polygons [2, p. 130].

Let us consider a geodesic circle of radius R; we divide it into $2n+1$ equal parts and through the division points we draw the tangent geodesics. If two consecutive tangents intersect each other we shall have a geodesic regular polygon of an odd number of sides. Taking each vertex as center let us draw the arc of the geodesic circle which joins the two opposite vertices. These arcs form a Reuleaux polygon of $2n+1$ sides, and it is easily seen that this polygon has constant breadth.

The necessary relation can easily be found between the radius R of the geodesic circle and the number $2n+1$ of sides so that two consecutives tangent geodesics intersect. Let A_i and A_{i+1} be two consecutive division points and OH the geodesic which halves the angle A_iOA_{i+1}. The condition that the geodesic OH be cut for the tangent geodesic at A_i is that the angle $A_iOH=\pi/2n+1$ be smaller than the angle of parallelism corresponding to the tangent geodesic at A_i and to the center O, hence [1, p. 621]

$$\tan(\pi/(2n+1)) < 2e^{R}/(e^{2R}-1).$$

That is, to have a Reuleaux polygon of $2n+1$ sides we must start from a geodesic circle with a radius R which satisfies the inequality

$$R < \log \frac{1}{\tan(\pi/2(2n+1))}.$$

7. **Properties of the convex curves of constant breadth.** The convex curves of constant breadth on the surfaces of constant negative curvature (or on the hyperbolic plane) have analogous properties to the curves of constant breadth on the plane. For example it is easily seen that the constant breadth α equals the diameter δ. Therefore if A is a point of C the orthogonal geodesic g to C at A will cut C at the point A' and at this point g will also be orthogonal to C. The points A and A' can be called opposite points.

We shall prove that if C is of constant breadth any orthogonal geodesic to C, say AA', will intersect a neighbouring orthogonal geodesic at a point contained in the arc AA'.

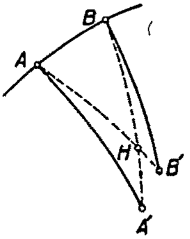

FIG. 2

Let $B \equiv A(\tau+d\tau)$ and BB' be the orthogonal geodesic to C at B (Fig. 2). If BB' does not intersect the arc AA' the point B' must have the position indicated in Fig. 2. But in this position we have

$$AA' < AH + HA', \qquad BB' < BH + HB' \tag{7.1}$$

and by addition

$$AA' + BB' < AB' + BA'. \tag{7.2}$$

Since C is of constant breadth we have $AA' = BB' > AB'$, BA' and therefore $AA'+BB' > AB'+BA'$ which gives a contradiction with (7.2).

From this we deduce that the pseudospherical radius of curvature R is always finite and not greater than α. Moreover since the orthogonal geodesics at A and B are also orthogonal geodesics at A' and B' the pseudospherical center of curvature will be the same at A and A'. Hence: *the sum of the pseudospherical radii of curvature corresponding to opposite points equals the breadth* α.

Since $R < \infty$, we shall have $\kappa_g > 1$; therefore we can apply the formula (4.5) which gives, according to (2.1),

$$L = (2\pi + F) \sinh \alpha - L \cosh \alpha,$$

that is,

$$L = (2\pi + F) \tanh (\alpha/2). \tag{7.3}$$

We conclude the following theorem: *For any convex curve of constant breadth on the surface of constant negative curvature* $K = -1$, *the equation* (7.3) *holds.*

If $K = -1/a^2$ instead of -1, (7.3) gives

$$L/a = (2\pi + F/a^2) \tanh (\alpha/2a).$$

Muliplying both sides by a and letting $a \to \infty$ we find

$$L = \pi \alpha,$$

which is a well known relation between the length and breadth of the curves of constant breadth on the plane [2, p. 131].

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A FIXED-POINT THEOREM

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Introduction. The purpose of this note is to sharpen a recent result of G. E. Schweigert [4].[1] It will be shown that the condition of semi local-connectedness may be dropped. However, if this is strengthened to local-connectedness, then the conclusion asserts the existence of a fixed point. Further, though perhaps of less interest, it is shown that separability is not necessary.

In a second section we give a somewhat more abstract version which is valid for certain partially ordered topological spaces. So far as is known this is the first result of this type to appear in the literature.

1. Schweigert's theorem.

It is assumed that S is a compact (that is, bicompact) Hausdorff space, connected and nondegenerate. Moreover T is a topological transformation of S onto itself, $TS=S$.

THEOREM. *If e is an end point of S fixed under T, then there exists a continuum $K\subset S-e$ invariant under T. Further, no point of S separates any pair of points of K in S.*

PROOF. Since e is an end point it is readily seen that we can find a point y such that

$$(1)\quad S = A + B, \qquad e \in A, \qquad y + T^{-1}y \subset B, \qquad A\cdot B = z \in S,$$

with A and B nondegenerate proper subcontinua of S. We infer that

$$(2)\quad S = TA + TB, \qquad e \in TA, \qquad y \in TB, \qquad TA\cdot TB = Tz,$$

so that $S=(A+TA)+B\cdot TB$. Clearly $A+TA$ is a continuum and hence [5] so is

$$(A + TA)\cdot B\cdot TB = z\cdot TB + Tz\cdot B.$$

We then have (supposing that $z\neq Tz$) either (a) $z\in TB$ and $Tz\in S-B$ or (b) $Tz\in B$ and $z\in S-TB$. In the first case $T^{-1}z\in B$ and we are able to apply the same argument that we use in the second case if we use T^{-1} in place of T. We therefore assume (b) and readily verify that $A\subset TA$, $TB\subset B$.

Using induction it follows that we may write

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

$$S = T^nA + T^nB, \quad e \in T^nA, \quad T^{n+1}z \in T^nB, \quad T^nz = T^nA \cdot T^nB, \tag{3}$$
$$T^{n-1}A \subset T^nA, \quad T^nB \subset T^{n-1}B.$$

If X is the union of the sets T^nA and Y is the intersection of the sets T^nB it may be shown that X is connected and $K = \overline{X} \cdot Y$ is a continuum [5]. Clearly $T\overline{X} = \overline{TX}$, $TY = Y$ so that K is an invariant continuum contained in $S - e$. Suppose that p and q are points of K separated by r in S:

$$S - r = U + V, \qquad p \in U, \qquad q \in V, \qquad U \mid V.$$

We may admit that e is in U. For some m and all $n \geqq m$ it is clear that both U and V intersect T^nA. Thus, this set being connected, we infer that r is a point of T^nA and hence that Y meets T^nA and consequently Y has a point in common with X. But this would imply that for some n, $T^{n+1}z = T^nz$, giving $Tz = z$.

COROLLARY. *There exists a continuum $H \subset S - e$, having no cut point and invariant under T.*

This follows at once if we observe that an argument advanced by Kelley [2] to improve a theorem of Ayres [1] is unnecessarily restricted by the assumption of separability. The basic material necessary to modify Kelley's proof will be found in §§1 and 3 of [5]. In particular the generalized Brouwer theorem is given on p. 488. See also the references to Milgram, Moore, and Tukey and, further, Kuratowski [3].

COROLLARY. *If S is locally connected there exists a fixed point distinct from e.*

PROOF. With the notation employed in the proof of the theorem suppose that p and q are distinct points and let P and Q be disjoint open connected sets containing p and q and lying in $S - e$. For some m and all $n \geqq m$ both P and Q intersect T^nA. Thus P meets both T^nA and T^nB and so contains T^nz. Similarly Q contains T^nz. This is a contradiction.

It is a simple matter to show that Schweigert's theorem follows from the one given above. We need only use the same argument he gives to pass from the invariant node to the case of the fixed end point.

2. **A generalization.** In this section it is assumed that P is a compact Hausdorff space containing more than one point and, in addition, that we are given a binary relation Δ on P. We suppose that Δ is reflexive and transitive so that we have (a) $x\Delta x$ for each x in P and

(b) $x\Delta y$ and $y\Delta z$ imply $x\Delta z$. On the topological side let $M(a)$ $[N(a)]$ denote the set of all x such that $x\Delta a$ $[a\Delta x]$ holds. We then assume that Δ is semi-continuous (that is, both lower and upper semi-continuous) in the sense that

(i) *For each a in P the sets $M(a)$ and $N(a)$ are closed.*

A non-empty set Q is termed Δ-simple if for any pair q_1, q_2 of Q we have either $q_1\Delta q_2$ or $q_2\Delta q_1$. In virtue of a result due to Hausdorff (or directly by transfinite induction) it is readily seen that there exists a maximal Δ-simple set containing any given Δ-simple set. An element p is called a maximal element for a set R if $r\in R$ and $p\Delta r$ imply $r\Delta p$.

Every maximal Δ-simple set is closed.

For if Q is such a set then $Q(q)=M(q)+N(q)$ is closed for each $q\in Q$ and it is not difficult to see that Q is the intersection of all the sets $Q(q)$ for all $q\in Q$.

Each non-void closed subset of P contains a maximal element.

It is legitimate to suppose that the set concerned is P itself, so let Q be a maximal Δ-simple set. For q_1, q_2 in Q one set of the pair $N(q_1)$, $N(q_2)$ is a subset of the other. For each q the set $N(q)$ is closed and in virtue of the compactness of P there is a point p common to all these sets. The point p is in Q. If x is a point of Q we have $x\Delta p$ since p is in the set $N(x)$. Suppose that for x in $P-Q$ we have $p\Delta x$. Since Q is maximal Δ-simple there is a point y in Q such that both $x\Delta y$ and $y\Delta x$ fail to hold. But by transitivity $p\Delta x$ and $y\Delta p$ imply $y\Delta x$, a contradiction.

Three further assumptions must be made about P:

(ii) *There exists a unique element e in P such that we have $e\Delta x$ for each x.*

(iii) *Each set $M(a)$ is Δ-simple.*

(iv) *For any x, y distinct from e there is a z such that $z\Delta x$, $z\Delta y$ hold but $z\neq e$.*

It follows at once that

If Q is maximal Δ-simple then $a\in Q$ implies $M(a)\subset Q$.

For if $q\in Q$ and $x\in M(a)$ then $a\Delta q$ and transitivity imply $x\Delta q$. If $q\Delta a$ holds then $q\in M(a)$ and by (iii) we have either $x\Delta q$ or $q\Delta x$ so that x is an element of Q by maximality.

From (iv) we see that

Any pair of maximal Δ-simple sets X, Y have in common an element $z\neq e$.

To prove this let x, y be elements of $X-e$, $Y-e$ respectively. Let z be given by (iv). Then z is in $M(x)$ and so in X. Similarly z is in Y.

THEOREM. *If T is a homeomorphism of P onto itself and both T and*

T^{-1} preserve the relation Δ, then there exists an $f\neq e$ such that both $Tf\Delta f$ and $f\Delta Tf$ hold.

PROOF. Let C be a maximal Δ-simple set. Then TC is a set of the same type. Let z be an element common to C and TC and distinct from e. Both z and Tz are in TC so that we must have either $z\Delta Tz$ or $Tz\Delta z$. Now the argument we employ for the former case can be used in the latter provided we replace T by T^{-1} so that we assume that we have $z\Delta Tz$. Let Z be the set $z, Tz, T^2z, \cdots$. It is clear that if D is any maximal Δ-simple set containing Z, then TD has these same properties. Let X be the intersection of all the maximal Δ-simple sets containing Z. Then X is closed, Δ-simple, and further satisfies the condition $TX=X$. Let f be a maximal element of X. Both Tf and $T^{-1}f$ are in X and so we have $Tf\Delta f$ and $T^{-1}f\Delta f$. The latter implies $f\Delta Tf$. This completes the proof.

COROLLARY. *If Δ is symmetric then there is a fixed point distinct from e.*

The second corollary to the theorem of §1 follows if we define Δ as follows: $x\Delta y$ holds if $x=e$, $x=y$ or x separates y from e in S.

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THE MANIFOLDS OF LINEAR ELEMENTS OF AN n-SPHERE

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1. **Introduction.** The 3-manifolds of oriented and non-oriented linear elements of closed surfaces have been investigated by Nielsen,[1] Hotelling,[2] Threlfall,[3] van der Waerden and others.[4] In the present paper we take up the case of the space M of oriented linear elements, and the space M' of non-oriented linear elements, of an n-sphere, $n \geqq 1$. The chief tools in the present investigation are certain orthogonal transformations (§§3–4) and theorems on addition of complexes.[5] Our success in the determination of certain homology classes (§§7–8, 14) leads to complete determination of (integral) Betti groups of M and M'. Our results may be summarized as follows:

(M1) For $n>1$, M is an orientable $(2n-1)$-manifold. Its Betti groups, which are not the null groups, are the following: For even n, B^0 and $B^{2n-1} \approx G_0$ (AH, p. 556) and $B^{n-1} \approx G_2$; for odd n, B^0, B^{2n-1}, B^{n-1}, and $B^n \approx G_0$.

(M2) For $n=2$, M is the projective space. For $n>2$, its fundamental group is the identity.

(M3) For $n=1, 3, 7$, M is the topological product of an n-sphere and an $(n-1)$-sphere.

(M′1) For $n>1$, M' is an orientable or a non-orientable $(2n-1)$-manifold according as n is even or odd. Its Betti groups, which are not the null, are the following: For even n, B^0 and $B^{2n-1} \approx G_0$, $B^{n-1} \approx G_4$, and $B^r \approx G_2$, $r=1, 3, \cdots, n-3; n+1, n+3, \cdots, 2n-3$. For odd n, B^0 and $B^n \approx G_0$, and $B^r \approx G_2$, $r=1, 3, \cdots, n-2; n+1, n+3, \cdots, 2n-2$.

(M′2) For $n=2$, M' is the lens space (Linsenraum) (4,1).[6] For $n>2$, its fundamental group is the cyclic group of order 2.

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[5] Alexandroff-Hopf, *Topologie* I, Berlin (1935), pp. 287–293. This book will be referred to as AH.

[6] Seifert-Threlfall, *Lehrbuch der Topologie*, Leipzig, 1934, p. 210. This book will be referred to as ST.

(M′3) For $n=1, 3, 7$, M' is the topological product of an n-sphere and an $(n-1)$-dimensional projective space.

I. THE MANIFOLD M OF ORIENTED LINEAR ELEMENTS

2. **The representation of M.** The n-sphere, $n\geqq 1$, may be represented in the Euclidean $(n+1)$-space X of the points $(x_0, x_1, \cdots, x_n)$ by the unit sphere: $x'x=x_0^2+x_1^2+\cdots+x_n^2=1$, where x denotes the matrix of one column of the coordinates of the point, and x' the transposed of the matrix x. Similarly we have y, y' and the space Y. The space M is then represented by the subspace of the topological product $R=X\times Y$, defined by the equations

$$M \qquad x'x = 1, \qquad x'y = 0, \qquad y'y = 1.$$

Evidently this subspace is a closed $(2n-1)$-manifold when it is connected (AH, p. 404). Henceforth we shall take this subspace as the space M, and speak of the points of M instead of oriented linear elements.

3. **The two "halves" of M.** The decomposition of the n-sphere $x'x=1$ into the two n-cells corresponding to $x_0\geqq 0$ and $x_0\leqq 0$ gives rise to that of M into the two halves:

$$M_1 \qquad x_0 \geqq 0, \qquad x'x = 1, \qquad x'y = 0, \qquad y'y = 1;$$

$$M_2 \qquad x_0 \leqq 0, \qquad x'x = 1, \qquad x'y = 0, \qquad y'y = 1.$$

Let e denote the matrix of one column of the $n+1$ elements $1, 0, \cdots, 0$. The matrix

$$A_1 = (x+e)(x+e)'/(x_0+1) - I,$$

where $x'x=1$, $x_0+1\neq 0$, and the matrix

$$A_2 = (x-e)(x-e)'/(x_0-1) + I,$$

where $x'x=1$, $x_0-1\neq 0$, are symmetric and orthogonal, and have x' and x as their first rows and columns.[7] Let $u^{(i)}=(u_0^{(i)}, u_1^{(i)}, \cdots, u_n^{(i)})$ and $v^{(i)}=(v_0^{(i)}, v_1^{(i)}, \cdots, v_n^{(i)})$ denote points in the Euclidean $(n+1)$-spaces U_i and V_i respectively, $i=1, 2$. Form the mappings[8]

$$t_i \qquad u^{(i)} = x, \qquad v^{(i)} = A_i y.$$

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[7] H. W. Turnbull and A. C. Aitken, *An introduction to the theory of canonical matrices*, London, 1932, Lemma II, p. 104.

[8] The author wishes to express his gratitude to his colleague Professor P. L. Hsu for the construction of the matrices B_{n+1} and the mapping t_0 in §10 and for the reference in footnote 7. The construction of t_i is then immediate.

t_1 and t_2 are topological mappings of M_1 and M_2 respectively on

$$K_1 \qquad u_0^{(1)} \geqq 0, \qquad u^{(1)\prime}u^{(1)} = 1, \qquad v_0^{(1)} = 0, \qquad v^{(1)\prime}v^{(1)} = 1$$

in $R_1 = U_1 \times V_1$ and

$$K_2 \qquad u_0^{(2)} \leqq 0, \qquad u^{(2)\prime}u^{(2)} = 1, \qquad v_0^{(2)} = 0, \qquad v^{(2)\prime}v^{(2)} = 1$$

in $R_2 = U_2 \times V_2$. Let E_1 be the n-cell: $u_0^{(1)} \geqq 0$, $u^{(1)\prime}u^{(1)} = 1$ in U_1, E_2 the n-cell: $u_0^{(2)} \leqq 0$, $u^{(2)\prime}u^{(2)} = 1$ in U_2, and S_i the $(n-1)$-sphere: $v_0^{(i)} = 0$, $v^{(i)\prime}v^{(i)} = 1$ in V_i. Evidently $K_i = E_i \times S_i$.

4. *M* **as sum. The case** $n=1$. Let us denote the subspace: $x_0 = 0$, $x'x = 1$, $x'y = 0$, $y'y = 1$ in R by $\mathrm{Bd}(M_1) = \mathrm{Bd}(M_2)$, and the subspace: $u_0^{(i)} = 0$, $u^{(i)\prime}u^{(i)} = 1$ in U_i by $\mathrm{Bd}(E_i)$, and finally set $\mathrm{Bd}(K_i) = \mathrm{Bd}(E_i) \times S_i$, the topological product of two $(n-1)$-spheres. Let P be any point of $\mathrm{Bd}(M_i)$. Then $t_i(P) = P_i$ of $\mathrm{Bd}(K_i)$, and $P_2 = t_2 t_1^{-1}(P_1)$ is a topological mapping of $\mathrm{Bd}(K_1)$ on $\mathrm{Bd}(K_2)$. Since the symmetric and orthogonal matrix $A_1 = A_1' = A_1^{-1}$, the topological mapping $t = t_2 t_1^{-1}$ of $\mathrm{Bd}(K_1)$ on $\mathrm{Bd}(K_2)$ is given by $A_2 A_1$, with x in the two factor matrices replaced by $u^{(1)}$. Let $\bar{u}^{(1)}$ denote the matrix of one column of the $n+1$ elements: $0, u_1^{(1)}, \cdots, u_n^{(2)}$. Then evidently

$$t \qquad u^{(2)} = u^{(1)}, \qquad v^{(2)} = \left\{2(\bar{u}^{(1)}\bar{u}^{(1)\prime} + ee') - I\right\}v^{(1)}.$$

Through identification of all pairs of points of $\mathrm{Bd}(K_1)$ and $\mathrm{Bd}(K_2)$, corresponding under t, there results from K_1 and K_2 a *sum*[5] $K_1 + K_2$. Since M is homeomorphic with $K_1 + K_2$, we shall write $M = K_1 + K_2$.

For $n=1$, K_i consists of two semicircles. By means of t, the two end points of each semicircle of K_1 or K_2 are identified with the two end points of one and only one semicircle of K_2 or K_1 respectively. Hence M is the topological product of a 1-sphere and a 0-sphere.

5. **The fundamental group. The case** $n=2$. For $n>2$, all the fundamental groups of the topological products K_i and $\mathrm{Bd}(K_i)$ are the identity (ST, p. 156), and therefore that of M is also the identity (ST, p. 179).

For general n, let ϵ denote the point $(0, \cdots, 0, 1)$ in a Euclidean $(n+1)$-space. Let m_i and b_i denote respectively the $(n-1)$-spheres $\mathrm{Bd}(E_i) \times \epsilon$ and $\epsilon \times S_i$ on $\mathrm{Bd}(K_i)$, and be called the *meridian* and *latitude* of $\mathrm{Bd}(K_i)$.

In the remaining part of this section we confine ourselves exclusively to the case $n=2$. Denote by O_i the point $(\epsilon; \epsilon)$ of the torus $\mathrm{Bd}(K_i)$ and take it as the initial point of closed oriented curves on K_i. Obviously $t(O_1) = O_2$. The meridian and latitude circles m_i and b_i will be regarded as with definite orientations, fixed as follows. First, take the senses of increasing θ and ϕ in the parametric representations

$$\begin{array}{ll} m_1 & u_0^{(1)} = 0,\ u_1^{(1)} = \sin\theta,\ u_2^{(1)} = \cos\theta;\ v_0^{(1)} = 0,\ v_1^{(1)} = 0,\ v_2^{(1)} = 1, \\ b_1 & u_0^{(1)} = 0,\ u_1^{(1)} = 0,\ u_2^{(1)} = 1;\ v_0^{(1)} = 0,\ v_1^{(1)} = -\sin\phi,\ v_2^{(1)} = \cos\phi \end{array}$$

as the orientations of m_i and b_i respectively. Now

$$\begin{array}{llll} t(m_1) & u_0^{(2)} = 0, & u_1^{(2)} = \sin\theta, & u_2^{(2)} = \cos\theta; \\ & v_0^{(2)} = 0, & v_1^{(2)} = \sin 2\theta, & v_2^{(2)} = \cos 2\theta, \\ t(b_1) & u_0^{(2)} = 0, & u_1^{(2)} = 0, & u_2^{(2)} = 1; \\ & v_0^{(2)} = 0, & v_1^{(2)} = \sin\phi, & v_2^{(2)} = \cos\phi. \end{array}$$

Let $(u^{(i)}; v^{(i)})$ denote a variable point of a subspace Γ. We shall call the subspace of $(u^{(i)}; \epsilon)$ and that of $(\epsilon; v^{(i)})$ the *projections* $g_1(\Gamma)$ and $g_2(\Gamma)$. g_i are evidently continuous mappings of Γ. The projections $g_1t(m_1)$ of $t(m_1)$ and $g_2t(b_1)$ of $t(b_1)$ are respectively m_2 and b_2. Moreover g_1t is a topological mapping of m_1 on m_2, and $g_2t = t$ a topological mapping of b_1 on b_2. Now fix the orientations of m_2 and b_2 by demanding that g_1t maps the oriented m_1 on the oriented m_2 and that g_2t maps the oriented b_1 on the oriented b_2.

Let $[m_i]$, $[b_i]$ denote the classes of closed oriented curves through O_i on K_i, homotopically deformable into m_i and b_i when the point O_i is kept fixed. The fundamental group of K_i is given by the two generators $[m_i]$, $[b_i]$ and the defining relation $[m_i] = 1$. From the representation of $t(m_1)$ it is obvious that, when a variable point P_1 starts from O_1 and describes the oriented m_1 once, the point $t(P_1)$ starts from O_2 and describes a closed oriented curve on $\mathrm{Bd}(K_2)$ in such a way that its projection $g_1t(P_1)$ describes the oriented m_2 once and its projection $g_2t(P_1)$ the oriented b_2 twice. Furthermore t maps the oriented b_1 on the oriented b_2. From the usual consideration of the Euclidean plane as the universal covering of $\mathrm{Bd}(K_2)$, we conclude that in the fundamental group of K_2,

$$[t(m_1)] = [m_2]\,[b_2^2], \qquad [t(b_1)] = [b_2].$$

Hence the fundamental group of M is given by the single generator $[b_1]$ and the defining relation $[b_1^2] = 1$ (ST, pp. 177–178), or is the cyclic group of order 2.

In fact, K_1, K_2 and the mapping t of $\mathrm{Bd}(K_1)$ on $\mathrm{Bd}(K_2)$ define the Heegaard diagram of the 3-manifold M. From the enumeration of all the 3-manifolds whose Heegaard diagrams lie on a torus (ST, pp. 210, 220), M is the 3-dimensional projective space. This is a new proof of a well known result.[9]

6. **The projections of $t(m_1)$.** These projections were used in the preceding section for the determination of the fundamental group of M for $n=2$, and will be investigated here and in the next two sections for the determination in §9 of $B^{n-1}(M)$ and $B^n(M)$ for $n \geqq 2$. Henceforth suppose always $n \geqq 2$. Now

$$(tm_1)\quad \begin{aligned} &u_0^{(2)} = 0, \quad u_1^{(2)} = u_1^{(1)}, \cdots, u_n^{(2)} = u_n^{(1)};\\ &v_0^{(2)} = 0, \quad v_1^{(2)} = 2u_1^{(1)}u_n^{(1)}, \cdots, v_{n-1}^{(2)} = 2u_{n-1}^{(1)}u_n^{(1)},\\ &v_n^{(2)} = 2(u_n^{(1)})^2 - 1, \end{aligned}$$

$$t(b_1)\quad \begin{aligned} &u_0^{(2)} = 0, \quad u_1^{(2)} = 0, \cdots, u_{n-1}^{(2)} = 0, \quad u_n^{(2)} = 1;\\ &v_0^{(2)} = 0, \quad v_1^{(2)} = -v_1^{(1)}, \cdots, v_{n-1}^{(2)} = -v_{n-1}^{(1)}, \quad v_n^{(2)} = v_n^{(1)}. \end{aligned}$$

Again obviously g_1t is a topological mapping of m_1 on m_2 and $g_2t=t$ a topological mapping of b_1 on b_2. The most important projection $g_2t(m_1)$ is given by

$$g_2t(m_1)\quad \begin{aligned} &u_0^{(2)} = 0, \quad u_1^{(2)} = 0, \cdots, u_{n-1}^{(2)} = 0, \quad u_n^{(2)} = 1,\\ &v_0^{(2)} = 0, \quad v_1^{(2)} = 2u_1^{(1)}u_n^{(1)}, \cdots, v_{n-1}^{(2)} = 2u_{n-1}^{(1)}u_n^{(1)},\\ &v_n^{(2)} = 2(u_n^{(1)})^2 - 1. \end{aligned}$$

g_2t maps m_1 continuously in b_2. It maps all the points $(0, u_1^{(1)}, \cdots, u_{n-1}^{(1)}, 0) \times \epsilon$ of m_1 on the single point $\epsilon \times (0, \cdots, 0, -1)$ of b_2. When $v_n^{(2)} \neq -1$, that is, when $u_n^{(1)} \neq 0$, the last n equations above can be solved for $\bar{u}^{(1)}$ in terms of $v_1^{(2)}, \cdots, v_n^{(2)}$. Hence g_2t maps m_1 on b_2. Moreover, on the same point $\epsilon \times (0, v_1^{(2)}, \cdots, v_n^{(2)})$ of b_2, g_2t maps the pair of points $(0, \pm u_1^{(1)}, \cdots, \pm u_n^{(1)}) \times \epsilon$ of m_1, but no other point of m_1 when $v_n^{(2)} \neq -1$. In fact, g_2t maps continuously each of the closed halves, m_1' and m_1'', of m_1 corresponding to $u_n^{(1)} \geqq 0$ and $u_n^{(1)} \leqq 0$, on b_2; and it maps topologically each of the open halves of m_1, corresponding to $u_n^{(1)} > 0$ and $u_n^{(1)} < 0$, on b_2 minus the point corresponding to $v_n^{(2)} = -1$.

Take a simplicial decomposition of the meridian m_1, symmetric with respect to the center and to the topological product of the hyperplane $u_n^{(1)} = 0$ and the point ϵ. Take a coherent orientation of this simplicial m_1. Denote respectively again by m_1, m_1', m_1'' the $(n-1)$-cycle and the $(n-1)$-complexes, which are the sums of all the

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[9] See, for example, Seifert's *Solution of problem* 124, mentioned in footnote 4; and C. Weber's *Solution of problem* 84, ibid. pp. 5–6.

$(n-1)$-simplexes on m_1, m_1', m_1'', so oriented. By means of the mapping g_2t, the coherent orientation of m_1' induces an orientation of b_2. Take any simplicial decomposition of b_2 and a coherent orientation determined by this induced orientation. Denote again by b_2 the $(n-1)$-cycle which is the sum of all the $(n-1)$-simplexes on b_2 so oriented. Then, from the last paragraph, on $\mathrm{Bd}(K_2)$ the *homology class* of the singular cycle (ST, p. 97) $g_2t(m_1')$ and that of the cycle b_2 are the same: $(g_2t(m_1'))^* = b_2^*$ on $\mathrm{Bd}(K_2)$.

Now let $Q_0Q_1 \cdots Q_{n-1}$ be an oriented simplex in the coherent orientation of the simplicial m_1. Let Q_i' be the diametrically opposite point of Q_i on m_1. Then either the oriented simplex $Q_0'Q_1' \cdots Q_{n-1}'$ or the oppositely oriented simplex $-Q_0'Q_1' \cdots Q_{n-1}'$ is in the coherent orientation of m_1 according as n is even or odd. Hence, on $\mathrm{Bd}(K_2)$, $(g_2t(m_1''))^* = b_2^*$ for even n, but $= -b_2^*$ for odd n. Hence, on $\mathrm{Bd}(K_2)$, and therefore also on K_2, $(g_2t(m_1))^* = 2b_2^*$ for even n, but $=0$ for odd n.

7. **The homology classes of $t(m_1)$ and $t(b_1)$ on K_2.** To determine the homology class of $t(m_1)$ on K_2, let us regard the $2n+2$ equations of $t(m_1)$ in §6 as parametric equations of $t(m_1)$ with the n parameters $\bar{u}^{(1)}$, which are the coordinates of a variable point of the $(n-1)$-sphere S: $\bar{u}^{(1)\prime}\bar{u}^{(1)} = 1$. Introduce as another parameter the point τ of the interval T: $0 \leqq \tau \leqq 1$. After proper simplicial decomposition and coherent orientation of $D = T \times S$, we have as boundary $\mathrm{Bd}(D)$ of D: $\mathrm{Bd}(D) = S_0' - S_1'$, where S_τ' denotes the topological product of the point τ and S. Now map D continuously in R_2 by the mapping:

$$f_1 \quad \begin{aligned} &u_0^{(2)} = -(1-\tau^2)^{1/2}, \quad u_1^{(2)} = \tau u_1^{(1)}, \cdots, u_n^{(2)} = \tau u_n^{(1)}; \\ &v_0^{(2)} = 0, \quad v_1^{(2)} = 2u_1^{(1)}u_n^{(1)}, \cdots, v_{n-1}^{(2)} = 2u_{n-1}^{(1)}u_n^{(1)}, \\ &\qquad\qquad\qquad\qquad v_n^{(2)} = 2(u_n^{(1)})^2 - 1. \end{aligned}$$

Obviously $f_1(D)$ is on K_2, and $f_1 = t$ on $S_1' = m_1$. Hence

$$t(m_1) \sim f_1(S_0') \quad \text{on} \quad K_2.$$

Again after proper simplicial decomposition and coherent orientation of $H = \Sigma \times S$, where Σ is the interval: $0 \leqq \sigma \geqq 1$, we have $\mathrm{Bd}(H) = S_0'' - S_1''$ where S_σ'' denotes the topological product of the point σ and S. Map H in R_2 by the continuous mapping

$$f_2 \quad \begin{aligned} &u_0^{(2)} = -\cos(\sigma\pi/2), \quad u_1^{(2)} = 0, \cdots, u_{n-1}^{(2)} = 0, \quad u_n^{(2)} = \sin(\sigma\pi/2); \\ &v_0^{(2)} = 0, \quad v_1^{(2)} = 2u_1^{(1)}u_n^{(1)}, \cdots, v_{n-1}^{(2)} = 2u_{n-1}^{(1)}u_n^{(1)}, \quad v_n^{(2)} = 2(u_n^{(1)})^2 - 1. \end{aligned}$$

Obviously $f_2(H)$ is on K_2, $f_2=f_1$ on $S_0''=S_0'$, and $f_2(S_1'')=g_2t(m_1)$. Hence

$$f_1(S_0') = f_2(S_0'') \sim f_2(S_1'') = g_2t(m_1) \quad \text{on} \quad K_2.$$

This homology together with the preceding one gives

$$t(m_1) \sim g_2t(m_1) \quad \text{on} \quad K_2.$$

From the result at the end of §6, we have finally that, on K_2, $(t(m_1))^* = 2b_2^*$ for even n, but $=0$ for odd n.

Obviously $(t(b_1))^*=b_2^*$ on K_2.

On the sum K_1+K_2, formed by means of the mapping t, $m_1^*=2b_2^*$ for even n, but $=0$ for odd n, and $b_1^*=b_2^*$.

8. The homology classes of $t(m_1)$ and $t(b_1)$ on Bd(K_2). From Künneth's theorem on the Betti groups of topological product of complexes (AH, p. 308), the Betti groups of K_i and Bd(K_i) together with their bases can be easily determined. All K_i and Bd(K_i) have no torsion coefficients. Their only Betti numbers, which are not zero, are $p^0(K_i) = p^{n-1}(K_i) = 1$, and $p^0(\mathrm{Bd}(K_i)) = p^{2n-2}(\mathrm{Bd}(K_i)) = 1$, $p^{n-1}(\mathrm{Bd}(K_i))=2$.

Let m_i^* and b_i^* denote respectively the homology classes of the $(n-1)$-cycles on K_i or Bd(K_i), which are the sums of all the coherently oriented $(n-1)$-simplexes on certain simplicial decompositions of m_i and b_i.[10] Then the $(n-1)$-dimensional homology basis of K_i consists of b_i^* only, while that of Bd(K_i), of m_i^* and b_i^*.

The $(n-1)$-dimensional Betti group $B^{n-1}(\mathrm{Bd}(K_i))$ is the free Abelian group with the two free generators m_i^* and b_i^*. The topological mapping t of Bd(K_1) on Bd(K_2) induces an isomorphism of $B^{n-1}(\mathrm{Bd}(K_1))$ on $B^{n-1}(\mathrm{Bd}(K_2))$ (ST, p. 98). This isomorphism must be given by

$$(t(m_1))^* = \alpha m_2^* + \beta b_2^*, \qquad (t(b_1))^* = \gamma m_2^* + \delta b_2^*,$$

on Bd(K_2), where the coefficients are integers and $\alpha\delta-\beta\gamma=\pm 1$.[11] From the equations of $t(b_1)$ in §6, $\gamma=0$; and therefore $\delta=\pm 1$. Hence $\alpha=\pm 1$. Take the simplicial decompositions and coherent orientations of m_1 and b_2 as given in §6. We can then, and shall, assign coherent orientations to m_2 and b_1 such that $\delta=1$ and $\alpha=1$. Hence $(t(m_1))^* = m_2^*+\beta b_2^*$, and $(t(b_1))^*=b_2^*$ on Bd(K_2). The first equation implies

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[10] The two interpretations of the symbols m_i^* and b_i^* (namely, on K_i or on Bd(K_i)) will give rise to no confusion, as we shall always state explicitly the complex on which the homology classes are considered.

[11] K. Reidemeister, *Einfuhrung in die kombinatorische Topologie*, Braunschweig, 1932, p. 95.

$(t(m_1))^* = \beta b_2^*$ on K_2, since $m_2^* = 0$ on K_2. From the result at the end of §7, $\beta = 2$ for even n, but $= 0$ for odd n. Hence, on $\mathrm{Bd}(K_2)$, $(t(m_1))^* = m_2^* + 2b_2^*$ for even n, but $= m_2^*$ for odd n.

Let $K_1 \cdot K_2$ denote either of the isomorphic sub-complexes of K_1 and K_2, which are to be identified in forming the sum $K_1 + K_2$: $K_1 \cdot K_2 = \mathrm{Bd}(K_1) = \mathrm{Bd}(K_2)$. On $K_1 \cdot K_2$, $m_1^* = m_2^* + 2b_2^*$ for even n, but $= m_2^*$ for odd n, and $b_1^* = b_2^*$.

9. **The Betti groups of** M. It is obvious that $B^0(M) \approx G_0$.

Let $N^r(K_1 \cdot K_2)$ be the subgroup of $B^r(K_1 \cdot K_2)$, whose elements are the homology classes of those r-cycles, which are null-homologous on both K_1 and K_2; $S^r(K_1 + K_2)$ the sub-group of $B^r(K_1 + K_2)$, whose elements are the homology classes of those r-cycles, each of which is the sum of one r-cycle on K_1 and one r-cycle on K_2. We shall determine as follows $B^r(M)$ by means of the theorem:

$$B^r(M) - S^r(K_1 + K_2) \approx N^{r-1}(K_1 \cdot K_2)$$

for every $r \geqq 1$ (AH, p. 293).

Let n be even. We say that $S^{n-1}(K_1 + K_2) \approx G_2$, $N^{2n-2}(K_1 \cdot K_2) \approx G_0$, and $S^r(K_1 + K_2)$, $N^{r-1}(K_1 \cdot K_2)$ are the null for all other values of $r \geqq 1$. Let us prove first for example $S^{n-1}(K_1 + K_2) \approx G_2$. Any $(n-1)$-cycle on K_i is homologous on K_i to an integral multiple of b_i. Hence any $(n-1)$-cycle on $K_1 + K_2$, which is a sum of one $(n-1)$-cycle on K_1 and one $(n-1)$-cycle on K_2, is homologous on $K_1 + K_2$ to a linear combination of b_1 and b_2 with integral coefficients. From the results in §7, $2b_2^* = 0$ and $b_1^* = b_2^*$ on $K_1 + K_2$. Moreover, $b_1^* \neq 0$ on $K_1 + K_2$. For, if otherwise, let C be a complex on $K_1 + K_2$, such that $\mathrm{Bd}(C) = b_1$. Let $C = C_1 + C_2$, where C_i is on K_i. Then $b_1 - \mathrm{Bd}(C_1) = \mathrm{Bd}(C_2)$. This shows that the right member would be on K_2, while the left member would be on K_1. Hence $\mathrm{Bd}(C_2)$ and $b_1 - \mathrm{Bd}(C_1)$, and therefore $\mathrm{Bd}(C_1)$, would be all on $K_1 \cdot K_2$. Being on $K_1 \cdot K_2$ and null-homologous on K_i, $\mathrm{Bd}(C_i)$ would be homologous on $K_1 \cdot K_2$ to an integral multiple of m_i, say $\lambda_i m_i$. Hence $b_1^* = \lambda_1 m_1^* + \lambda_2 m_2^*$ on $K_1 \cdot K_2$, which is impossible from §8. Hence $b_1^* \neq 0$ on $K_1 + K_2$, and $S^{n-1}(K_1 + K_2) \approx G_2$. Next $N^{n-1}(K_1 \cdot K_2) \approx G_1$ follows from §8. The other isomorphisms stated are obvious, and we shall omit the proof.

Hence for every $r \geqq 1$, at least one of the two groups $S^r(K_1 + K_2)$ and $N^{r-1}(K_1 \cdot K_2)$ is the null, and, from the theorem stated above, $B^r(M)$ is isomorphic with the other group. Consequently $B^{n-1}(M) \approx G_2$, $B^{2n-1}(M) \approx G_0$, and $B^r(M) \approx G_1$ for all other values of $r \geqq 1$.

Similarly, when n is odd, $S^{n-1}(K_1 + K_2)$, $N^{n-1}(K_1 \cdot K_2)$, $N^{2n-2}(K_1 \cdot K_2)$ all $\approx G_0$, and $S^r(K_1 + K_2)$, $N^{r-1}(K_1 \cdot K_2) \approx G_1$ for all other values of $r \geqq 1$. Consequently $B^{n-1}(M)$, $B^n(M)$ and $B^{2n-1}(M)$ all $\approx G_0$, and $B^r(M) \approx G_1$ for all other values of $r \geqq 1$.

Thus the proof of the statements in (M1) and (M2) in §1 is completed.

10. The cases $n=1, 3, 7$.[8] Let

$$B_8 = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ x_1 & -x_0 & x_3 & -x_2 & x_5 & -x_4 & -x_7 & x_6 \\ x_2 & -x_3 & -x_0 & x_1 & x_6 & x_7 & -x_4 & -x_5 \\ x_3 & x_2 & -x_1 & -x_0 & x_7 & -x_6 & x_5 & -x_4 \\ x_4 & -x_5 & -x_6 & -x_7 & -x_0 & x_1 & x_2 & x_3 \\ x_5 & x_4 & -x_7 & x_6 & -x_1 & -x_0 & -x_3 & x_2 \\ x_6 & x_7 & x_4 & -x_5 & -x_2 & x_3 & -x_0 & -x_1 \\ x_7 & -x_6 & x_5 & x_4 & -x_3 & -x_2 & x_1 & -x_0 \end{pmatrix}.$$

Let B_4 be the four-rowed square matrix at the upper left corner of B_8, and B_2 the two-rowed square matrix at the upper left of B_4. For $n=1, 3, 7$, B_{n+1} has x' and x as the first row and column, and is orthogonal. The mapping

$$t_0 \qquad\qquad u = x, \qquad v = B_{n+1}y$$

is a topological mapping of the *whole* M, for $n=1, 3, 7$, on

$$u'u = 1, \qquad v_0 = 0, \qquad v'v = 1,$$

the topological product of an n-sphere and an $(n-1)$-sphere. Hence we have our statement (M3) in §1.

II. THE MANIFOLD M' OF NON-ORIENTED LINEAR ELEMENTS

11. M' as sum. The cases $n=1, 3, 7$. By the symbol $(x; y) \rightleftarrows (x; -y)$ we mean that the two points $(x; y)$ and $(x; -y)$ are to be identified. The space M' of non-oriented linear elements of an n-sphere is then represented by the equations of M and the additional condition $(x; y) \rightleftarrows (x; -y)$. In this sense we shall say for simplicity that M' is M *after diametrical identification* $(x; y) \rightleftarrows (x; -y)$. From the symmetry of x and y in the equations of M, M' is homeomorphic with the space given by the equations of M and the additional condition $(x; y) \rightleftarrows (-x; y)$, and therefore is also the space of oriented linear elements of an n-dimensional projective space. M' is obviously a closed $(2n-1)$-manifold if it is connected.

Let M' be decomposed also into two halves M_i', $i=1, 2$, namely the M_i after diametrical identification $(x;\ y)\rightleftarrows(x;\ -y)$. From the nature of the equations of the topological mapping t_i on M_i, t_i carries $(x;\ y)$ and $(x^*;\ y^*)$ respectively into $(u^{(i)};\ v^{(i)})$ and $(u^{(i)};\ -v^{(i)})$ when and only when $(x^*;\ y^*)=(x;\ -y)$. Hence t_i is a topological mapping of M_i' on K_i', namely the K_i after diametrical identification $(u^{(i)};\ v^{(i)})\rightleftarrows(u^{(i)};\ -v^{(i)})$. Let π_i be the S_i in $\dot{V}_i$ (§3) after diametrical identification $v^{(i)}\rightleftarrows -v^{(i)}$, and $\text{Bd}(M_i')$, $\text{Bd}(K_i')$ the $\text{Bd}(M_i)$, $\text{Bd}(K_i)$ in §4 after diametrical identifications. Then we have here $K_i'=E_i\times\pi_i$, $\text{Bd}(M_1')=\text{Bd}(M_2')$ and $\text{Bd}(K_i')=\text{Bd}(E_i)\times\pi_i$. The mapping t is a topological mapping of $\text{Bd}(K_1')$ on $\text{Bd}(K_2')$. Thus, in the sense of homeomorphism, $M'=K_1'+K_2'$, the sum being defined by means of t.

In the particular cases $n=1,\ 3,\ 7$, from the mapping t_0 in §10, topological on the whole M', we conclude that M' is the topological product of an $(n-1)$-sphere and an $(n-1)$-dimensional projective space.

12. **The fundamental group. The case** $n=2$. Let us denote by π_i^r the r-dimensional projective space: $v_0^{(i)}=0$, $v^{(i)\prime}v^{(i)}=1$, $v_{r+2}^{(i)}=\cdots=v_n^{(i)}=0$, $v^{(i)}\rightleftarrows -v^{(i)}$. In particular, π_i^{n-1} is our π_i previously defined. Let $\sigma_i^r=\epsilon\times\pi_i^r$. For $n>2$, the fundamental group of K_i' or $\text{Bd}(K_i')$ is the cyclic group of order 2 with the class of the oriented σ_i^1 as the generator (ST, p. 156). From the equations of $t(b_1)$ in §6, we know that t maps the oriented σ_1^1 into properly oriented σ_2^1. Therefore the fundamental group of M' is the cyclic group of order 2 (ST, pp. 177–178).

When $n=2$, $\text{Bd}(K_i')$ is the torus. From §5 together with a necessary change of notations, we know that the fundamental group of K_i' is given by two generators $[m_i]$ and $[\sigma_i]$ and the defining relation $[m_i]=1$, and that

$$([t(m_1)]) = [m_2]([\sigma_2^1])^4, \qquad ([t(\sigma_1^1)]) = [\sigma_2^1].$$

Hence the fundamental group of M' is the cyclic group of order 4 (ST, pp. 177–178), in agreement with Threlfall's result,[8] and M' is the Linsenraum (4, 1) (ST, pp. 219–220, 210, 215).

13. **The Betti groups of** K_i' **and** $\text{Bd}(K_i')$. Henceforth we always assume $n\geqq 2$. For simplicity, let our σ_i^{n-1} previously defined be denoted by σ_i.

The Betti groups of the factors of K_i' and $\text{Bd}(K_i')$ together with their bases are well known. The Betti groups of K_i' and $\text{Bd}(K_i')$ together with their bases can be easily determined from Künneth's theorem. The Betti groups B^0 of K_i' and $\text{Bd}(K_i')$ are all $\approx G_0$, and the others are listed as follows.

When n is even, $B^r(K_i') \approx G_2, r = 1, 3, \cdots, n-3$, with the basis $(\sigma_i^r)^*$, $B^{n-1}(K_i') \approx G_0$ with the basis σ_i^*, $B^r(\mathrm{Bd}(K_i'))$ and $B^{n-1+r}(\mathrm{Bd}(K_i'))$ $\approx G_2$ with the respective bases $(\sigma_i^r)^*$ and $(\mathrm{Bd}(E_i) \times \pi_i^r)^*$, $B^{n-1}(\mathrm{Bd}(K_i'))$ $\approx G_0 + G_0$ with the basis m_i^* and σ_i^*, $B^{2n-2}(\mathrm{Bd}(K_i')) \approx G_0$ with the basis $(\mathrm{Bd}(E_i) \times \pi_i)^*$, and all the rest are the null.

When n is odd, $B^r(K_i') \approx G_2$, $r = 1, 3, \cdots, n-2$, with the basis $(\sigma_i^r)^*$, $B^r(\mathrm{Bd}(K_i'))$ and $B^{n-1+r}(\mathrm{Bd}(K_i')) \approx G_2$ with the respective bases $(\sigma_i^r)^*$ and $(\mathrm{Bd}(E_i) \times \pi_i^r)^*$, $B^{n-1}(\mathrm{Bd}(K_i')) \approx G_0$ with the basis m_i^*, and all the rest are the null.

14. The homology classes of $t(m_1)$, $t(\sigma_1)$ and $t(\mathrm{Bd}(E_1) \times \pi_1^r)$ on $\mathrm{Bd}(K_2')$. We shall assume as in §6 that, for even n, σ_i has a simplicial decomposition and a proper orientation. For odd n, the projective space σ_i is non-orientable and there is no longer any non-null (integral) $(n-1)$-cycle on it. We assume only that σ_i in this case has a simplicial decomposition and that all its $(n-1)$-simplexes are oriented so that t carries the integral complex σ_1 into the integral complex σ_2. Regarding σ_i as b_i after diametrical identification, we can apply to M' the discussions in §§6–8 with only slight modification in arguments and with necessary change of notations. From these sections we have evidently the following results: On K_i', $(t(m_1))^* = 4\sigma_2^*$ for even n, but $= 0$ for odd n, and $(t(\sigma_1))^* = \sigma_2^*$ for even n. Hence on the sum $M' = K_1' + K_2'$, $m_1^* = 4\sigma_2^*$ for even n, but $= 0$ for odd n, and $\sigma_1^* = \sigma_2^*$ for even n. On $\mathrm{Bd}(K_2')$, $(t(m_1))^* = m_2^* + 4\sigma_2^*$ for even n, but $= m_2^*$ for odd n, and $(t(\sigma_1))^* = \sigma_2^*$ for even n. Hence on $K_1' \cdot K_2'$, $m_1^* = m_2^* + 4\sigma_2^*$ for even n, but $= m_2^*$ for odd n, and $\sigma_1^* = \sigma_2^*$ for even n.

Suppose that σ_i^r and $\mathrm{Bd}(E_i) \times \pi_i^r$, $r = 1, 3, \cdots, n-3, n-1$ for even n and $r = 1, 3, \cdots, n-2$ for odd n, has simplicial decompositions and proper coherent orientations. From the nature of the equations of $t(b_1)$ in §6,

$$(t(\overset{r}{\sigma}_1))^* = \pm (\overset{r}{\sigma}_2)^* ;$$

and since t is a topological mapping of $\mathrm{Bd}(K_1')$ on $\mathrm{Bd}(K_2')$, it induces an isomorphism of $B^{n-1+r}(\mathrm{Bd}(K_1'))$ on $B^{n-1+r}(\mathrm{Bd}(K_2'))$:

$$(t(\mathrm{Bd}\ (E_1) \times \overset{r}{\pi}_1))^* = \pm (\mathrm{Bd}\ (E_2) \times \overset{r}{\pi}_2)^* .$$

We can, and shall, assume that the coherent orientations of σ_i^r and $\mathrm{Bd}(E_i) \times \pi_i^r$ have been so chosen that the signs in the above equations are all positive. Hence on $K_1' \cdot K_2'$, $(\sigma_1^r)^* = (\sigma_2^r)^*$, $(\mathrm{Bd}(E_1) \times \pi_1^r)^*$ $= (\mathrm{Bd}(E_2) \times \pi_2^r)^*$.

15. The Betti groups of M'. Obviously $B^0(M')$ is the infinite cyclic group, and M' is connected.

Let n be even. We say that $S^r(K_1'+K_2')\approx G_2$, $r=1, 3, \cdots, n-3$, $S^{n-1}(K_1'+K_2')\approx G_4$, $N^{r-1}(K_1'\cdot K_2')\approx G_2$, $r=n+1, n+3, \cdots, 2n-3$, $N^{2n-2}(K_1'\cdot K_2')\approx G_0$, and $S^r(K_1'+K_2')$, $N^{r-1}(K_1'\cdot K_2')$ are the null for all other values of $r\geqq 1$. The proof of these isomorphisms is based on the results in §§13–14 and an argument similar to that used in §9. Then, as in §9, we have $B^r(M')\approx G_2$, $r=1, 3, \cdots, n-3; n+1, n+3, \cdots, 2n-3$, $B^{n-1}(M')\approx G_4$, $B^{2n-1}(M')\approx G_0$, and $B^r(M')$ are the null for all other values of $r\geqq 1$. M' is therefore orientable, when n is even.

Similarly, when n is odd, $S^r(K_1'+K_2')\approx G_2$, $r=1, 3, \cdots, n-2$, $N^{r-1}(K_1'\cdot K_2')\approx G_2$, $r=n+1, n+3, \cdots, 2n-2$, $N^{n-1}(K_1'\cdot K_2')\approx G_0$, and $S^r(K_1'+K_2')$, $N^{r-1}(K_1'\cdot K_2')$ are the null for all other values of $r\geqq 1$. Hence $B^r(M')\approx G_2, r=1,3, \cdots, n-2; n+1, n+3, \cdots, 2n-2$, and $B^n(M')\approx G_0$, and $B^r(M')$ are the null for all other values of $r\geqq 1$. M' is therefore non-orientable when n is odd.

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ON TOPOLOGIES FOR FUNCTION SPACES

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Given topological spaces X, T, and Y and a function h from $X\times T$ to Y which is continuous in x for each fixed t, there is associated with h a function h^* from T to $F=Y^X$, the space whose elements are the continuous functions from X to Y. The function h^* is defined as follows: $h^*(t)=h_t$, where $h_t(x)=h(x, t)$ for every x in X. The correspondence between h and h^* is obviously one-to-one.

Although the continuity of any particular h depends only on the given topological spaces X, T, and Y, the topology of the function space F is involved in the continuity of h^*. It would be desirable to so topologize F that the functions h^* which are continuous are precisely those which correspond to continuous functions h. It has been known for a long time that this is possible if X satisfies certain conditions, chief among which is the condition of *local compactness* (Theorem 1). This condition is often felt to be too restrictive (since it practically excludes the possibility of X itself being a function space), and several years ago, in a letter, Hurewicz proposed to me the problem of defining such a topology for F when X is not locally compact. At that time I showed by an example (essentially Theorem 3) that this is not generally possible. Recently I discovered that, by restricting the range of T in a very reasonable way, one of the standard topologies for F has the desired property even for spaces X which are not locally compact (Theorem 2). In this last result the condition of local compactness is replaced by the first countability axiom and this appeals to me as a less troublesome condition.

It should be pointed out that the problem is motivated by the special case in which T is the unit interval. When T is the unit interval, h is a homotopy and h^* is a path in the function space; in the topology of deformations, equivalence of the concepts of "homotopy" and of "function-space path" is usually required.

Among the various possible topologies for F there is one, which I shall call the compact-open[1] (co.o.) topology, which seems to be the most natural. For any two sets, A in X and W in Y, let $M(A, W)$ denote the set of mappings $f\in F$ for which $f(A)\subset W$. The co.o. topology is defined by selecting as a sub-basis for the open sets of F the

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[1] Terminology followed in this note is generally that of Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, New York, 1942.

sets $M(A, W)$ where A ranges over the compact subsets of X and W ranges over the open subsets of Y.

THEOREM 1. *If X is regular and locally compact, Y an arbitrary topological space, and if F has the co.o. topology, then continuity of h is equivalent to continuity of h^* for any topological space T.*

THEOREM 2. *If X is a space which satisfies the first countability axiom, Y an arbitrary topological space, and if F has the co.o. topology, then continuity of h is equivalent to continuity of h^* for any T which satisfies the first countability axiom.*

THEOREM 3. *If X is separable metrizable and Y is the real line, then in order that it be possible to so topologize F that continuity of h and of h^* are equivalent, it is necessary and sufficient that X be locally compact.*

LEMMA 1. *If F has the co.o. topology, then continuity of h implies continuity of h^* under no restrictions on the topological spaces X, T, and Y.*

PROOF. Let W be an open set in Y and A a compact set in X and let t_0 be a point in $h^{*-1}(M(A, W))$. Then $A\times t_0 \subset h^{-1}(W)$. Since $h^{-1}(W)$ is open it is the union of open sets $U_\alpha \times V_\alpha$. Since A is compact, $A\times t_0$ is contained in a finite union $\bigcup_{i=1}^n U_i\times V_i$ with each V_i a neighborhood of t_0. Then $\bigcap_{i=1}^n V_i$ is an open neighborhood of t_0 and is contained in $h^{*-1}(M(A, W))$.

PROOF OF THEOREM 1. In view of the lemma it is sufficient to prove that continuity of h^* implies continuity of h. Let W be an open set in Y and let (x_0, t_0) be a point in $h^{-1}(W)$. Since $h^*(t_0)$ is continuous in x there exists an open neighborhood U of x_0 such that $h^*(t_0)\in M(U, W)$. Because of the conditions on X there is an open neighborhood R of x_0 such that $\bar{R}$ is compact and contained in U. Since $M(\bar{R}, W)$ is open and contains $h^*(t_0)$ there is an open neighborhood V of t_0 such that $h^*(V)\subset M(\bar{R}, W)\subset M(R, W)$. Thus $R\times V$ is an open neighborhood of (x_0, t_0) which is contained in $h^{-1}(W)$.

PROOF OF THEOREM 2. As before we have to prove that continuity of h^* implies continuity of h. Let W be an open set in Y and suppose that $h^{-1}(W)$ is not open. Then there is a point (x_0, t_0) in $h^{-1}(W)$ which is also in the closure of the complement of $h^{-1}(W)$. Let $\{G_n\}$ be a base for the open sets of $X\times T$ at the point (x_0, t_0) and choose, for each integer n, a point (x_n, t_n) in the intersection of $\bigcap_{i\leq n} G_i$ and the complement of $h^{-1}(W)$. Since $h^*(t_0)$ is continuous in x there exists an open neighborhood U of x_0 such that $h^*(t_0)\in M(U, W)$. Let

$A = U \cap \bigcup_{n=0}^{\infty} x_n$. Since A is compact, $M(A, W)$ is open and since h^* is continuous and $t_0 \in h^{*-1}(M(A, W))$, there is a neighborhood V of t_0 such that $h^*(V) \subset M(A, W)$. There is an integer N such that $x_n \in U$ and $t_n \in V$ whenever $n > N$. Hence $h(x_n, t_n) \in W$ for every n greater than N. This contradiction with the choice of the points (x_n, t_n) proves that $h^{-1}(W)$ is open. Thus h is continuous.

LEMMA 2. *Let X be a separable metrizable space, let Y be the real line, and suppose that the topology of F is such that continuity of h for $T = [0, 1]$ implies the continuity of h^*. Let $W = (a, b)$ be a finite open interval in Y and let A be a closed subset of X which is not compact. Then the set $M(A, W)$ has no interior points.*

PROOF. Since A is not compact there is a sequence $\{x_n\}$ in A such that $\bigcup_{n=1}^{\infty} x_n$ is closed in X. Given any element $h^*(0)$ of the set $M(A, W)$ let us define

$$h_t(x_n) = \min \{1 + b, h_0(x_n) + nt\}.$$

Since the function h is defined over the closed set $X \times [0] \cup (\bigcup_{n=1}^{\infty} x_n) \times [0, 1]$ it may be extended continuously over the normal space $X \times [0, 1]$. If $t > 0$ there is an integer n such that $a + nt > 1 + b$; hence $h^*(t)$ is in the complement of $M(A, W)$ for every positive t. By hypothesis the topology of F is such that h^* is continuous. Hence $h^*(0)$ belongs to the closure of the complement of $M(A, W)$.

LEMMA 3. *If the topology for F is such that continuity of h^* always implies continuity of h then, given a point x_0 in X, an open set W in Y, and an element f_0 in $M(x_0, W)$, there is a neighborhood R of x_0 such that $M(R, W)$ is a neighborhood of f_0 in F.*

PROOF. Define $\phi(x, f) = f(x)$ for every $(x, f) \in X \times F$. Since $\phi^*(f) = f$, ϕ^* is continuous and hence ϕ is also continuous. Since $\phi^{-1}(W)$ is therefore open there must be a neighborhood R of x_0 and a neighborhood V of f_0 such that $\phi(R, V) \subset W$. Thus $f_0 \in V \subset M(R, W)$ and hence $M(R, W)$ is a neighborhood of f_0 in F.

PROOF OF THEOREM 3. Let W be the finite open interval (a, b) and suppose that the topology of F is such that continuity of h and of h^* are equivalent for every T. From Lemma 3 it follows that, given any point x_0 in X and any element f_0 in $M(x_0, W)$, there is a neighborhood R of x_0 such that $M(R, W)$ is a neighborhood of f_0 in F. Since X is regular there is a neighborhood U of x_0 whose closure is contained in R, so that $M(\overline{U}, W)$ is also a neighborhood of f_0. Since f_0 is an interior point of $M(\overline{U}, W)$ it follows from Lemma 2 that $\overline{U}$ is compact. Thus X must be locally compact. This proves the necessity of the condition; sufficiency is a consequence of Theorem 1.

COROLLARY. *If Y is the real line and X is separable metrizable but not locally compact, then F does not satisfy the first countability axiom in the co.o. topology.*

PROOF. Let W be the finite open interval (a, b). If F satisfied the first countability axiom then Theorem 2 would apply to yield the continuity of the function ϕ defined above. If x_0 is a point at which X is not locally compact and f_0 any element in $M(x_0, W)$, then it follows from the proof of Lemma 3 that there is a neighborhood R of x_0 such that $M(R, W)$ is a neighborhood of f_0 in F. Let U be a neighborhood of x_0 whose closure is contained in R, so that f_0 is an interior point of $M(\overline{U}, W)$. Since $\overline{U}$ is not compact this is not in agreement with Lemmas 1 and 2. This contradiction shows that F does not satisfy the first countability axiom in the co.o. topology.

SYRACUSE UNIVERSITY

ON CERTAIN VARIATIONS OF THE HARMONIC SERIES

PAUL ERDÖS AND IVAN NIVEN

Consider any block of terms from the harmonic series

$$\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{n+k-1}=S(n,k) \qquad (n \geqq 1,\ k \geqq 1).$$

Define the integer k_2 by the relation[1]

$$S(n+k, k_2) < S(n,k) < S(n+k, k_2+1), \tag{1}$$

and similarly $k_3, k_4, \cdots$ by

$$S(n+k+k_2, k_3) < S(n+k, k_2) < S(n+k+k_2, k_3+1), \tag{2}$$

$$\begin{aligned} S(n+k+k_2+k_3, k_4) &< S(n+k+k_2, k_3) \\ &< S(n+k+k_2+k_3, k_4+1), \end{aligned} \tag{3}$$

and so on. We shall study the series

$$\begin{aligned} S(n,k) &- S(n+k, k_2) + S(n+k+k_2, k_3) \\ &- S(n+k+k_2+k_3, k_4) + \cdots . \end{aligned} \tag{4}$$

THEOREM. *The series* (4) *is convergent if and only if* $k=k_2$.

LEMMA 1. $\log(1+k/n) < S(n,k) < \log(1+2k/(2n-1))$.

The inequality on the left arises from the usual comparison of the harmonic series with the integral of the function $1/x$. To prove the other inequality, we note that the convexity of the function $1/x$ implies

$$\int_{n-1/2}^{n+1/2}\frac{dx}{x} > \frac{1}{n} \quad \text{or} \quad \log\frac{2n+1}{2n-1} > \frac{1}{n}\cdot$$

We replace n by $n+1, n+2, \cdots, n+k-1$ in the latter inequality and add the results.

LEMMA 2. *If* $k=k_2$, *then* $k=k_j$ *with* $j>2$, *and the series* (4) *converges.*

We need prove only that $k=k_3$. Since $k>k_3$ is not possible, let us

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[1] It is not possible that $S(n,k)=S(n+k,k_2)$. For let h be the unique integer in the range $(n, n+k+k_2-1)$ which is divisible by the highest power of 2, say 2^r. Let m be the l.c.m. of all the odd divisors of $n, n+1, \cdots, n+k+k_2-1$. Then $2^{r-1}mS(n,k) = 2^{r-1}mS(n+k,k_2)$ is an equation involving $k+k_2-1$ integers and the one fraction $m2^{r-1}/h$.

assume that $k<k_3$, say $k_3=k+r$. Then by (1) and (2)

$$S(n, k) < S(n+k, k+1) \quad \text{and} \quad S(n+k, k) > S(n+2k, k+r).$$

By Lemma 1 these imply

$$k/n < (2k+2)/(2n+2k-1)$$

and

$$2k/(2n+2k-1) > (k+r)/(n+2k),$$

which reduce to the incompatible inequalities

$$2k^2 < 2n+k \quad \text{and} \quad 2k^2 > k(2r-1)+2nr-r.$$

LEMMA 3. *If $k_2>k$, then $k_{j+1}>k_j$, and moreover $k_{j+1}-k_j \geqq k_2-k$.*

We prove that $k_3-k_2 \geqq k_2-k$. Suppose that $k_2=k+r$, but that $k_3<k+2r$. Then by (1) and (2)

$$S(n, k) > S(n+k, k+r)$$

and

$$S(n+k, k+r) < S(n+2k+r, k+2r).$$

Again we use Lemma 1 as in the proof of Lemma 2 to get

$$2k^2+k+r > 2nr \tag{5}$$

and

$$2nr+k+r > 2k^2+2kr+2r^2.$$

These add to give the impossible result

$$2k+2r > 2kr+2r^2.$$

LEMMA 4. *If $k_2>k$, say $k_2=k+r$, then there exists an s such that $k_s>k_{s-1}+r$.*

Suppose that $k_s=k_{s-1}+r$ for all s, so that $k_s=k+(s-1)r$. Then the inequality

$$S(n+k+k_2+\cdots+k_s, k_{s+1}) < S(n+k+k_2+\cdots+k_{s+1}, k_{s+2}+1)$$

can be written as

$$S\left(n+sk+\frac{s^2-s}{2}\cdot r, k+sr\right)$$
$$< S\left(n+sk+k+\frac{s^2+s}{2}\cdot r, k+sr+r+1\right).$$

We apply Lemma 1 as before to reduce this to

$$2nr + 2n + 2sk + s^2r + k > 2ksr + s^2r^2 + sr^2 + 2k^2.$$

This is impossible for $r>1$, since s^2r^2 is the dominating term for large s. If $r=1$ the inequality reduces to

$$4n + k > 2k^2 + s.$$

We are now in a position to prove that (4) diverges if $k<k_2$. Using Lemma 4 we may assume that $r=k_2-k$ is as large as we please. By definition we have

$$S(n + k + k_2 + \cdots + k_j,\ k_{j+1}) - S(n + k + k_2 + \cdots + k_{j+1},\ k_{j+2}) < \frac{1}{n + k + k_2 + \cdots + k_{j+2}}.$$

Summing these for $j=1, 2, \cdots, s-1$ (where $k_1=k$), and adding

$$S(n, k) - S(n + k, k_2) < 1/(n + k + k_2),$$

we obtain

$$\begin{aligned} S(n, k) - S(n + k + k_2 + \cdots + k_s,\ k_{s+1}) & \\ < \sum_{j=2}^{s+1} (n + k + k_2 + \cdots + k_j)^{-1} & \\ < \sum_{j=2}^{s+1} \{n + C_{j,2}r\}^{-1} < \sum_{x=1}^{s}\left(n + \frac{x^2r}{2}\right)^{-1} & \\ < \int_0^\infty \left(n + \frac{x^2r}{2}\right)^{-1} dx = \frac{\pi}{(2rn)^{1/2}} < \frac{1}{2n^{1/2}}. & \end{aligned}$$

Thus we have

$$S(n + k + k_2 + \cdots + k_s,\ k_{s+1}) > S(n, k) - 1/2n^{1/2},$$

and we shall complete the proof by showing that the right side of this inequality, which is independent of s, is positive.

With r sufficiently large, inequality (5) implies that $k>n^{1/2}$. And since $S(n, k)$ exceeds k times its smallest term, we have

$$S(n, k) > k/(k + n) > 1/2n^{1/2}.$$

In conclusion we shall prove that if $S(n, k)$ were defined as $\sum_{j=0}^{k-1}(n+j)^{-\alpha}$, then the series (4) would converge for $0<\alpha<1/2$. (We are unable at present to state a theorem for the cases $1/2\leqq\alpha<1$.) In (1), (2), (3), and so on, the first inequality sign should be altered to include the possibility of equality. We shall prove that for n sufficiently large

$$(6)\qquad \begin{gathered} \sum_{j=0}^{k-1} (n+j)^{-\alpha} < \sum_{j=k}^{2k} (n+j)^{-\alpha} \quad \text{or} \\ \sum_{j=0}^{k-1} \left\{ (n+j)^{-\alpha} - (n+k+j)^{-\alpha} \right\} < (n+2k)^{-\alpha}, \end{gathered}$$

that is, that the number of terms in each block of the series is eventually a constant, which implies convergence. Since the first term of the sum on the left of the latter inequality (6) is the largest (as can be shown by elementary calculus), this inequality is implied by

$$n^{-\alpha} - (n+k)^{-\alpha} < (n+2k)^{-\alpha}/k.$$

By examination of the graph of $y=x^{-\alpha}$ we see that the term on the left of the latter inequality is less than k times the negative of the derivative at $x=n$, and hence we need merely prove that

$$k\alpha n^{-\alpha-1} < \frac{1}{k}(n+2k)^{-\alpha} \quad \text{or} \quad \alpha < \frac{n}{k^2}\left(\frac{n}{n+2k}\right)^{\alpha}.$$

This in turn is implied by $k^2 < n$, as is shown by

$$\alpha < \frac{1}{2} < \frac{n}{n+2k} < \left(\frac{n}{n+2k}\right)^{\alpha} < \frac{n}{k^2}\left(\frac{n}{n+2k}\right)^{\alpha}.$$

We prove that $k^2 < n$ for n sufficiently large by using the fact that every term of series (4) is less than the first one, say c. Thus we have

$$k(n+k-1)^{-\alpha} < \sum_{j=0}^{k-1} (n+j)^{-\alpha} < c \quad \text{or} \quad k < c(n+k-1)^{\alpha}.$$

When we write $k=n^{\beta}$ this becomes $n^{\beta} < c(n+n^{\beta}-1)^{\alpha}$. For n sufficiently large this implies $\beta < 1/2$, that is, $k < n^{1/2}$.

PURDUE UNIVERSITY

NOTE ON THE PRECEDING PAPER

IRVING KAPLANSKY AND HARRY POLLARD

The sufficiency portion of the theorem on the harmonic series proved by Erdös and Niven in the preceding paper hinges on the fact that (in their notation) $k_2=k$ implies $k_j=k$ for $j>2$. We shall show that this is true more generally for any series $\sum u_n$ such that $\{u_n\}$ is completely monotonic. The result follows at once from the theorem below.

In the case $k_2>k$, the method has thus far not yielded any result of the kind obtained by Erdös and Niven.

THEOREM. *Let* $u_n \neq 0$ $(n=1, 2, \cdots)$ *be a sequence such that*

$$(-1)^k \Delta^k u_n \geqq 0 \qquad (k = 0, 1, \cdots; n = 1, 2, \cdots), \tag{1}$$

that is, $\{u_n\}$ *is completely monotonic, and*

$$\lim_{n\to\infty} u_{n+1}/u_n = 1. \tag{2}$$

Define

$$S(n, k) = u_n + u_{n+1} + \cdots + u_{n+k-1},$$
$$f(n, k) = S(n + k, k + 1) - S(n, k).$$

Then $f(n, k)>0$ *implies* $f(n+1, k)>0$.

We require the following lemma, which is a consequence of a theorem of D. V. Widder.[1]

LEMMA. *Let* $\phi(t)$ *be a function continuous in* $(0, 1)$ *and having at most one change of sign in this interval. If* $\alpha(t)$ *is non-decreasing in* $(0, 1)$, *then the sequence* v_n *defined by*

$$v_n = \int_0^1 t^n \phi(t) d\alpha(t), \qquad n = 1, 2, \cdots,$$

has at most one change of sign.

PROOF. If $\phi(t)$ is of constant sign in $(0, 1)$ there is nothing to prove. Suppose then that it changes sign at $t=t_0$. Define $\psi(t)=\int_{t_0}^t \phi(t)d\alpha(t)$. Then $\psi(t)$ has at most one change of trend[2] in $(0, 1)$. Since

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[1] D. V. Widder, *The inversion of the Laplace integral and the related moment problem*, Trans. Amer. Math. Soc. vol. 36 (1934) p. 195.

[2] Loc. cit. p. 155.

$v_n=\int_0^1 t^n d\psi(t)$, the result follows from the theorem of Widder.[1]

PROOF OF THE THEOREM. A well known result of Hausdorff states that a sequence satisfying (1) admits the representation

$$u_n = \int_0^1 t^n d\alpha(t),$$

where $\alpha(t)$ is non-decreasing.[3] It follows that

$$f(n, k) = \int_0^1 t^n \phi(t) d\alpha(t),$$

where $\phi(t)=(t^{2k+1}-2t^k+1)/(t-1)$. It is readily verified that $\phi(t)$ has exactly one change of sign in (0, 1); hence the same is true for the function $\phi(t)-\epsilon$, for sufficiently small $\epsilon>0$, say $\epsilon<\epsilon_0$. Thus by the lemma the sequence

$$f(n, k) - \epsilon u_n = \int_0^1 t^n [\phi(t) - \epsilon] d\alpha(t)$$

has at most one change of sign for $\epsilon<\epsilon_0$.

Suppose now that $f(n, k)>0$, while $f(n+1, k)\leqq 0$. From (2) it follows that $f(N, k)>0$ for a large enough $N>n+1$. Choose ϵ_1, $0<\epsilon_1<\epsilon_0$, so small that

$$f(n, k) - \epsilon_1 u_n > 0,$$
$$f(n + 1, k) - \epsilon_1 u_{n+1} < 0,$$
$$f(N, k) - \epsilon_1 u_N > 0.$$

Then the sequence $f(n, k)-\epsilon_1 u_n$ has at least two changes of sign. But this contradicts the remark above that it can have at most one change of sign. This completes the proof.

NEW YORK CITY

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[3] Loc. cit., p. 191.

PROOF OF A THEOREM OF LITTLEWOOD AND PALEY

A. ZYGMUND

1. **Introduction.** In recent years, important results in the theory of Fourier series were obtained by Littlewood and Paley [3].[1] They used complex methods, and their main tool was an auxiliary function, $g(\theta)$, which they themselves had introduced.

Let $\phi(z)$ be any function regular for $|z|<1$. The real-valued and non-negative function $g(\theta)=g(\theta;\phi)$ is defined by the formula

$$g(\theta) = \left\{\int_0^1 (1-\rho)\,|\,\phi'(\rho e^{i\theta})\,|^2 d\rho\right\}^{1/2}, \qquad 0 \leqq \rho < 1. \tag{1.1}$$

The integral on the right is finite or infinite, but always has meaning.

Let $f(\theta)$ be any L-integrable function of period 2π, and let $f(\rho,\theta)$ be the Poisson integral of f. Thus

$$f(\rho,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(u)P(\rho,\theta-u)du,$$

where $P(\rho,t)=(1-\rho^2)/2(1-2\rho\cos t+\rho^2)$ is the Poisson kernel. If $\bar{f}(\rho,\theta)$ is the harmonic function conjugate to $f(\rho,\theta)$ and vanishing at the origin, and if we set

$$\phi(z) = f(\rho,\theta) + i\bar{f}(\rho,\theta), \qquad z = \rho e^{i\theta},$$

the function (1.1) will sometimes be denoted by $g(\theta;f)$.

The function $g(\theta)$ is suggested by some heuristic argument (see [3, I]). It does not seem to possess any obvious geometric significance, although it has a majorant, $s(\theta)$, with a simple geometric meaning. The reader interested in this problem is referred to papers [4, 7]. In the present note we shall be exclusively concerned with the function $g(\theta)$.

As usual, by H^λ we denote the class of functions $\phi(z)$ regular in $|z|<1$ and satisfying

$$\int_0^{2\pi} |\,\phi(\rho e^{i\theta})\,|^\lambda d\theta = O(1), \qquad 0 \leqq \rho < 1. \tag{1.2}$$

As is well known, this condition implies almost everywhere the existence of the radial limit $\phi(e^{i\theta})=\lim_{\rho\to 1}\phi(\rho e^{i\theta})$.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

Basic for the Littlewood-Paley theory (see also [8][2]) is the following result.

THEOREM. *Suppose that* $\phi(z)\in H^\lambda$, $\lambda>0$, *and that* $g(\theta)=g(\theta;\phi)$. *Then*

$$\left\{\int_0^{2\pi} g^\lambda(\theta)d\theta\right\}^{1/\lambda} \leqq C_\lambda\left\{\int_0^{2\pi} |\phi(e^{i\theta})|^\lambda d\theta\right\}^{1/\lambda}, \tag{1.3}$$

with C_λ *depending on* λ *only.*

This result is proved in Littlewood and Paley [3, II] (where, unnecessarily, it is stated in a form valid for $\lambda>1$ only). The proof the authors give is far from simple. The inequality (1.3) is first established for $\lambda=2, 4, 6, \cdots$. In this case, the integral on the left of (1.3) is treated as a multiple (λ-uple) integral, whose complexity obviously increases with λ. On the other hand, the passage from these special values of λ to general $\lambda>0$ is simple.

The purpose of this note is to give another, perhaps slightly simpler proof of (1.3). The general idea of the proof will be as follows. (a) First we establish (1.3) for $\lambda=2$, which is immediate and quite familiar. (b) Next we show that the validity of (1.3) for any particular λ implies its validity for any smaller and positive λ. (c) Finally, by a certain conjugacy argument we pass from the case $\lambda\leqq 2$ to $\lambda>2$.

The details of the proof are developed in the second section of this note. Part (b) of the proof is borrowed from Littlewood and Paley, and whatever novelty of the subsequent argument is restricted to part (c).

We shall also need the following two lemmas.

LEMMA 1. *Let* $\phi(z)\in H^\lambda$, $\lambda>0$, *and let* $\Phi(\theta)=\sup_{0\leqq\rho<1}|\phi(\rho e^{i\theta})|$. *Then*

$$\left\{\int_0^{2\pi} \Phi^\lambda(\theta)d\theta\right\}^{1/\lambda} \leqq A_\lambda\left\{\int_0^{2\pi} |\phi(e^{i\theta})|^\lambda d\theta\right\}^{1/\lambda}.$$

LEMMA 2. *For any regular function* $\phi(z)$, $z=\rho e^{i\theta}$,

$$4|\phi'(z)|^2 = \Delta(|\phi|^2) \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}|\phi|^2\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}|\phi|^2.$$

Lemma 1 is a very well known result of Hardy and Littlewood (see [1] or [6, p. 247]). Lemma 2 is also very well known. Its use in the sequel is analogous to that made by Littlewood and Paley in their proof of the theorem.

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[2] I take this opportunity to correct a misprint in [8, II]. On p. 349, line 2, the denominator $(\log n)^{1/\alpha}$ should be replaced by $(\log n)^{1/\lambda}$.

2. **Proof of the theorem.** It is enough to prove (1.3) for $\phi(z)$ regular in $|z| \leqq 1$. For if $0<R<1$, and if $g_R = g(\theta; \phi(Rz))$, then

$$g_R^2(\theta) = R^2 \int_0^1 (1-\rho)\,|\phi'(R\rho e^{i\theta})|^2 d\rho \leqq \int_0^1 (1-\rho R)\,|\phi'(\rho R e^{i\theta})|^2 d\rho$$

$$= \int_0^R (1-\rho)\,|\phi'(\rho e^{i\theta})|^2 d\rho.$$

Thus $g_R(\theta) \leqq g(\theta)$, $g_R(\theta) \to g(\theta)$, as $R \to 1$. If (1.3) is valid with g, $\phi(z)$ replaced by g_R, $\phi(Rz)$, then on making R tend to 1 we get (1.3) in full generality.

(a) To prove (1.3) for $\lambda = 2$, let $\phi(z) = c_0 + c_1 z + c_2 z^2 + \cdots$. The series is absolutely and uniformly convergent for $|z| \leqq 1$, and

$$\frac{1}{2\pi}\int_0^{2\pi} g^2(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta \left\{ \int_0^{2\pi} (1-\rho)\,|\phi'(\rho e^{i\theta})|^2 d\rho \right\}$$

$$= \int_0^1 (1-\rho)d\rho \,\frac{1}{2\pi}\int_0^{2\pi} |\phi'(\rho e^{i\theta})|^2 d\theta$$

$$= \int_0^1 (1-\rho) \sum_{\nu=0}^{\infty} \nu^2 |c_\nu|^2 \rho^{2\nu-2} \phi d\rho$$

$$= \sum_{\nu=1}^{\infty} \nu^2 |c_\nu|^2 \int_0^1 (1-\rho)\rho^{2\nu-2} d\rho = \sum_{\nu=1}^{\infty} \frac{\nu^2}{2\nu(2\nu-1)} |c_\nu|^2$$

$$\leqq \frac{1}{2}\sum_{\nu=1}^{\infty} |c_\nu|^2 \leqq \frac{1}{2}\cdot\frac{1}{2\pi}\int_0^{2\pi} |\phi(e^{i\theta})|^2 d\theta.$$

(b) Suppose that (1.3) is established for some particular value of λ and let $0<\kappa<\lambda$. Suppose that $\phi(z) \in H^\kappa$ and assume for a moment that $\phi(z)$ has no zeros for $|z|<1$. Let $\psi = \phi^{\kappa/\lambda}$. Thus $\phi^\kappa = \psi^\lambda$, and $\psi \in H^\lambda$. If we set $\Psi(\theta) = \sup_{0 \leqq \rho < 1} |\psi(\rho e^{i\theta})|$, $g(\theta) = g(\theta; \phi)$, $G(\theta) = g(\theta; \psi)$, then

$$g^2(\theta) = (\lambda/\kappa)^2 \int_0^1 (1-\rho)\,|\psi|^{2(\lambda-\kappa)/\kappa} |\psi'|^2 d\rho = (\lambda/\kappa)^2 \Psi^{2(\lambda-\kappa)/\kappa}(\theta) G^2(\theta),$$

$$\int_0^{2\pi} g^\kappa d\theta \leqq (\lambda/\kappa)^\kappa \int_0^{2\pi} \Psi^{\lambda-\kappa} G^\kappa d\theta$$

$$\leqq (\lambda/\kappa)^\kappa \left\{ \int_0^{2\pi} \Psi^\lambda d\theta \right\}^{(\lambda-\kappa)/\lambda} \left\{ \int_0^{2\pi} G^\lambda d\theta \right\}^{\kappa/\lambda}$$

by the inequality of Hölder. Since we assumed the validity of (1.3) for our particular λ,

$$\left\{\int_0^{2\pi} g^\kappa d\theta\right\}^{1/\kappa} \leqq (\lambda/\kappa)\left\{A_\lambda\left(\int_0^{2\pi} |\psi(e^{i\theta})|^\lambda\right)^{1/\lambda}\right\}^{(\lambda-\kappa)/\kappa}$$

$$\cdot C_\lambda\left\{\int_0^{2\pi} |\psi(e^{i\theta})|^\lambda d\theta\right\}^{1/\lambda}$$

$$\leqq (\lambda/\kappa)A_\lambda^{(\lambda-\kappa)/\kappa} C_\lambda\left\{\int_0^{2\pi} |\psi(e^{i\theta})|^\lambda\right\}^{1/\kappa}$$

$$= (\lambda/\kappa)A_\lambda^{(\lambda-\kappa)/\kappa} C_\lambda\left\{\int_0^{2\pi} |\phi(e^{i\theta})|^\kappa d\theta\right\}^{1/\kappa}.$$

This is (1.3), with κ instead of λ, and with $C_\kappa = (\lambda/\kappa)^{1/2}A_\lambda^{(\lambda-\kappa)/\kappa}C_\lambda$.

Suppose now that $\phi(z)$ $(\neq 0)$ does have zeros in $|z| < 1$. It is well known (see [2]) that then $\phi(z) = \phi_1(z) + \phi_2(z)$ $(|z| < 1)$, where ϕ_1 and ϕ_2 have no zeros, and $|\phi_\kappa(z)| \leqq 2|\phi(z)|$, for $k = 1, 2$. If $g_k = g(\theta; \phi_\kappa)$, then $g \leqq g_1 + g_2$, by the inequality of Minkowski. Thus

$$\int_0^{2\pi} g^\kappa d\theta \leqq 2^\kappa\left\{\int_0^{2\pi} g_1^\kappa d\theta + \int_0^{2\pi} g_2^\kappa d\theta\right\}$$

$$\leqq 2^\kappa\left\{C_\kappa^\kappa\left(\int_0^{2\pi} |\phi_1|^\kappa d\theta + \int_0^{2\pi} |\phi_2|^\kappa d\theta\right)\right\}$$

$$\leqq 2^{2\kappa+1}C_\kappa^\kappa \int_0^{2\pi} |\phi(e^{i\theta})|^\kappa d\theta.$$

This completes the proof of (b). In particular, (1.3) is established for $0 < \lambda \leqq 2$.

(c) On account of (b), it is enough to prove (1.3) for $\lambda \geqq 4$. Let μ be defined by

$$1/(\lambda/2) + 1/\mu = 1.$$

Thus $1 < \mu \leqq 2$, since $\lambda \geqq 4$. Let $\xi(\theta)$ denote any non-negative function such that $M_\mu[\xi] = (\int_0^{2\pi} \xi^\mu d\theta)^{1/\mu} \leqq 1$. Then

$$(2.1) \qquad M_\lambda^2[g] = \left\{\int_0^{2\pi} (g^2)^{\lambda/2} d\theta\right\}^{2/\lambda} = \sup_\xi \int_0^{2\pi} g^2\xi d\theta,$$

where $g = g(\theta; \phi)$. We may even restrict ξ to trigonometric polynomials, without invalidating (2.1). Let us fix any such polynomial $\xi(\theta)$, and let us write $\gamma(\theta) = g(\theta; \xi)$. If $\zeta(z)$ denotes the analytic function whose real part is the Poisson integral $\xi(\rho, \theta)$ of $\xi(\theta)$ and whose imaginary part vanishes at the origin, then $M_\mu[\zeta(e^{i\theta})] \leqq R_\mu M_\mu[\xi] \leqq R_\mu$ by the very well known result of M. Riesz [5]. Here R_μ depends on μ only. Thus, observing that (1.3) has already been proved for λ between 1 and 2, we get

$$M_\mu[\gamma] \leqq CM_\mu[\zeta] \leqq CR_\mu M_\mu[\xi] \leqq CR_\mu, \tag{2.2}$$

where $C=\sup_{1\leqq\lambda\leqq 2}C_\lambda$ is an absolute constant.

We write

$$\begin{aligned}\int_0^{2\pi} g^2(\theta)\xi(\theta)d\theta &= \int_0^1 (1-\rho)\left\{\int_0^{2\pi} |\phi'(\rho e^{i\theta})|^2\xi(\theta)d\theta\right\}d\rho\\ &= 2\int_0^1 (1-\rho^2)\left\{\int_0^{2\pi} |\phi'(\rho^2 e^{i\theta})|^2\xi(\theta)d\theta\right\}\rho d\rho\\ &\leqq 4\int_0^{2\pi}(1-\rho)\left\{\int_0^{2\pi} |\phi'(\rho^2 e^{i\theta})|^2\xi(\theta)d\theta\right\}d\rho.\end{aligned}$$

The function $w(z)=|\phi'(z)|^2$ being subharmonic,

$$w(\rho^2 e^{i\theta}) \leqq \frac{1}{\pi}\int_0^{2\pi} w(\rho e^{iu})P(\rho, \theta-u)du,$$

$$\begin{aligned}\int_0^{2\pi} w(\rho^2 e^{i\theta})\xi(\theta)d\theta &\leqq \frac{1}{\pi}\int_0^{2\pi}\xi(\theta)\left\{\int_0^{2\pi} w(\rho e^{iu})P(\rho, \theta-u)du\right\}d\theta\\ &= \int_0^{2\pi} w(\rho e^{iu})\xi(\rho, u)du,\end{aligned}$$

$$\int_0^{2\pi} g^2(\theta)\xi(\theta)d\theta \leqq 4\int_0^1 (1-\rho)\left\{\int_0^{2\pi} |\phi'(\rho e^{i\theta})|^2\xi(\rho, \theta)d\theta\right\}d\rho. \tag{2.3}$$

By Lemma 2, $4|\phi'|^2=\Delta|\phi|^2$. The formula

$$\Delta U = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 U}{\partial\theta^2} = \rho^{-1}U_\rho + U_{\rho\rho} + \rho^{-2}U_{\theta\theta} \tag{2.4}$$

implies for any functions $a(\rho, \theta)$, $b(\rho, \theta)$

$$\Delta(ab) = a\Delta b + b\Delta a + 2(a_\rho b_\rho + \rho^{-2}a_\theta b_\theta).$$

Thus, taking $a=|\phi|^2$, $b=\xi(\rho, \theta)$, and observing that $\Delta\xi=0$, we get

$$4|\phi'|^2\xi = \Delta(|\phi|^2\xi) - 2\{|\phi|^2_\rho\xi_\rho + \rho^{-2}|\phi|^2_\theta\xi_\theta\}.$$

It follows that the right-hand side of (2.3), which is non-negative, does not exceed

$$\int_0^{2\pi}\int_0^1 (1-\rho)\Delta(|\phi|^2\xi)\rho d\rho d\theta$$

$$+4\int_0^{2\pi}\int_0^1 (1-\rho)|\phi|(|\phi_\rho||\xi_\rho|+\rho^{-2}|\phi_\theta||\xi_\theta|)d\rho d\theta = A+B$$

say. Since $|\phi|\leqq\Phi$, $|\phi_\rho|=|\phi'|$, $|\rho^{-1}\phi_\theta|=|\phi'|$, $|\xi_\rho|\leqq|\zeta'|$, $|\rho^{-1}\xi_\theta|\leqq|\zeta'|$, we find

$$\begin{aligned} B &\leqq 8\int_0^{2\pi}\Phi(\theta)d\theta\left\{\int_0^1 (1-\rho)|\phi'||\zeta'|d\rho\right\} \\ &\leqq 8\int_0^{2\pi}\Phi(\theta)g(\theta)\gamma(\theta)d\theta \qquad (2.5) \\ &\leqq 8M_\lambda[\Phi]M_\lambda[g]M_\mu[\gamma] \leqq 8CA_\lambda R_\mu M_\lambda[g]M_\lambda[\phi(e^{i\theta})]. \end{aligned}$$

Here we use Schwarz's inequality, Hölder's inequality with the three indices λ, λ, μ, and the inequality (2.2). To A we apply the first formula (2.4) and note that, owing to periodicity, the integral of $(|\phi|^2\xi)_{\theta\theta}$ over $(0, 2\pi)$ is zero. Thus

$$\begin{aligned} A &= \int_0^{2\pi}d\theta\int_0^1 (1-\rho)\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}|\phi|^2\xi\right)d\rho \\ &= \int_0^{2\pi}d\theta\int_0^1 \rho\frac{\partial}{\partial\rho}(|\phi|^2\xi)d\rho \\ &= \int_0^{2\pi}\left\{|\phi(e^{i\theta})|^2\xi(\theta)d\theta - \int_0^1 |\phi(\rho e^{i\theta})|^2\xi(\rho,\theta)d\rho\right\}d\theta \qquad (2.6) \\ &\leqq \int_0^{2\pi}|\phi(e^{i\theta})|^2\xi(\theta)d\theta \leqq M_\lambda^2[\phi(e^{i\theta})]M_\mu[\xi] \leqq M_\lambda^2[\phi(e^{i\theta})]. \end{aligned}$$

From (2.1), (2.3), (2.5) and (2.6) follows

$$M_\lambda^2[g] \leqq 8CA_\lambda R_\mu M_\lambda[g]M_\lambda[\phi(e^{i\theta})] + M_\lambda^2[\phi(e^{i\theta})],$$

so that $X = M_\lambda[g]/M_\lambda[\phi(e^{i\theta})]$ does not exceed the largest root of the equation $X^2 = 8CA_\lambda R_\mu X+1$. This proves (1.3) for $\lambda\geqq 4$, and so also for all $\lambda > 0$.

3. **Additional remark.** The constant A_λ in Lemma 1 is bounded for $\lambda\geqq\epsilon>0$ [1; 6, p. 247]. On the other hand, $R_\mu = O(\mu-1)^{-1} = O(\lambda)$ as $\mu\rightarrow 1$ (that is, $\lambda\rightarrow\infty$) [5; 6, p. 149]. Hence the largest root of the equation $X^2 = 8CA_\lambda R_\mu X+1$ is $O(\lambda)$ for $\lambda\rightarrow\infty$. In other words, the constant C_λ in (1.3) is bounded in every interval $0<\epsilon\leqq\lambda\leqq 1/\epsilon$, and

is $O(\lambda)$ for $\lambda\to\infty$. Thus $C_\lambda \leqq K\lambda$ for $\lambda\geqq 1$, where K is an absolute constant.

Suppose that $\phi(z)$ is regular for $|z|\leqq 1$, and does not exceed 1 in absolute value there. Let α be any positive constant less than $1/Ke$. Summing the inequalities

$$\frac{\alpha^n}{n!}\int_0^{2\pi} g^n d\theta \leqq \frac{\alpha^n}{n!}(Kn)^n\int_0^{2\pi} |\phi(e^{i\theta})|^n d\theta \leqq 2\pi(\alpha K)^n n^n/n!$$

for $n=0, 1, 2, \cdots$ we get

$$\int_0^{2\pi}\exp\{\alpha g(\theta)\}d\theta \leqq \beta. \tag{3.1}$$

This inequality may be considered as the limiting case of (1.3) for $\lambda=\infty$. It may even be slightly strengthened. *If* $\phi(z)=u+iv$ *is regular for* $|z|<1$, *and if* $|u|\leqq 1$, *then we have* (3.1) *where* α *and* β *are positive absolute constants.*

To show this, we observe that the proof of §2 could be slightly modified, by using instead of Lemma 2 the relation

$$2|\phi'|^2 = \Delta u^2 \qquad (\phi = u + iv)$$

and instead of Lemma 1 the inequality

$$\left\{\int_0^{2\pi} U^\lambda(\theta)d\theta\right\}^{1/\lambda} \leqq B_\lambda\left\{\int_0^{2\pi} |u(1,\theta)|^\lambda d\theta\right\}^{1/\lambda}, \qquad \lambda > 1,$$

where $u(\rho,\theta)$ is any harmonic function satisfying $\int_0^{2\pi}|u(\rho,\theta)|^\lambda d\theta = O(1)$, and where $u(1,\theta)=\lim_{\rho\to 1}u(\rho,\theta)$, $U(\theta)=\sup_{0\leqq\rho<1}u(\rho,\theta)$. The coefficient B_λ depends on λ only and it is $O(1)$ as $\lambda\to\infty$. (Unlike A_λ, B_λ tends to infinity as $\lambda\to 1$.) This modification of the argument of §2 leads to the following analogue of (1.3)

$$\left\{\int_0^{2\pi} g^\lambda(\theta)d\theta\right\}^{1/\lambda} \leqq C'_\lambda\left\{\int_0^{2\pi} |u(1,\theta)|^\lambda d\theta\right\}^{1/\lambda} \tag{3.2}$$

$$(\lambda > 1,\ \phi = u + iv)$$

and here again $C'_\lambda = O(\lambda)$ as $\lambda\to\infty$. Let $C'_\lambda \leqq K\lambda$ for $\lambda\geqq 2$. If $|u|\leqq 1$, and if we consider (3.2) for $\lambda=2, 4, 6, \cdots$, we immediately obtain

$$\int_0^{2\pi}\exp \alpha g d\theta < 2\int_0^{2\pi}\cosh \alpha g d\theta \leqq 4\pi\sum_0^\infty \frac{\alpha^{2n}}{(2n)!}(K\cdot 2n)^{2n} \leqq \beta < \infty.$$

This completes the proof.

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MT. HOLYOKE COLLEGE

REPRESENTATION OF FOURIER INTEGRALS AS SUMS. I

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Let $\phi(x)$ be an arbitrary function and let the functions $F(x)$ and $G(x)$ be defined by the series:

$$F\left(x\left(\frac{\pi}{2}\right)^{1/2}\right) = \phi(x) - \frac{1}{3}\phi\left(\frac{x}{3}\right) + \frac{1}{5}\phi\left(\frac{x}{5}\right) - \cdots,$$

$$G\left(x\left(\frac{\pi}{2}\right)^{1/2}\right) = \frac{1}{x}\phi\left(\frac{1}{x}\right) - \frac{1}{x}\phi\left(\frac{3}{x}\right) + \frac{1}{x}\phi\left(\frac{5}{x}\right) - \cdots.$$

Then $G(x)$ is the Fourier sine *transform of $F(x)$; that is,*

$$G(x) = \left(\frac{\pi}{2}\right)^{1/2}\int_0^\infty \sin xtF(t)dt.$$

The purpose of this paper is to give restrictions on $\phi(x)$ so that this relation is valid in some sense.

It will be shown here that the restrictions on $\phi(x)$ are closely related to restrictions which insure the existence and inversion of $\int_0^\infty \sin xt\phi(t)dt$. Well known and important cases for the inversion of the Fourier transform are: 1. $\phi(t)\subset L_2$, 2. $\phi(t)\subset L_1$, and 3. $\phi(t)$ of bounded variation. The analogous cases will be considered.

It is convenient to employ the following notation: $\text{sn } x=\sin(\pi/2)x$, $\text{cs } x=\cos(\pi/2)x$, and $\alpha_n=\sin(\pi/2)n$; $n=0, 1, 2, \cdots$. Thus we are trying to justify the relation

$$\sum_1^\infty \frac{\alpha_n}{x}\phi\left(\frac{n}{x}\right) = \int_0^\infty \text{sn } xt\sum_1^\infty \frac{\alpha_n}{n}\phi\left(\frac{t}{n}\right)dt.$$

We shall call this in what follows the sine transform. (No confusion should result from the fact that sn x has a different meaning in the theory of elliptic functions.)

The proofs given here do not assume any previous knowledge of Fourier integrals although certain elementary properties of Fourier series are employed.

1. L_2 theory.

We make the restriction on $\phi(x)$ not only that it belong to L_2 but also that at least one of the series converges suitably to a function in L_2.

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THEOREM 1a. *Let* $\phi(x) \subset L_2$ $(0, \infty)$ *and suppose as* $N \rightarrow \infty$ *that* $\sum(\alpha_n/n)\phi(x/n)$, *where* n *runs from* 1 *to* N, *converges in mean square to a function* $f(x)$ *in* $L_2(0, \infty)$. *Then* $\sum(\alpha_n/x)\phi(n/x)$ *converges in mean square to a function* $g(x)$ *in* $L_2(0, \infty)$ *and*

$$g(x) = \frac{d}{dx} \frac{2}{\pi} \int_0^\infty \frac{1 - \text{cs } xt}{t} f(t) dt$$

almost everywhere.

PROOF. We shall employ a simple lemma of some interest in itself.

LEMMA 1. *Let* $\phi(x) \subset L_2(0, \infty)$ *and suppose for some sequence of constants* $\{b_n\}$ *that as* $N \rightarrow \infty$, $\sum(b_n/n)\phi(x/n)$, *where* n *runs from* 1 *to* N, *converges in mean square to* $f(x)$, $L_2(0, \infty)$. *Then* $\sum(b_n/x)\phi(n/x)$ *converges in mean square to a function* $g_1(x)$ *in* $L_2(0, \infty)$ *and* $\int_0^\infty |f(x)|^2 dx = \int_0^\infty |g_1(x)|^2 dx$.

If $tx = n\nu$ then

$$\begin{aligned} \int_0^\infty \left| \sum_{N_1}^{N_2} \frac{b_n}{x} \phi\left(\frac{n}{x}\right) \right|^2 dx &= \sum_{N_1}^{N_2} \sum_{N_1}^{N_2} \int_0^\infty \frac{b_n b_\nu}{x^2} \phi\left(\frac{n}{x}\right) \phi\left(\frac{\nu}{x}\right) dx \\ &= \sum_{N_1}^{N_2} \sum_{N_1}^{N_2} \int_0^\infty \frac{b_n b_\nu}{n\nu} \phi\left(\frac{t}{\nu}\right) \phi\left(\frac{t}{n}\right) dt \\ &= \int_0^\infty \left| \sum_{N_1}^{N_2} \frac{b_n}{n} \phi\left(\frac{t}{n}\right) \right|^2 dt. \end{aligned}$$

But as N_1 and N_2 approach ∞ the latter integral approaches zero.

LEMMA 2. *For* $0 \leqq t \leqq 1$; $N = 1, 2, 3, \cdots$, *the partial sums* $\sum_1^N \alpha_n(1 - \cos nt)/nt$ *are uniformly bounded.*

Let

$$S(x) = \sum_1^N \alpha_n \frac{1 - \cos nxt}{nt},$$

so

$$S'(x) = \sum_1^N \alpha_n \sin nxt.$$

But $\sin\theta - \sin 3\theta + \cdots \pm \sin(2m+1)\theta = (-1)^m \sin(2m+2)\theta/2\cos\theta$. So $S'(x) = \pm\sin N'xt/2\cos xt$, $0 \leqq x \leqq 1$, where N' is an integer. By the mean value theorem $S(1) = S(0) + S'(x_1)$, $0 < x_1 < 1$. Since $S(0) = 0$ we have $|S(1)| \leqq 1/2 \cos 1$.

Let

$$Y = \frac{2}{\pi}\int_0^\infty \frac{1-\mathrm{cs}\, xt}{t} f(t)dt = \lim_{N\to\infty}\frac{2}{\pi}\int_0^\infty \frac{1-\mathrm{cs}\, xt}{t}\sum_1^N \frac{\alpha_n}{n}\phi\left(\frac{t}{n}\right)dt$$

$$= \lim_{N\to\infty}\frac{2}{\pi}\int_0^\infty \sum_1^N \alpha_n\frac{1-\mathrm{cs}\, nxt}{n}\phi(t)\frac{dt}{t}\cdot$$

Split this integral into two parts, $\int_0^K+\int_K^\infty$; $K=2/\pi x$.

The passage to the limit under the integral sign is justified in the first integral by Lemma 2. In the second integral the series is known to be uniformly bounded as it is the Fourier series of a function of bounded variation. Since $\int_K^\infty|\phi(t)|dt/t$ exists, passage to the limit under the integral sign is justified in the second integral. As is well known,

$$\frac{2}{\pi}\sum_1^\infty \alpha_n\frac{1-\mathrm{cs}\, nx}{n} = \begin{cases} 0; & 4m-1<x<4m+1,\\ 1; & 4m+1<x<4m+3,\\ 1/2; & 2m+1=x.\end{cases}$$

Let

$$h(x) = \begin{cases} 0; & |x|>1,\\ 1; & |x|<1,\\ 1/2; & |x|=1.\end{cases}$$

With $h(x)$ so defined it is obvious that the following key lemma is valid:

LEMMA 3. $(2/\pi)\sum\alpha_n(1-\mathrm{cs}\, nx)/n=\sum\alpha_n h(n/x)$, *where n runs from* 1 *to* ∞ *in both sums.*

Thus, using this relation, we obtain

$$Y = \int_0^\infty\sum_1^\infty \alpha_n h\left(\frac{n}{tx}\right)\phi(t)\frac{dt}{t}\cdot$$

The partial sums of this series are bounded by 1 and are zero near the origin so

$$Y = \sum_1^\infty\int_0^\infty \alpha_n h\left(\frac{n}{tx}\right)\phi(t)\frac{dt}{t} = \sum_1^\infty\int_0^x \frac{\alpha_n}{t}\phi\left(\frac{n}{t}\right)dt = \int_0^x g(t)dt.$$

This last step is justified by Lemma 1. Thus $g(x)=dY/dx$ almost everywhere.

THEOREM 1b. *Under the hypotheses of Theorem* 1a *almost everywhere*

$$f(x) = \frac{d}{dx}\frac{2}{\pi}\int_0^\infty \frac{1-\mathrm{cs}\, xt}{t}g(t)dt.$$

Let $\Phi(x) = \phi(1/x)/x$; then, by Lemma 1, Φ satisfies Theorem 1a. But

$$\sum_1^N \frac{\alpha_n}{n} \phi\left(\frac{x}{n}\right) \equiv \sum_1^N \frac{\alpha_n}{x} \Phi\left(\frac{n}{x}\right)$$

and

$$\sum_1^N \frac{\alpha_n}{x} \phi\left(\frac{n}{x}\right) \equiv \sum_1^N \frac{\alpha_n}{n} \Phi\left(\frac{x}{n}\right).$$

This observation completes the proof without recourse to Plancherel's theorem.

2. L_1 **theory.** Besides assuming that $\phi(x) \subset L_1$ we also assume that $\phi(x)$ is "small" near the origin.

THEOREM 2a. *Let* $(1+1/x)\phi(x) \subset L_1(0, \infty)$; *then*

$$f(x) = \sum_1^\infty \frac{\alpha_n}{n} \phi\left(\frac{x}{n}\right) \quad \textit{and} \quad g(x) = \sum_1^\infty \frac{\alpha_n}{x} \phi\left(\frac{n}{x}\right)$$

exist for almost all x *and*

$$g(x) = \frac{d}{dx} \frac{2}{\pi} \int_0^{\to\infty} \frac{1 - \mathrm{cs}\, xt}{t} f(t)dt$$

almost everywhere.

Since $\phi(x)$ and $\phi(1/x)/x \subset L_1(0, \infty)$ the first part of the theorem is a consequence of this lemma:

LEMMA 4. *Let* $f(x) \subset L_1(1, \infty)$ *and let* $\{b_n\}$ *be a sequence of constants such that* $|b_n| < B$. *Then* $\sum b_n f(nx)$, *where* n *runs from* 1 *to* ∞, *exists for almost all* $x > 1$ *and* $\int_1^\infty \left| \sum_1^\infty b_n f(nx) \right| dx/x \leqq B \int_1^\infty |f(x)| dx$.

Note that $\sum_1^N |f(nx)|$ and $\sum_1^N h(n/x)$ are monotone in N so by the Lesbegue integral theory of monotone sequences

$$\begin{aligned}
\int_1^\infty \sum_1^\infty | f(nx) | \frac{dx}{x} &= \sum_1^\infty \int_1^\infty | f(nx) | \frac{dx}{x} \\
&= \sum_1^\infty \int_0^\infty | f(x) | \, h(n/x) \frac{d\alpha}{x} \\
&= \int_0^\infty | f(x) | \sum_1^\infty h(n/x) \frac{dx}{x} \leqq \int_1^\infty | f(x) | \, dx.
\end{aligned}$$

Thus $\sum_1^\infty |f(nx)|$ converges for almost all x.

Since $\left|\sum_1^\infty b_n f(nx)\right| \leqq B \sum_1^\infty |f(nx)|$, the proof is complete.

The lemma implies $\int_0^\lambda \sum_1^\infty |\alpha_n \phi(t/n)/n| dt$ exists so

$$\frac{2}{\pi}\int_0^{\lambda}\frac{1-\text{cs}\,xt}{t}\sum_1^{\infty}\frac{\alpha_n}{n}\phi\left(\frac{t}{n}\right)dt$$
$$=\lim_{N\to\infty}\frac{2}{\pi}\int_0^{\infty}\sum_1^{N}\alpha_n\frac{1-\text{cs}\,nxt}{n}\,h\left(\frac{nt}{\lambda}\right)\phi(t)\,\frac{dt}{t}\cdot$$

For each value of t the series is some partial sum of the series $\sum_1^{\infty}\alpha_n(1-\text{cs}\,nxt)/n$ and since these partial sums are bounded the limit may be taken under the integral sign, giving

$$\frac{2}{\pi}\int_0^{\infty}\sum_1^{\infty}\alpha_n\frac{1-\text{cs}\,nxt}{n}\,h\left(\frac{nt}{\lambda}\right)\phi(t)\,\frac{dt}{t}\cdot$$

For the same reason the limit of this integral as $\lambda\to\infty$ is

$$\frac{2}{\pi}\int_0^{\infty}\sum_1^{\infty}\alpha_n\frac{1-\text{cs}\,nxt}{n}\,\phi(t)\,\frac{dt}{t}=\int_0^{\infty}\sum_1^{\infty}\alpha_n h\left(\frac{n}{xt}\right)\phi(t)\,\frac{dt}{t}$$
$$=\sum_1^{\infty}\int_0^{\infty}\alpha_n h\left(\frac{n}{xt}\right)\phi(t)\,\frac{dt}{t}$$
$$=\sum_1^{\infty}\int_0^{x}\frac{\alpha_n}{t}\phi\left(\frac{n}{t}\right)dt$$
$$=\int_0^{x}\sum_1^{\infty}\frac{\alpha_n}{t}\phi\left(\frac{n}{t}\right)dt.$$

The interchange of limits is justified by the dominated convergence of the series.

THEOREM 2b. *Under the hypotheses of Theorem* 2a *almost everywhere*

$$f(x)=\frac{d}{dx}\frac{2}{\pi}\int_0^{\to\infty}\frac{1-\text{cs}\,xt}{t}g(t)dt.$$

Let $\Phi(x)=\phi(1/x)/x$; then if $xt=1$,

$$\int_0^{\infty}|\Phi(x)|(1+1/x)dx=\int_0^{\infty}|\phi(t)|(1+1/t)dt.$$

Thus $\Phi(x)$ satisfies the conditions of Theorem 2a but

$$\sum_1^{\infty}\frac{\alpha_n}{n}\Phi\left(\frac{x}{n}\right)\equiv\sum_1^{\infty}\frac{\alpha_n}{x}\phi\left(\frac{x}{n}\right)$$

and

$$\sum_1^{\infty}\frac{\alpha_n}{x}\Phi\left(\frac{n}{x}\right)\equiv\sum_1^{\infty}\frac{\alpha_n}{n}\phi\left(\frac{n}{x}\right).$$

This observation completes the proof.

3. Bounded variation theory. Here our restrictions on $\phi(x)$ are weaker than the restrictions usually stated for the existence and inversion of the Fourier transform in the corresponding case. The Stieltjes integral is found to be the natural tool for the proof in this case. The more transcendental Lesbegue integral is not needed at all.

THEOREM 3a. *Let* $x\phi(x)/1+x$ *be of bounded variation in* $(0, \infty)$ *and tend to zero at zero and infinity. Then*

$$f(x) = \sum_1^\infty \frac{\alpha_n}{2n}\left\{\phi\left(\frac{x}{n}+\right)+\phi\left(\frac{x}{n}-\right)\right\}$$

and

$$g(x) = \sum_1^\infty \frac{\alpha_n}{2x}\left\{\phi\left(\frac{n}{x}+\right)+\phi\left(\frac{n}{x}-\right)\right\}$$

are convergent for positive x *and* $g(x)=\int_0^{\to\infty} \mathrm{sn}\ xtf(t)dt$.

Without loss of generality we shall suppose that $\phi(x)$ is normalized; that is, $\phi(x+)+\phi(x-)=2\phi(x)$. For any positive a, $\phi(x)$ is of bounded variation in (a, ∞) and $x\phi(x)$ is of bounded variation in $(0, a)$. Therefore $\phi(x)=\phi_1(x)-\phi_2(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are positive and monotone and tend to 0 at ∞. Clearly then

$$\left|\sum_1^\infty \frac{\alpha_n}{x}\phi\left(\frac{n}{x}\right)\right| \leqq \frac{1}{x}\phi_1\left(\frac{1}{x}\right)+\frac{1}{x}\phi_2\left(\frac{1}{x}\right).$$

Also $x\phi(x)=\psi_1(x)+\psi_2(x)$ where $\psi_1(x)$ and $\psi_2(x)$ are positive and monotone and tend to 0 at 0. Then

$$\left|\sum_1^\infty \frac{\alpha_n}{n}\phi\left(\frac{x}{n}\right)\right| \leqq \frac{1}{x}\psi_1(x)+\frac{1}{x}\psi_2(x).$$

This proves the first part of the theorem. To prove the second part, first suppose $\phi(x)$ constant for $0\leqq x<\beta\leqq 1/2$, then

$$\begin{aligned}\int_0^\lambda \mathrm{sn}\ t\sum_1^N \frac{\alpha_n}{n}\phi\left(\frac{t}{n}\right)dt &= \int_0^\infty \sum_1^N \alpha_n\ \mathrm{sn}\ tnh\left(\frac{nt}{\lambda}\right)\phi(t)dt\\ &= -\int_0^\infty\int_0^t \sum_1^N \alpha_n\ \mathrm{sn}\ pnh\left(\frac{np}{\lambda}\right)dpd\phi(t)\\ &= -\int_{\beta-}^\infty\int_0^t \sum_1^N \alpha_n\ \mathrm{sn}\ pnh\left(\frac{np}{\lambda}\right)dpd\phi(t).\end{aligned}$$

Suppose $N\geqq\lambda/\beta$ and $t\geqq\beta$; then $N>[\lambda/t]$ so

$$\int_0^t \sum_1^N \alpha_n \operatorname{sn} pnh\left(\frac{np}{\lambda}\right) dp = \frac{2}{\pi} \sum_1^{[\lambda/t]} \alpha_n \frac{1 - \operatorname{cs} nt}{n} + (1 - \operatorname{cs} \lambda) \frac{2}{\pi} \sum_{[\lambda/t]+1}^{N} \frac{\alpha_n}{n}.$$

This series converges uniformly to a continuous uniformly bounded function $Q_\lambda(t)$ as $N \to \infty$. Therefore

$$\int_0^\lambda \operatorname{sn} t \sum_1^\infty \frac{\alpha_n}{n} \phi\left(\frac{t}{n}\right) dt = - \int_0^\infty Q_\lambda(t) d\phi(t).$$

As $\lambda \to \infty$, $Q_\lambda(t)$ is uniformly bounded and approaches $\sum_1^\infty \alpha_n 2 \cdot (1 - \operatorname{cs} nt)/\pi n$ uniformly, excepting in neighborhoods of the odd integer points. Let $\phi(t) = \phi_c(t) + \phi_s(t)$ where $\phi_c(t)$ is continuous and $\phi_s(t)$ is a step function, constant except at the odd integer points.

$$- \int_0^\infty Q_\lambda(t) d\phi = - \int_0^\infty Q_\lambda(t) d\phi_c - \sum_1^\infty Q_\lambda(2n - 1) \delta\phi_s(2n - 1).$$

The limit of this expression as $\lambda \to \infty$ is

$$\sum_1^\infty \left\{ \phi_c(4n - 3) - \phi_c(4n - 1) \right\} - \frac{1}{2} \sum_1^\infty \delta\phi_s(2n - 1).$$

But

$$\sum_1^\infty \delta\phi_s(2n - 1)$$

$$= \sum_1^\infty \left\{ \phi_s(4n - 1 +) + \phi_s(4n - 1 -) - \phi_s(4n - 3 +) - \phi_s(4n - 3 -) \right\}.$$

Substituting this latter expression gives finally $\sum_1^\infty \alpha_n \phi(n)$.

LEMMA 5. *Suppose* $\psi(x)$ *is a nondecreasing function for* $0 \leqq x < \beta \leqq 1/2$, $\psi(x) = 0$ *for* $x > \beta$, *and* $\psi(0+) = 0$. *Then*

$$\left| \int_0^\lambda \frac{\operatorname{sn} t}{t} \sum_1^\infty \alpha_n \psi\left(\frac{t}{n}\right) dt \right| \leqq A \psi(\beta -)$$

where A *is independent of* λ *and* β.

Let N be such that for a given λ

$$\left| \int_0^\lambda \frac{\operatorname{sn} t}{t} \sum_{N+1}^\infty \alpha_n \psi\left(\frac{t}{n}\right) dt \right| \leqq \psi(\beta -).$$

Now

$$\int_0^\lambda \frac{\operatorname{sn} t}{t}\sum_1^N \alpha_n\psi\left(\frac{t}{n}\right)dt = \int_0^\infty \psi(t)\sum_1^N \alpha_n \operatorname{sn} tnh\left(\frac{tn}{\lambda}\right)\frac{dt}{t}$$

$$= \psi(\beta -)\int_\theta^\beta \sum_1^N \alpha_n \operatorname{sn} tnh\left(\frac{tn}{\lambda}\right)\frac{dt}{t}\cdot$$

The second law of the mean defines θ. If $N \leqq \lambda/\beta$ then

$$\int_0^\beta \sum_1^N \alpha_n \operatorname{sn} tnh\left(\frac{tn}{\lambda}\right)\frac{dt}{t} = \int_0^\beta \sum_1^N \alpha_n \operatorname{sn} tn\,\frac{dt}{t}$$

$$= \pm\int_0^\beta \frac{\operatorname{sn} N't}{2\operatorname{cs} t}\,\frac{dt}{t}$$

$$= \pm\frac{1}{2\operatorname{cs}\beta}\int_{\theta_2}^\beta \operatorname{sn} N't\,\frac{dt}{t}\cdot$$

This last expression is bounded independently of N, λ, and β. If $N > \lambda/\beta$ then a similar evaluation gives

$$\pm\int_0^\beta \frac{\operatorname{sn} M't}{2\operatorname{cs} t}\,\frac{dt}{t} + \int_0^\lambda \frac{\sin t}{t}\,dt\sum_{M+1}^N \alpha_n; \qquad M = [\lambda/\beta].$$

This expression is also bounded independently of N, λ, and β. The same considerations apply to the integral $\int_0^\theta$ and the lemma is proved.

We are now able to handle the general case. Define $\phi_i(x)+\phi_e(x) = \phi(x)$ where $\phi_i(x)=0$ for $x<\beta$ and $\phi_e(x)=0$ for $x>\beta$. Then $x\phi_e(x) = \psi_1(x)-\psi_2(x)$ where $\psi_1(x)$ and $\psi_2(x)$ satisfy the conditions of Lemma 5.

Let β satisfy $A\psi_1(\beta)\leqq\epsilon$, $A\psi_2(\beta)\leqq\epsilon$, and $0<\beta\leqq 1/2$. There is a λ_0 such that for $\lambda\geqq\lambda_0$

$$\left|\int_0^\lambda \operatorname{sn} t\sum_1^\infty \frac{\alpha_n}{n}\phi_i\left(\frac{t}{n}\right)dt - \sum_1^\infty \alpha_n\phi(n)\right| < \epsilon.$$

Therefore by Lemma 5

$$\left|\int_0^\lambda \operatorname{sn} t\sum_1^\infty \frac{\alpha_n}{n}\phi\left(\frac{t}{n}\right)dt - \sum_1^\infty \alpha_n\phi(n)\right| < 3\epsilon.$$

The observation that $\phi(ct)$ for $c>0$ satisfies the conditions of the theorem completes the proof.

THEOREM 3b. *If* $\phi(x)$ *satisfies the conditions of Theorem* 3a *then* $f(x)=\int_0^{\to\infty}\operatorname{sn} xtg(t)dt$.

Let $\Phi(x)=\phi(1/x)/x$. Then $x\Phi(x)/1+x=\phi(1/x)/1+x$, so clearly $\Phi(x)$ satisfies the conditions of Theorem 3a which implies Theorem 3b for $\phi(x)$.

4. **Remarks.** It will be noted that the proofs for the three cases are independent; however, the methods of proof are similar. The three cases are given in the apparent order of increasing difficulty of proof.

If $x\phi(x)\equiv\phi(1/x)$ then $f(x)\equiv g(x)$ and we arrive at a representation of self-reciprocal functions as sums.

While much has been written on self-reciprocal functions, not so much attention has been given to function spaces or classes which are self-reciprocal as a whole. The notable exception to this statement is, of course, the space L_2. Note then, that the functions $f(x)$ and $g(x)$ defined by Theorem 2a belong to the same linear function space. Again the functions $f(x)$ and $g(x)$ of Theorem 3a belong to another linear function space.

The theorems here show that under quite general conditions two operations of different form give rise to the same functional transformation. It is natural then to attempt to extend the range of the transformation by using either operation alone where both are not applicable. For example, if $\phi(x)$ is different from zero at only one point the series for $f(x)$ and $g(x)$ are, of course, convergent and $g(x)$ may be defined as the "Fourier transform" of $f(x)$. However since $f(x)$ and $g(x)$ are zero except in a set of measure zero the ordinary integral definition of the Fourier transform is silent since a set of measure zero is disregarded in integration.

The following examples of the theory are of some interest. If $\phi(x)=1/(x+1)$ then

$$f(x) = g(x) = 1/(x+1) - 1/(x+3) + 1/(x+5) - \cdots ;$$

clearly $g(x)=\int_0^1(t^x/(1+t^2))dt$.

Again if $\phi(x)=x^{-s}$, $0<s<1$, then $f(x)=x^{-s}L(1-s)$ and $g(x)=x^{s-1}L(s)$. Here $L(s)=1-3^{-s}+5^{-s}- \cdots$. Evaluation of the integral $\int_0^\infty \text{sn}\, xtf(t)dt$ gives the well known identity, $L(s)=(\pi/2)^s \cdot\cos(s\pi/2)\Gamma(1-s)L(1-s)$.

The second part of this paper treats other conditions on $\phi(x)$ and also the direct representation of the sine transform as a double sum

$$g(x) = \sum_1^\infty \sum_1^\infty \frac{\mu_n\alpha_n\alpha_m}{nx} f\left(\frac{n}{mx}\right).$$

Here μ_n is the well known Möbius symbol.

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THE ASYMPTOTIC DEVELOPMENTS OF A CLASS OF ENTIRE FUNCTIONS

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1. Introduction. In an earlier paper [2],[1] the author established a general theorem concerning the asymptotic behavior in the neighborhood of the point at infinity of an entire function $f(z)$ defined by a Maclaurin series of the form

$$\sum_{n=0}^{\infty} g(n)z^n, \tag{1}$$

where the coefficient $g(n)$ of z^n satisfies certain conditions. In the present paper, we wish to show an application of this theorem. We shall use it to find the asymptotic expansions of an important class of entire functions, namely those defined by the series

$$\sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha_1 n + s_1)\Gamma(\alpha_2 n + s_2) \cdots \Gamma(\alpha_m n + s_m)} \qquad (m \geqq 2), \tag{2}$$

where $\alpha_j > 0$ $(j = 1, 2, \cdots, m)$ and s_j are any constants, real or complex. Functions of this type are frequently connected with the solutions of linear differential equations. Moreover, a special case of (2) is the so-called generalized Bessel function, namely

$$\phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(ln + p)} \qquad (l > 0),$$

which has important applications in partition theory.

Throughout the paper, we shall denote by $f(z)$ the function defined by series (2), and by $g(n)$ the coefficient of z^n in (2). Moreover, we shall adopt single symbols to represent certain combinations of the quantities m, α_j, and s_j, these being as follows:[2]

$$\begin{gathered} \alpha = \sum_{j=1}^{m} \alpha_j; \qquad s = \sum_{j=1}^{m} s_j; \qquad \sigma = \alpha^{\alpha} \prod_{j=1}^{m} (\alpha_j)^{-\alpha(j)}; \\ \gamma = \prod_{j=1}^{m} (\alpha_j)^{s(j)-1/2}; \qquad t = s + \frac{1}{2} - \frac{1}{2}m; \qquad c = \frac{\alpha^{t-3/2}}{(2\pi)^{(m-1)/2} \cdot \gamma}. \end{gathered} \tag{3}$$

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[1] Numbers in square brackets apply to the references at the end of the paper.

[2] We use the symbol $\alpha(j)$ to denote α_j when the latter occurs as an exponent. The symbols $s(j)$ in (3) and $\beta(j)$, and so on, in (6) have a similar meaning.

These six quantities enter prominently into the results to be established, and whenever any of the symbols α, s, σ, γ, t, or c appears in a formula, it will mean the corresponding quantity defined above.

2. **Statement of results.** It will be observed that the function $g(w)$, where $w=x+iy$, is single-valued and analytic in the finite w-plane. Furthermore, in §4, we shall show that, when considered throughout the arbitrary right half-plane $x>x_0$, $g(w)$ can be expressed in the form

$$(4)\qquad \begin{aligned} g(w) = c\alpha\sigma^{w}\left\{\frac{1}{\Gamma(\alpha w+t)}+\frac{c_1}{\Gamma(\alpha w+t+1)}+\cdots\right. \\ \left.+\frac{c_q+\delta(\alpha w, q)}{\Gamma(\alpha w+t+s)}\right\},\end{aligned}$$

where c_1, c_2, c_3, $\cdots$ are determinate constants, and where $\lim_{|w|\to\infty}\delta(\alpha w, q)=0$; $q=0, 1, 2, \cdots$. Hence, according to the theorem proved in [2], *the function $f(z)$, when considered for values of z of large modulus such that $-\pi<\arg z\leqq\pi$, can be developed asymptotically in the form*

$$(5)\qquad f(z)\sim c\sum_{\mu}\left\{Y_{\mu}^{(1-t)}\exp Y_{\mu}\sum_{n=0}^{\infty}c_n(Y_{\mu})^{-n}\right\}-\sum_{n=1}^{\infty}g(-n)z^{-n},$$

where $c_0=1$, $Y_{\mu}=(\sigma z)^{1/\alpha}e^{2\pi i\mu/\alpha}$, and $\sum_{\mu}$ denotes summation over those integral values of μ which satisfy the condition $|\arg z+2\pi\mu|\leqq\pi\alpha/2$.

If $0<\alpha<2$ and either $|\arg z|<\pi\alpha/2$ or $|\arg z|>\pi\alpha/2$, then (5) assumes a simpler form, as is shown in [2, §4].

3. **A special lemma.** From what has been said in §2, it is clear that the problem of determining the asymptotic expansion (5) of the function $f(z)$ is reduced to showing that the function $g(w)$ can be expressed in the form (4). For this purpose, we employ a certain lemma which will be introduced in this section.

Let $\Omega(w)$ represent the function

$$\prod_{j=1}^{m}\left\{\frac{\Gamma(\beta_j w+b_j)}{\Gamma(\rho_j w+r_j)}\right\}\qquad (m\geqq 2),$$

where ρ_j and β_j are positive $(1\leqq j\leqq m)$, while b_j and r_j are any constants, real or complex. Let ρ, β, r, and b denote the respective sums $\sum_{j=1}^{m}\rho_j$, $\sum_{j=1}^{m}\beta_j$, $\sum_{j=1}^{m}r_j$, and $\sum_{j=1}^{m}b_j$, and let

$$(6)\qquad \theta=\prod_{j=1}^{m}\frac{(\beta_j)^{\beta(j)}}{(\rho_j)^{\rho(j)}};\qquad \lambda=\prod_{j=1}^{m}\frac{(\beta_j)^{b(j)-1/2}}{(\rho_j)^{r(j)-1/2}}.$$

Then the lemma with which we are concerned is as follows:

LEMMA. *If $\rho=\beta$ and $r=b$, then for values of w of large modulus such that $|\arg w|<\pi$, the function $\Omega(w)$ can be developed asymptotically in the form*

$$\Omega(w) \sim \lambda\theta^{w}\left\{1+\frac{c_1}{(w+1)}+\frac{c_2}{(w+1)(w+2)}+\cdots\right\} \tag{7}$$

where $c_1, c_2, c_3, \cdots$ are determinate constants.

This lemma is a generalization of a simpler one employed by Ford [1, p. 74] in his discussion of functions of Bessel type, and its proof follows the same line of reasoning.[3] In fact, if we let

$$\Psi(w) = \Omega(w)/\lambda\theta^{w},$$

take the logarithm of both sides, and proceed as in the reference, we arrive at the relation

$$\Psi(w) \sim 1+\frac{c_1}{(w+1)}+\frac{c_2}{(w+1)(w+2)}+\cdots, \tag{8}$$

which is valid for large $|w|$ such that $|\arg w|<\pi$, and from which the lemma follows immediately. As for the values of the c's, they can be calculated from the formula

$$c_n = \lim_{|w|\to\infty} (w+1)(w+2)\cdots(w+n)R_n(w), \tag{9}$$

where

$$R_n(w) = \Psi(w) - 1 - \sum_{q=1}^{n-1}\frac{c_q}{(w+1)(w+2)\cdots(w+q)}.$$

4. **Application to the function** $g(w)$. In this section, we shall employ the lemma just established to prove relation (4). We shall denote by ω_1 the product appearing in the denominator of $g(w)$; that is,

$$\omega_1 = \prod_{j=1}^{m}\Gamma(\alpha_j w + s_j).$$

We begin by writing ω_1 in the special form

$$\omega_1 = \omega_1(w') = \prod_{j=1}^{m}\Gamma\left(\frac{\alpha_j w' + s_j\alpha + \alpha_j - \alpha_j t}{\alpha}\right),$$

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[3] Ford's lemma has been generalized in another manner by H. Van Engen. See *Concerning gamma function expansions*, Tohoku Math. J. vol. 45 (1939) pp. 124–129.

where α and t are as defined in (3), $w'=\alpha w+t-1$, and where the notation $\omega_1(w')$ means that w', rather than w, is being regarded as the argument. Then we form the function

$$W(w') = \Gamma(w'+1)/\omega_1(w'). \tag{10}$$

Applying Gauss' multiplication theorem for the gamma function to $\Gamma(w'+1)$, and substituting in (10), we have

$$\begin{aligned} W(w') &= \frac{m^{w'+1/2}}{(2\pi)^{(m-1)/2}} \frac{\omega_2(w')}{\omega_1(w')}; \\ \omega_2(w') &= \prod_{j=1}^{m} \Gamma\left(\frac{w'+j}{m}\right). \end{aligned} \tag{11}$$

It is easily verified that the fraction $\omega_2(w')/\omega_1(w')$ when regarded as a function $\Omega(w')$ of w' satisfies the hypothesis of the lemma in §3. Hence it can be represented asymptotically when $|\arg w'| < \pi$ in the form

$$\Omega(w') \sim \lambda\theta^{w'}\left[1 + \frac{c_1}{(w'+1)} + \frac{c_2}{(w'+1)(w'+2)} + \cdots\right], \tag{12}$$

where the c's are determined as indicated in (8). In (12), λ and θ have special values which are given by the products (6) in which ρ_j, β_j, r_j, b_j are now equal to α_j/α, $1/m$, $s_j-\alpha_j(t-1)/\alpha$ and j/m respectively. They are found to be

$$\lambda = \left(\frac{\alpha}{m}\right)^{1/2} \prod_{j=1}^{m} (\alpha_j)^{-p(j)}; \qquad p(j) = s_j - 1/2 - \alpha_j(t-1)/\alpha,$$

$$\theta = \frac{\alpha}{m} \prod_{j=1}^{m} (\alpha_j)^{-\alpha(j)/\alpha}.$$

Having established (12), we next recall from (10) that

$$g(w) = 1/\omega_1 = \frac{W(w')}{\Gamma(w'+1)}; \qquad w' = \alpha w + t - 1. \tag{13}$$

From (11), (12), and (13), it follows that $g(w)$ can be represented asymptotically in terms of w' in the form

$$g(w) \sim C(m\theta)^{w'} \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(w'+n+1)}; \qquad c_0 = 1,$$

where C denotes the constant $m^{1/2}\lambda/(2\pi)^{(m-1)/2}$. If we now replace w'

by its value $w' = \alpha w + t - 1$, and make simple reductions, the last relation above assumes the form

$$g(w) \sim c\alpha\sigma^{w}\left\{\frac{1}{\Gamma(\alpha w + t)} + \frac{c_1}{\Gamma(\alpha w + t + 1)} + \cdots\right\}.$$

This relation is valid when $|\arg w| < \pi$; hence relation (4) is seen to hold throughout any right half-plane $x > x_0$, and this fact is what we set out to prove.

As was remarked previously, the asymptotic expansion (5) of the function $f(z)$ defined by series (2) follows at once as soon as (4) has been established.

5. **Applications to special functions.** The general result stated in §2 and established in the succeeding sections can be applied to find the asymptotic expansions of the function

$$\phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(ln + p)} \qquad (l > 0)$$

which was mentioned in §1. It will be noticed that the value of $g(-n)$ $(n = 1, 2, 3, \cdots)$ is zero in this case. The values of the quantities α and t are found by (3) to be $1+l$ and $p+1/2$ respectively, while σ and c are as follows:

$$\sigma = \frac{(1+l)^{1+l}}{l^l}, \qquad c = \frac{(1+l)^{p-1}}{(2\pi)^{1/2}l^{p-1/2}}.$$

The function $\phi(z)$ has been studied previously by E. M. Wright [4] whose results appear comparable with the present results if only the dominant terms in (5) are retained. They involve a sequence of constants $\{a_n\}$ analogous to the sequence $\{c_n\}$ in (5), but which are found by a considerably different process.

As a second application of the present results, we may consider an important class of integral functions recently studied by C. V. Newsom [3], namely

$$F(z) = \sum_{n=0}^{\infty} g(n)z^{mn}$$

where m is a positive integer and $g(n)$ has the form

$$g(n) = 1 \Big/ \prod_{j=1}^{m} \Gamma(n + s_j).$$

In applying the present result to $F(z)$, we must replace z in (5) by z^m, and it is seen that the resulting expansion holds in the sector

$-\pi/m < \arg z \leqq \pi/m$. In this case, we have $\alpha = m, t = \sum_{j=1}^{m} s_j + 1/2 - m/2$, $\sigma = m^m$ and $c = m^{t-3/2}(2\pi)^{1/2-m/2}$. The result is substantially the same as that obtained by Newsom.

In conclusion, we may remark that certain extensions of the results established in the present paper are evidently possible. For example, the method can apparently be applied if a function $h(n)$, properly restricted, is put in the numerator of the coefficient of z^n in (2).[4] Moreover, an extension of both the method and the result would seem possible in case some of the numbers α_j in (2) were negative, sufficient of them remaining positive so that $f(z)$ is entire.

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[4] See for example [1, Arts. 20 and 26].

A NEW SOLUTION FOR LINEAR DIFFERENCE EQUATIONS

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We shall give a solution of the fundamental difference equation:

$$F(t+1)-F(t)=\phi(t). \tag{1}$$

As to our method, we stress that the theory of Fourier series is used. Accordingly, the variable t is assumed to be real. We suppose that $\phi(t)$ is integrable and has bounded variation in every finite interval of t or satisfies any other condition sufficient for expansion in a Fourier series. For simplicity we at first assume that $\phi(t)$ is continuous. Our solution is

$$F(t)=-\frac{\phi(t)}{2}+\int_a^t\phi(\tau)d\tau+2\sum_{k=1}^{\infty}\int_a^t\phi(\tau)\cos 2\pi k(t-\tau)d\tau, \tag{2}$$

where a is constant.

The above series is not a Fourier series in the usual sense, since the upper limit of the integrals is not constant, but the variable t itself.

To prove the truth of our statement we compute the difference

$$\begin{aligned}F(t+1)-F(t)=&-\frac{\phi(t+1)-\phi(t)}{2}+\int_t^{t+1}\phi(\tau)d\tau\\&+2\sum_{k=1}^{\infty}\int_t^{t+1}\phi(\tau)\cos 2\pi k(t-\tau)d\tau.\end{aligned} \tag{3}$$

Now, expansion of $\phi(t)$ in a Fourier series in any interval of length 1, say in the interval $c\cdots c+1$, c meaning an arbitrary real constant, gives

$$\phi(t)=\int_c^{c+1}\phi(\tau)d\tau+2\sum_{k=1}^{\infty}\int_c^{c+1}\phi(\tau)\cos 2\pi k(t-\tau)d\tau.$$

The series represents $\phi(t)$ in the interior of the said interval, but it represents the value $(\phi(c)+\phi(c+1))/2$ at either end point, say at the left end point c. Therefore

$$\frac{\phi(c)+\phi(c+1)}{2}=\int_c^{c+1}\phi(\tau)d\tau+2\sum_{k=1}^{\infty}\int_c^{c+1}\phi(\tau)\cos 2\pi k(c-\tau)d\tau.$$

This holds for every c and we can write t instead of c, giving

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$$(4)\quad \frac{\phi(t)+\phi(t+1)}{2}=\int_t^{t+1}\phi(\tau)d\tau+2\sum_{k=1}^{\infty}\int_t^{t+1}\phi(\tau)\cos 2\pi k(t-\tau)d\tau.$$

From (3) and (4) it follows immediately that

$$F(t+1)-F(t)=-\frac{\phi(t+1)-\phi(t)}{2}+\frac{\phi(t)+\phi(t+1)}{2}=\phi(t).$$

Now we allow $\phi(t)$ to become discontinuous. We suppose that at every point of discontinuity the limits $\phi(t+0)$ and $\phi(t-0)$ exist, which is certainly true in case of functions of bounded variation, and that the relation

$$(5)\qquad \frac{\phi(t+0)+\phi(t-0)}{2}=\phi(t)$$

holds. The solution is then given by

$$(2^*)\ F(t)=-\frac{\phi(t-0)}{2}+\int_a^t\phi(\tau)d\tau+2\sum_{k=1}^{\infty}\int_a^t\phi(\tau)\cos 2\pi k(t-\tau)d\tau.$$

It follows indeed that

$$(3^*)\quad \begin{aligned} F(t+1)-F(t)&=-\frac{\phi(t+1-0)}{2}+\frac{\phi(t-0)}{2}\\ &\quad+\int_t^{t+1}\phi(\tau)d\tau+2\sum_{k=1}^{\infty}\int_t^{t+1}\cdots.\end{aligned}$$

But the series on the right represents now the value

$$(4^*)\quad \begin{aligned}&\int_t^{t+1}\phi(\tau)d\tau+2\sum_{k=1}^{\infty}\int_t^{t+1}\phi(\tau)\cos 2\pi k(t-\tau)d\tau.\\ &\qquad=\frac{\phi(t+0)+\phi(t+1-0}{2}\cdot\end{aligned}$$

Therefore, considering (5),

$$F(t+1)-F(t)=\frac{\phi(t-0)+\phi(t+0)}{2}=\phi(t).$$

Plana (1820) and, independent of him, Abel (1823) were the first to solve equation (1). Their starting point was the Euler's "sum formula," and they found the same solution

$$F(t)=-\frac{\phi(t)}{2}+\int^t\phi(\tau)d\tau+2\int_0^{\infty}\frac{q(t,\tau)}{e^{2\pi\tau}-1}d\tau,$$

where

$$q(t, \tau) = [\phi(t + \tau i) - \phi(t - \tau i)]/2i \quad \text{and} \quad i = (-1)^{1/2}$$

and the lower limit of the first integral is a constant. The above solution is not identical with our solution, as given by (2), since, although the first two terms agree, the third ones do not. Take, for instance, $\phi(t) \equiv 1$. Then $q(t, \tau) \equiv 0$, and the integral

$$\int_0^\infty \frac{q(t, \tau)}{e^{2\pi\tau} - 1} d\tau$$

vanishes identically in t, but the series

$$\sum_{k=1}^{\infty} \int_a^t \cos 2\pi k(t - \tau) d\tau$$

does not.

The proofs of Plana and Abel were not exact. Only Cauchy gave (1826) an exact derivation of the above solution, applying the theory of functions of a complex variable and, in particular, his calculus of residues. For more details see Ernst Lindelöf, *Le calcul des residues*, Paris, 1905, p. 68.

Later on H. Poincaré established the concept of asymptotic representation of a function and applied it to difference equations, Amer. J. Math. vol. 7 (1885) pp. 213–217 and 237–258. Out of several more recent authors we name in particular N. E. Nörlund, R. D. Carmichael and G. D. Birkhoff. They proved, by using different methods, the existence of analytic solutions of linear difference equations and investigated their properties.

We finally mention two books: Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, Julius Springer, 1924, which includes a very extended list of the literature, and P. M. Batchelder, *An introduction to linear difference equations*, Cambridge, Mass., The Harvard University Press, 1927, which presents the methods of both Nörlund and Birkhoff, and moreover gives some important new results.

CORNELL UNIVERSITY

THE EXISTENCE OF ANORMAL CHAINS

DAVID BLACKWELL

1. Introduction. Let $\bar{B}$ be a Borel field of subsets of a space X, and let $P(x, E)$ be for fixed x a probability measure on $\bar{B}$ and for fixed E a $\bar{B}$-measurable function of x. $P(x, E)$ may be considered as representing the transition probability of going from x into E in a single trial. Denote by Ω the space of sequences $\omega: (x_0, x_1, \cdots)$ where $x_i \in X$ and by $\bar{E}$ the Borel field of subsets of Ω determined by all sets

$$\{x_i \in E\}, \quad \text{where} \quad E \in \bar{B}, \qquad i = 1, 2, \cdots.$$

Doob [2, pp. 102–103][1] has shown that there exists for each $x \in X$ a probability measure $P_x(S)$ defined on $\bar{E}$ such that for every P_x-integrable function $f(x_1, \cdots, x_n)$

$$(1) \int f(\omega) dP_x = \int\int \cdots \int f(x_1, \cdots, x_n) dP(x_{n-1}, x_n) \cdots dP(x, x_1),$$

that Ω with the measure P_x is a Markoff process, that is, $E(x_1, \cdots, x_n; g) = E(x_n; g)$ where $g = g(x_{n+1}, x_{n+2}, \cdots)$ and the E's denote conditional expectations with respect to the indicated variables, and that $E(x_1, \cdots, x_r; f)$ is the function obtained by carrying out the first $n-r$ integrations in (1).

Write $Q(x, E) = P_x(\limsup \{x_i \in E\})$, so that $Q(x, E)$ represents the probability of entering E infinitely often, starting from x. Following Doblin [1, p. 68 et seq.] we make the following definitions for sets of $\bar{B}$: E is *inessential* if $Q(x, E) = 0$ for all x, and *essential* otherwise. An essential set is *improperly essential* if it is a denumerable sum of inessential sets, and *absolutely essential* otherwise. A finite or denumerable sum of improperly essential sets is consequently improperly essential. E is *closed* if $P(x, E) = 1$ for all $x \in E$, and a closed set is indecomposable if it does not contain two disjunct non-empty closed subsets. An absolutely essential indecomposable set is said to be *normal* if it contains a closed set which contains no improperly essential subsets and *anormal* otherwise. If X is a normal set, we shall say that the Markoff chain determined by $P(x, E)$ is a normal chain.

Doblin [1] has obtained for normal chains many elegant results which are considerably more complicated for the anormal case. For example [1, p. 81] in the normal case there exists a closed set G such

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[1] Numbers in brackets refer to the references cited at the end of the paper.

that $Q(x, E)=1$ for every essential subset E of G and every $x\in G$; in the anormal case G can no longer be chosen independently of E. It is consequently of interest to investigate conditions for normality. Doblin [1, p. 82] has given a quite general sufficient condition for the occurrence of the normal case, and no example of an anormal chain has hitherto been given. The purpose of this paper is to give a simple necessary and sufficient condition for the occurrence of the normal case, consisting merely in the measurability of the function $f(x)$ which represents the probability that, starting from x, the point remains indefinitely in an improperly essential set. An example of an anormal chain is also given.

2. **The normal case.** We restate a result of Doblin [1, p. 80] in the following theorem.

THEOREM 1. *If X is indecomposable and absolutely essential, the closed set $\{Q(x, E)=1\}$ is non-empty if and only if E is absolutely essential.*

The following lemma asserts that the probability of entering E infinitely often, starting from y at the jth trial, is independent of j:

LEMMA. *With respect to any P_x measure,*

$$P_x(x_j = y;\ \limsup \{x_i \in E\}) = Q(y, E). \tag{2}$$

PROOF. If $f_{m,n}$ is the characteristic function of the set $E(m, n) =\sum_{m\leqq i\leqq n}\{x_i\in E\}$, it follows from (1) that

$$P_y(E(m, n)) = \int\int \cdots \int f_{m,n}dP(x_{n-1}, x_n) \cdots dP(y, x_1)$$
$$=P_x(x_j = y;\ E(m + j, n + j)).$$

Letting first n, then m, become infinite, we obtain (2).

The following theorem, which is new and of some independent interest, asserts that for any two sets T and E, unless there are points of T which are practically certain to enter E infinitely often, it is impossible for a point to enter both T and E infinitely often.

THEOREM 2. *If $Q(x, E)\leqq a<1$ for all $x\in T$, then for all x*

$$h(x) = P_x(\limsup \{x_i \in E\}\cdot\limsup \{x_i \in T\}) = 0. \tag{3}$$

PROOF. Define $E(N, n)=\{x_i\notin E \text{ for } N\leqq i<n,\ x_n\in E\}$, $T(N, n) = \{x_i\notin T \text{ for } N\leqq i<n,\ x_n\in T\}$. Now for fixed N,

$$h(x) \leqq \sum_{N<n<r} P_x(E(N, n)T(n, r) \limsup \{x_i \in E\}). \tag{4}$$

But

$$P_x(E(N, n)T(n, r) \limsup \{x_i \in E\})$$
$$= \int_{E(N,n)T(n,r)} P_x(x_1, \cdots, x_r; \limsup \{x_i \in E\} dP_x$$
$$= \int_{E(N,n)T(n,r)} Q(x_r, E) dP_x \leqq aP_x(E(N, n)T(n, r),$$

by the lemma and the fact that x_r belongs to T in the domain of integration. Using the last inequality in (4), we obtain

$$h(x) \leqq a \sum_{N<n<r} P_x(E(N, n)T(n, r)) = aP_x\left(\sum_{N<n<r} E(N, n)T(n, r)\right).$$

Letting N become infinite we obtain $h(x) \leqq ah(x)$, which implies (3).

Our necessary and sufficient condition for the occurrence of the normal case, given in Theorem 3, is an easy consequence of Theorems 1 and 2.

THEOREM 3. *Let X be absolutely essential and indecomposable. Then X is normal if and only if $f(x) = \text{l.u.b.}_{\text{imp.ess.}E}\, Q(x, E)$ is $\overline{B}$-measurable.*

PROOF. By Theorem 1, for every improperly essential E and every x we have $Q(x, E) < 1$. It follows that $f(x) < 1$ for all x, since a denumerable sum of improperly essential sets is improperly essential. If $f(x)$ is measurable, there exists an $a < 1$ such that $T = \{f(x) < a\}$ is absolutely essential; for $X = \sum_{n=1}^{\infty} \{f(x) < 1 - 1/n\}$, and not all these sets can be improperly essential. Denoting by S the closed set $\{Q(x, T) = 1\}$, it follows from Theorem 1 that S is non-empty and from Theorem 2 that $Q(x, E) = 0$ for all $x \in S$ and all improperly essential E. In particular if $E \subset S$ and E is not absolutely essential, $Q(x, E) = 0$ for all $x \in E$, which by Theorem 2 in the special case $T = E$ implies that E is inessential. Thus S is a non-empty closed set containing no improperly essential subsets, and X is normal.

Conversely if there exists such a subset S it is easily verified that $f(x) = Q(x, X - S)$ and is therefore measurable.

3. **An anormal chain.** The space X is the semi-infinite interval $0 \leqq x < \infty$, and $\overline{B}$ is the Borel field of all finite or denumerable subsets of X and their complements. Let a_n be any sequence of numbers such that $0 < a_n < 1$, $\prod_0^{\infty} a_n > 0$. We define

$$P(x, E) = a_n f(x + 1, E) + (1 - a_n) d(E),$$

where n is the largest positive integer not exceeding x, $d(E)$ is 1 if E is non-denumerable and 0 otherwise, and $f(x, E)$ is the characteristic function of E. $P(x, E)$ is clearly a probability measure on $\overline{B}$ for fixed x. To verify that $P(x, E)$ is $\overline{B}$-measurable for fixed E, we may assume that E is at most denumerable, since $P(x, CE)=1-P(x, E)$. For this case $P(x, E)=0$ except on an at most denumerable set and is consequently measurable. Since the probability of going from x into $x+1$ is a_n for $n\leqq x<n+1$ and since $\prod_0^\infty a_n>0$, every set containing all points $x+n$, $n=1, 2, \cdots$, for some x is essential. The closed sets are those nondenumerable sets which contain with x all points $x+n$, so that X is indecomposable. Finally X is absolutely essential, since $\sum S_n=X$ implies that some S_n is nondenumerable and hence esssential. Thus X is anormal.

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HOWARD UNIVERSITY

AXIOMATIC CHARACTERIZATION OF FIELDS BY THE PRODUCT FORMULA FOR VALUATIONS

EMIL ARTIN AND GEORGE WHAPLES

Introduction. The theorems of class field theory are known to hold for two kinds of fields: algebraic extensions of the rational field and algebraic extensions of a field of functions of one variable over a field of constants. We shall refer to these fields as number fields and function fields, respectively. For class field theory, the function fields must indeed be restricted to those with a Galois field as field of constants; however, we make this restriction only in §5, and until then consider fields with an arbitrary field of constants.

In proving these theorems, the product formula for valuations plays an important rôle. This formula states that, for a suitable set of inequivalent valuations $|\ |_{\mathfrak{p}}$,

$$\prod_{\mathfrak{p}} |\alpha|_{\mathfrak{p}} = 1$$

for all numbers $\alpha \neq 0$ of the field. For fields of the types mentioned, this product formula is easy to prove. After reviewing this proof (§1), we shall show (§2) that, conversely, the number fields and function fields are characterized by their possession of a product formula. Namely, we prove that if a field has a product formula for valuations, and if one of its valuations is of suitable type, then it is either a function field or a number field.

This shows that the theorems of class field theory are consequences of two simple axioms concerning the valuations, and suggests the possibility of deriving these theorems directly from our axioms. We do this in the later sections of this paper for the generalized Dirichlet unit theorem, the theorem that the class number is finite, and certain others fundamental to class field theory. This axiomatic method has the decided advantage of uniting the two cases; also, it simplifies the proofs. For example, we avoid the use of either ideal theory or the Minkowski theory of lattice points. Thus these two theories are unnecessary to class field theory, since they are needed only to prove the unit theorem.

1. Preliminaries on valuations. If k is any field, then a function $|\alpha|$, defined for all $\alpha \in k$, is called a valuation of K if:

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(1) $|\alpha|$ *is a real number not less than* 0, *and* $|\alpha| = 0$ *only if* $\alpha = 0$,

$$|\alpha\beta| = |\alpha||\beta|, \tag{2}$$

$$|\alpha + \beta| \leqq |\alpha| + |\beta|. \tag{3}$$

We call a valuation nonarchimedean if in addition to (3) it satisfies

$$|\alpha + \beta| \leqq \max(|\alpha|, |\beta|). \tag{3'}$$

Note that the assumption that $|\alpha|$ is a real number eliminates the possibility of certain valuations discussed in various recent papers.

The theory of a field with respect to one given valuation is supposed to be known by the reader and shall be called the local theory. We review the most important facts. The valuation $|\alpha| = 1$ for all $\alpha \neq 0$ is called the trivial valuation. Two nontrivial valuations $|\alpha|_1$ and $|\alpha|_2$ are called equivalent when $|\alpha|_1 < 1$ implies $|\alpha|_2 < 1$, and it is easy to show[1] that there is a positive real number ρ such that $|\alpha|_1^\rho = |\alpha|_2$ for all $\alpha \in k$. If $|\alpha|$ is a nonarchimedean valuation, then $|\alpha|^\rho$ is an equivalent valuation for any $\rho > 0$. If $|\alpha|$ is archimedean and ρ positive, then $|\alpha|^\rho$ will be a valuation only for sufficiently small values of ρ. However, we shall in the remainder of this paper use the word "valuation" to mean any function $|\alpha|^\rho$ where ρ is any positive number and $|\alpha|$ is a true valuation.

A set of equivalent and nontrivial valuations of a field k is called a prime divisor of that field, and denoted by letters like p, $\mathfrak{p}$, $\mathfrak{P}$, q, $\mathfrak{q}$, $\mathfrak{Q}$, $\cdots$. If $\mathfrak{p}$ is a prime divisor, $|\alpha|_\mathfrak{p}$ stands for a particular, fixed valuation chosen from this set. The sign $\|\alpha\|_\mathfrak{p}$ will later be used to stand for another valuation of the same set.

If R is a subfield of k then each set $\mathfrak{p}$ of equivalent valuations of k is also a set of equivalent valuations of R. If these valuations are nontrivial on R then they define a prime divisor p of R, and $\mathfrak{p}$ is said to divide p: $\mathfrak{p}|p$. One p may be divisible by several $\mathfrak{p}$ of k. By well known methods,[2] the field k can be extended to the field $k_\mathfrak{p}$ which is completed with respect to the valuations of $\mathfrak{p}$. If R_p is the corresponding completion of R then R_p is a subfield of k and the degree $n(\mathfrak{p}) = (k_\mathfrak{p}/R_p)$ is called the local degree. If k itself is a finite extension of R of degree n it is easy to prove[3] the inequality

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[1] See van der Waerden [7, pp. 254–255], or Artin [1]. Numbers in brackets refer to the references cited at the end of the paper.

[2] van der Waerden [7, p. 250].

[3] To prove this, assume first that $k = R(\alpha)$, where α is a root of a polynomial $f(x)$, irreducible in R of degree n. Let $P(x)$ be the polynomial, irreducible in R_p of degree $n_\mathfrak{p}$, with root α. Since (van der Waerden [7, p. 264]) an extension field of R_p can be evaluated in only one way by a divisor $\mathfrak{p}$ of p, it follows that different divisors

$$\sum_{\mathfrak{p}|p} n(\mathfrak{p}) \leqq n \tag{4}$$

for each p of R.

In case $\mathfrak{p}$ is a discrete valuation, $n(\mathfrak{p})=e(\mathfrak{p})f(\mathfrak{p})$ where $e(\mathfrak{p})$ is the ramification number and $f(\mathfrak{p})$ the degree of the residue class field of k over that of R.

We proceed now to study a finite set of nontrivial inequivalent valuations $|\ |_1, |\ |_2, \cdots, |\ |_n$.

LEMMA 1. *If* $|\ |_1$ *and* $|\ |_2$ *are two nontrivial, inequivalent valuations of k, then there is a* $\gamma \in k$ *with* $|\gamma|_1<1$ *and* $|\gamma|_2>1$.

PROOF. Since the valuations are inequivalent there is an α with $|\alpha|_1<1$ and $|\alpha|_2 \geqq 1$ and a β with $|\beta|_1 \geqq 1$ and $|\beta|_2<1$. Take $\gamma=\alpha/\beta$.

LEMMA 2. *If* $|\ |_1, |\ |_2, \cdots, |\ |_n$ *are nontrivial and inequivalent there is an* $\alpha \in k$ *such that* $|\alpha|_1>1$ *and* $|\alpha|_\nu<1$ *for* $\nu=2, \cdots, n$.

PROOF. The lemma is true for $n=2$ by Lemma 1. We use induction, assuming that we have found a β such that $|\beta|_1>1$ and $|\beta|_\nu<1$ for $\nu=2, \cdots, n-1$. Choose γ so that $|\gamma|_1>1$ and $|\gamma|_n<1$. There are two cases:

Case 1. If $|\beta|_n \leqq 1$ let $\alpha=\beta^r\gamma$. Then $|\alpha|_1>1$ and, for r sufficiently large, $|\alpha|_2, |\alpha|_3, \cdots, |\alpha|_n$ are all less than 1.

Case 2. If $|\beta|_n>1$ let

$$\alpha = \frac{\beta^r}{\beta^r+1}\gamma$$

so that

$$|\alpha|_\nu = \frac{|\beta|_\nu^r\,|\gamma|_\nu}{|\beta^r+1|_\nu} \leqq \frac{|\beta|_\nu^r}{1-|\beta|_\nu^r}\,|\gamma|_\nu, \qquad \nu = 2, \cdots, n-1,$$

$$|\alpha|_n \leqq \frac{|\beta|_n^r}{|\beta|_n^r-1}\,|\gamma|_n.$$

Now $|\alpha|_\nu<1$ for r large; namely $\lim_{r\to\infty}|\beta|_\nu^r=0$ and $|\alpha|_n<1$ since

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$\mathfrak{p}_1, \mathfrak{p}_2, \cdots$ of p will lead to different irreducible polynomials $P(x)$. Since $f(x)$ is divisible by the product of the polynomials $P(x)$, this proves the inequality for this case. If several elements have to be adjoined to R in order to get k, we prove the theorem by repeated application of the simple case.

This proof shows also that in case of an inseparable extension k one can not expect to replace the inequality by an equality. If this can be done in a special case, it is a noteworthy property of the particular field. In §3 we shall find a class of fields with this property.

$$\lim_{r\to\infty}\frac{|\beta|_n^r}{|\beta|_n^r-1}=1.$$

For $\nu=1$ we find

$$|\alpha|_1 \geqq \frac{|\beta|_1^r}{1+|\beta|_1^r}|\gamma|_1,$$

so for large r, $|\alpha|_1>1$.

LEMMA 3. *If any n nontrivial inequivalent valuations of k are given, then for any positive ϵ there is an α such that*

$$|\alpha-1|_1 \leqq \epsilon, \quad |\alpha|_\nu \leqq \epsilon \quad \textit{for} \quad \nu>1.$$

PROOF. Choose β, by Lemma 2, so that $|\beta|_1>1$ and $|\beta|_\nu<1$ for $\nu>1$ and take

$$\alpha=\frac{\beta^r}{1+\beta^r}.$$

Then

$$|\alpha-1|_1=\frac{1}{|1+\beta^r|_1}\leqq\frac{1}{|\beta|^r-1}\leqq\epsilon$$

for r sufficiently large. For $\nu>1$,

$$|\alpha|_\nu=\frac{|\beta|_\nu^r}{|1+\beta|_\nu^r}\leqq\frac{|\beta|_\nu^r}{1-|\beta|_\nu^r}\leqq\epsilon$$

for r sufficiently large.

THEOREM 1 (APPROXIMATION THEOREM). *If we are given any n nontrivial inequivalent valuations $|\ |_\nu$ of k, an element α_ν of k for each valuation, and an $\epsilon>0$, then we can find an element α of k such that*

$$|\alpha-\alpha_\nu|_\nu \leqq \epsilon \quad \textit{for} \quad \textit{each} \quad \nu=1,2,\cdots,n.$$

PROOF. Let M be the maximum of the numbers $|\alpha_i|_j$ for all combinations of i and j and choose β_i $(i=1,2,\cdots,n)$ such that

$$|1-\beta_i|_i<\frac{\epsilon}{nM}, \qquad |\beta_i|_\nu<\frac{\epsilon}{nM} \quad \text{for} \quad \nu\neq i.$$

Let

$\alpha=\beta_1\alpha_1+\beta_2\alpha_2+\cdots+\beta_n\alpha_n$; then $|\alpha-\alpha_i|_i<\epsilon$ for each i.

COROLLARY. *If* $|\ |_1, |\ |_2, \cdots, |\ |_n$ *are nontrivial and inequivalent then a relation*

$$|x|_1^{\nu_1}\,|x|_2^{\nu_2}\cdots|x|_n^{\nu_n} = 1$$

is true for all $x \in k$, $x \neq 0$, *if and only if all* $\nu_i = 0$.

PROOF. If any $\nu_i \neq 0$, an x for which $|x|_i$ is sufficiently large and the other $|x|_\nu$ for $\nu \neq i$ are sufficiently near 1 gives a contradiction.

2. **The product formula.** Our corollary precludes the possibility that a finite number of valuations can be interrelated in a field. Such an interrelation may nevertheless happen for an infinite number of valuations. In case of the ordinary function fields and number fields that is not only the case but this fact may even be used to derive all the properties of these fields on a common basis.

We shall assume for our field k:

AXIOM 1. *There is a set* $\mathfrak{M}$ *of prime divisors* $\mathfrak{p}$ *and a fixed set of valuations* $|\ |_\mathfrak{p}$, *one for each* $\mathfrak{p} \in \mathfrak{M}$, *such that, for every* $\alpha \neq 0$ *of* k, $|\alpha|_\mathfrak{p} = 1$ *for all but a finite number of* $\mathfrak{p} \in \mathfrak{M}$ *and*

$$\prod_\mathfrak{p} |\alpha|_\mathfrak{p} = 1,$$

where this product is extended over all $\mathfrak{p} \in \mathfrak{M}$.

If this axiom is satisfied, $\mathfrak{M}$ can contain only a finite number of archimedean divisors: for $|1+1|_\mathfrak{p} > 1$ at all archimedean $\mathfrak{p}$. Suppose that Axiom 1 is satisfied and that $\mathfrak{M}$ contains no archimedean divisors at all; consider the set k_0 of all α for which $|\alpha|_\mathfrak{p} \leqq 1$ at all $\mathfrak{p} \in \mathfrak{M}$. Let α and β be two elements of k_0. It follows at once that $-\alpha$, $\alpha\beta$, and $\alpha+\beta$ are also in the set. If $\alpha \in k_0$ and $\alpha \neq 0$, the product formula gives at once $|\alpha|_\mathfrak{p} = 1$ for all $\mathfrak{p}$. It now follows that α^{-1} is in k_0. Thus k_0 forms a subfield of k, called the field of constants. It consists of 0 and those elements of k which satisfy $|\alpha|_\mathfrak{p} = 1$ for all $\mathfrak{p}$. It may also be defined as the largest subfield of k for which all $\mathfrak{p}$ reduce to the trivial valuation. If $\mathfrak{M}$ contains archimedean divisors, then there is no field of constants.

We associate with our set $\mathfrak{M}$ of valuations $\mathfrak{p}$ a certain space of vectors $\mathfrak{a}$ with one component $\alpha_\mathfrak{p}$ for each divisor $\mathfrak{p}$. The component $\alpha_\mathfrak{p}$ may range freely over the $\mathfrak{p}$-adic completion $k_\mathfrak{p}$ of k. If $\mathfrak{a}$ is such a vector we shall for brevity write $|\mathfrak{a}|_\mathfrak{p}$ instead of $|\alpha_\mathfrak{p}|_\mathfrak{p}$. The idèles of Chevalley[4] are special cases of these vectors; for an idèle we must have $\alpha_\mathfrak{p} \neq 0$ for all $\mathfrak{p}$ and $|\mathfrak{a}|_\mathfrak{p} = 1$ for all but a finite number of $\mathfrak{p}$.

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[4] See Chevalley [3, 4].

Our field k may be considered a subset of this space inasmuch as $\alpha \in k$ may also be considered as the vector whose $\mathfrak{p}$-coordinate is the element α of $k_{\mathfrak{p}}$.

With each idèle $\mathfrak{a}$ we associate the product

$$V(\mathfrak{a}) = \prod_{\mathfrak{p}} |\mathfrak{a}|_{\mathfrak{p}}$$

and think of it as something measuring the size of $\mathfrak{a}$. In a moment we shall see that it may be interpreted as a "volume."

For elements α of k the product formula yields

$$V(\alpha) = 1$$

so that for all idèles $\mathfrak{a}$ we get

$$V(\alpha\mathfrak{a}) = V(\mathfrak{a}).$$

If we select real numbers $x_{\mathfrak{p}} > 0$ for each $\mathfrak{p}$ and take care that $x_{\mathfrak{p}} \neq 1$ for a finite number of $\mathfrak{p}$ only, then we call the set of vectors $\mathfrak{c}$ satisfying

$$|\mathfrak{c}|_{\mathfrak{p}} \leqq x_{\mathfrak{p}} \quad \text{for all } \mathfrak{p}$$

a parallelotope of dimensions $x_{\mathfrak{p}}$.

We shall find later that every valuation is either archimedean or discrete. If this is true then there is an element $\alpha_{\mathfrak{p}}$ in $k_{\mathfrak{p}}$ whose value is maximal and not greater than $x_{\mathfrak{p}}$ so that it is no restriction of generality to start with a given idèle $\mathfrak{a}$ and to construct all vectors $\mathfrak{c}$ satisfying

$$|\mathfrak{c}|_{\mathfrak{p}} \leqq |\mathfrak{a}|_{\mathfrak{p}}.$$

We talk in this case of a parallelotope of size $\mathfrak{a}$. The product $V(\mathfrak{a})$ may then be interpreted as its volume.

Next we introduce the "order" of a given set of elements. It is a notion that shall unite different types of fields. If k has an archimedean valuation we mean by order the number of elements. Otherwise k has a field k_0 of constants: we let q stand for an arbitrarily selected but fixed number greater than 1 when the number of elements of k_0 is infinite, and for the number of elements of k_0 when this number is finite. By order of a set we mean in this case the number q^s where s is the number of elements in our set that are linearly independent with respect to k_0. Should k_0 contain q elements and our set be closed under addition and under multiplication by elements of k_0 then q^s is the number of elements in the set.

In the next section we shall be interested in the order of the set of elements α of k that are contained in a given parallelotope of size $\mathfrak{a}$. We denote this order by $M(\mathfrak{a})$. If $\theta \neq 0$ is in k than $M(\theta\mathfrak{a}) = M(\mathfrak{a})$.

Indeed multiplication by θ transforms the parallelotope of size $\mathfrak{a}$ into the parallelotope of size $\theta\mathfrak{a}$ and does not change the order.

In the next section it will be shown that $V(\mathfrak{a})$ and $M(\mathfrak{a})$ are related; namely that they are of the same order of magnitude.

If $\mathfrak{p}$ is a nonarchimedean prime divisor, the elements $\alpha \in k$ for which $|\alpha|_{\mathfrak{p}} \leqq 1$ form a ring $\mathfrak{o}_{\mathfrak{p}}$, called the ring of $\mathfrak{p}$-integers (or local integers). The elements α for which $|\alpha|_{\mathfrak{p}} < 1$ form a prime ideal in this ring and we denote this ideal by the same symbol $\mathfrak{p}$ as the prime divisor. If the residue class field $\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}$ is of finite order then we call this order the norm of $\mathfrak{p}$ and denote it by $N\mathfrak{p}$. We can talk of the order also in case of a constant field k_0 since k_0 may be considered as subfield of the field $\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}$. Thus, if f is the degree of $\mathfrak{o}_{\mathfrak{p}}/\mathfrak{p}$ over k_0, and if f is finite, we put $N\mathfrak{p} = q^f$.

AXIOM 2. *The set* $\mathfrak{M}$ *of Axiom* 1 *contains at least one prime* $\mathfrak{q}$, *which is of one of the following two types*:

1. *Discrete, with a residue class field of finite order* $N\mathfrak{q}$.
2. *Archimedean, with a completed field* $k_{\mathfrak{q}}$ *which is either the real or the complex field.*[5]

As mentioned before, there are an infinity of equivalent valuations belonging to one prime divisor $\mathfrak{p}$. One of them, $|\alpha|_{\mathfrak{p}}$, is singled out by our Axiom 1. For primes $\mathfrak{p}$ satisfying Axiom 2 we shall define another one that is singled out by inner properties. In case 1 of Axiom 2 we put (for $\alpha \neq 0$)

$$\|\alpha\|_{\mathfrak{p}} = \frac{1}{N\mathfrak{p}^{\nu}}$$

where ν is the ordinal number of α at $\mathfrak{p}$. In case 2 we take $\|\alpha\|_{\mathfrak{p}}$ to be ordinary absolute value when $k_{\mathfrak{p}}$ is the real field and the square of ordinary absolute value when $k_{\mathfrak{p}}$ is the complex field. Note that in the latter case $\|\alpha\|_{\mathfrak{p}}$ is not a true valuation. We call $\|\alpha\|_{\mathfrak{p}}$ the normed valuation at $\mathfrak{p}$.

THEOREM 2. *In case of the following special fields* k *we can construct a set* $\mathfrak{M}$ *of valuations such that our axioms hold, and the second one even holds for all* $\mathfrak{p}$ *of* $\mathfrak{M}$:

1. *Any finite algebraic number field* (*that is, a finite extension of the field of rational numbers*).
2. *Any field of algebraic functions over any given field* k_1 (*that is, a finite extension of the field* $k_1(z)$ *where* z *is transcendental with respect to* k_1).

---

[5] It is well known (Ostrowski [5]) that we could drop this condition on the completed field.

In case 2, the constant field k_0 of k with respect to $\mathfrak{M}$ consists of all elements of k that are algebraic with respect to k_1.

The proof is contained in the following chain of statements:

LEMMA 4. *Let k be a field for which Axiom 1 holds and R a subfield consisting not exclusively of constants of k. Let $\mathfrak{N}$ be the set of those nontrivial divisors p of R that are divisible by some $\mathfrak{p}$ of $\mathfrak{M}$. Then Axiom 1 holds in R for this set $\mathfrak{N}$.*

PROOF. Let p be any divisor of $\mathfrak{N}$ and a an element of R such that $|a|_p>1$. Then $|a|_\mathfrak{p}>1$ for all $\mathfrak{p}$ that divide p. Because of Axiom 1 there can be only a finite number of $\mathfrak{p}|p$. Let us now define

$$|b|_p = \prod_{\mathfrak{p}|p} |b|_\mathfrak{p} \quad \text{for all} \quad b \in R$$

and we have a set of valuations $|\ |_p$ for which Axiom 1 holds.

LEMMA 5. *Let k be a field for which Axiom 1 holds and K a finite algebraic extension of k. Let $\mathfrak{N}$ be the set of all divisors $\mathfrak{P}$ of K that divide some $\mathfrak{p}$ of $\mathfrak{M}$. Then Axiom 1 holds in K for some subset $\mathfrak{N}'$ of $\mathfrak{N}$.*

(It would not be difficult to show now that $\mathfrak{N}'=\mathfrak{N}$, but it is better to postpone this and other details until the next section.)

PROOF. 1. Let $A\neq 0$ be an element of K and $f(x)=0$ the equation for A with coefficients in k and with highest coefficient 1. If $\mathfrak{p}$ is a nonarchimedean valuation for which all coefficients in $f(x)$ have a value not greater than 1 and $\mathfrak{P}$ a divisor of $\mathfrak{p}$, then $|A|_\mathfrak{P}\leqq 1$ or else no cancellation could take place between the highest term in $f(A)=0$ and the others. So $|A|_\mathfrak{P}\leqq 1$ for all but a finite number of $\mathfrak{P}$. For the same reason $|1/A|_\mathfrak{P}\leqq 1$ for all but a finite number of $\mathfrak{P}$. Therefore $|A|_\mathfrak{P}\neq 1$ for only a finite number of $\mathfrak{P}$.

2. Let $F(x)$ be a polynomial in k that has the generators of K among its roots ($F(x)$ need not be irreducible). If K' is the splitting field of $F(x)$ we may first prove Lemma 5 for K' instead of K and then descend to the subfield K by use of Lemma 4. This shows that we may already assume that K is the splitting field of a polynomial $F(x)$ in k.

The algebraic structure of such a field is well known. If $\mathfrak{G}$ is the group of all its automorphisms σ and if we construct for any $A\in K$ the product

$$\prod_\sigma A^\sigma = \alpha$$

then α is invariant under $\mathfrak{G}$. Since we have to consider also the inseparable case we do not know that α is in k. But there is always a positive integer m such that

$$\prod_{\sigma} (\mathrm{A}^{\sigma})^{m} = a$$

is in k whatever A may be. Because of the product formula in k we get

$$\prod_{\mathfrak{p}} | a |_{\mathfrak{p}} = 1.$$

Now select, for each $\mathfrak{p}$, one divisor $\mathfrak{P}$ of K which divides $\mathfrak{p}$ and define $| \ |_{\mathfrak{P}}$ in such a way that $|b|_{\mathfrak{P}} = |b|_{\mathfrak{p}}$ for all b in k. Then

$$| a |_{\mathfrak{p}} = \prod_{\sigma} | \mathrm{A}^{\sigma} |_{\mathfrak{P}}^{\prime m}.$$

If we consider the expression $|\mathrm{A}^{\sigma}|_{\mathfrak{P}}$ as function of A, it is clearly a valuation of K that belongs to a divisor $\mathfrak{P}'$ which divides $\mathfrak{p}$ and may be equal to or different from $\mathfrak{P}$. If we change our notation slightly we obviously get

$$| a |_{\mathfrak{p}} = \prod_{\mathfrak{P}'} | \mathrm{A} |_{\mathfrak{P}'}',$$

where $\mathfrak{P}'$ runs through some divisors of $\mathfrak{p}$ and where $| \ |_{\mathfrak{P}'}'$ is a certain valuation belonging to $\mathfrak{P}'$.

If we substitute this in our product-formula we get

$$\prod_{\mathfrak{p}} \prod_{\mathfrak{P}'} | \mathrm{A} |_{\mathfrak{P}'}' = 1$$

and this proves Lemma 5.

Before we proceed with our next lemma let us consider the special field $R = k_1(z)$ of Theorem 2. If p is a nontrivial valuation of R that reduces to the trivial one on k_1 then p is nonarchimedean and we distinguish two cases:

1. If $|z|_p \leqq 1$ then $|f(z)|_p \leqq 1$ for every polynomial in z. Let $p(z)$ be a polynomial of lowest degree such that $|p(z)|_p < 1$. If $g(z)$ is another polynomial with $|g(z)|_p < 1$ then we divide:

$$g(z) = p(z)h(z) + r(z),$$

where the degree of $r(z)$ is lower than that of $p(z)$. From

$$r(z) = g(z) - p(z)h(z)$$

we get $|r(z)|_p < 1$; hence $r(z) = 0$.

Now let $\phi(z)$ be any element of R and put

$$\phi(z) = p(z)^{\nu} \cdot \psi(z),$$

where neither numerator nor denominator of $\psi(z)$ is divisible by $p(z)$. Then $|\psi(z)|_p=1$ so

$$|\phi(z)|_p = |p(z)|_p^\nu = c^\nu, \quad \text{where} \quad c = |p(z)|_p < 1.$$

$p(z)$ is obviously irreducible.

In order to find the normed valuation $\| \ \|_p$ in this case we have to determine the degree of the residue class field (mod $p(z)$). It is the degree f of $p(z)$ so that $Np=q^f$ and

$$\|\phi(z)\|_p = q^{-\nu f}.$$

2. If $|z|_p>1$ we replace z by $y=1/z$. Then $|y|_p<1$ and we have our previous case. The polynomial in y of lowest degree is y itself, so that there is only one prime divisor p of this kind. We denote it by p_∞.

Let $\phi(z)=g(z)/h(z)$ where $g(z)$ and $h(z)$ are polynomials of degrees m and n. Then

$$\phi(z) = y^{n-m}.\frac{g_1(y)}{h_1(y)}$$

where $g_1(y)$ and $h_1(y)$ are polynomials not divisible by y. Hence

$$\|\phi(z)\|_{p_\infty} = q^{m-n}.$$

A product formula connecting all these valuations or a subset of them can be written in the form

$$\prod_p \|\phi(z)\|_p^{\lambda(p)} = 1,$$

where $\lambda(p)$ are constants not less than 0. If we substitute for $\phi(z)$ the irreducible polynomials $p(z)$, then only two factors can possibly be different from 1: the valuations at the p belonging to $p(z)$ and at p_∞. This gives

$$q^{-f\lambda(p_\infty)}\cdot q^{f\lambda(p)} = 1$$

or $\lambda(p)=\lambda(p_\infty)$. So all $\lambda(p)$ are equal and may therefore be assumed to be equal to 1. In order to show that this product formula holds we put

$$V(\phi(z)) = \prod_p \|\phi(z)\|_p.$$

It is obvious that $V(\phi(z)\cdot\psi(z))=V(\phi(z))\cdot V(\psi(z))$ and that a similar rule holds for quotients.

We have just seen that $V(p(z))=1$ for any irreducible polynomial; it follows that $V(\phi(z))=1$ for any element $\phi(z)\neq 0$ of R.

In the same fashion we can discuss the field R of rational numbers.

It is well known that all valuations are either the single archimedean p_∞ for which $\|a\|_{p_\infty}$ is the ordinary absolute value or the p-adic valuations of R where p is a prime number. The normed valuation $|a|_p$ in this latter case is given by $1/p^\nu$, if ν is the ordinal number of a. Just as before we consider a hypothetical product formula

$$\prod_p \|a\|_p^{\lambda(p)} = 1.$$

Substituting for a a prime number p we get

$$\left(\frac{1}{p}\right)^{\lambda(p)} \cdot p^{\lambda(p_\infty)} = 1$$

or $\lambda(p) = \lambda(p_\infty)$. The numbers $\lambda(p)$ are therefore equal and may be considered equal to 1. The same method as before shows that the product formula really holds.

LEMMA 6. *Axiom 1 holds in the case of the field R of rational numbers and that of $R = k_1(z)$. If R is the rational field, $\mathfrak{M}$ is the set of all valuations; if $R = k_1(z)$, it is the set of all valuations that are trivial on k_1. The product formula itself takes on the form*

$$\prod_p \|a\|_p = 1$$

or a power of it and there is no other relation between these valuations.

Lemmas 5 and 6 already show that Axiom 1 holds for the fields mentioned in Theorem 2. That all valuations of $\mathfrak{M}$ satisfy Axiom 2 follows from the fact that this is true in R and consequently in a finite extension k.

It remains to prove the statement about the field k_0 of constants. If $\mathfrak{p}$ is trivial on k_1 it is also trivial on an algebraic extension of k_1. Hence we need only show that any constant c of k_0 is algebraic with respect to k_1. If on the contrary c were transcendental with respect to k_1 then from the equation c satisfied with respect to $k_1(z)$ it follows that z would be algebraic with respect to $k_1(c)$. Since $k_1(c)$ is in k_0, this would mean that z is in k_0. So all of k would be in k_0, contradicting the fact that no $\mathfrak{p}$ of $\mathfrak{M}$ is trivial on k.

More detailed information about the fields of Theorem 2 will follow from the next section.

3. **Characterization of fields by the product formula.** In this section we assume k to be any field for which the Axioms 1 and 2 hold and are going to prove that k is of the type described in Theorem 2.

For any prime $\mathfrak{p}$ that satisfies Axiom 2 we shall have to distinguish the valuation $|\alpha|_{\mathfrak{p}}$ of Axiom 1 and the equivalent normed valuation $\|\alpha\|_{\mathfrak{p}}$. We define the real number $\rho(\mathfrak{p})>0$ by

$$|\alpha|_{\mathfrak{p}} = \|\alpha\|_{\mathfrak{p}}^{\rho(\mathfrak{p})}.$$

By R we mean the following subfield of k:

1. If $\mathfrak{M}$ has archimedean valuations, R is the rational field. By $\|a\|_{p_\infty}$ we mean the ordinary absolute value in R.

2. In the other case k has a field k_0 of constants and cannot contain any algebraic extension of k_0 since our valuations are trivial on k_0 and would be trivial on that extension. Let z be any element of k not in k_0; then $R=k_0(z)$ is a transcendental extension of k_0. By integers we mean in this case the polynomials in z. By $\|a\|_{p_\infty}$ we mean the one valuation we found in proving Lemma 6 that has $\|z\|_{p_\infty}>1$.

In both cases we mean by $\mathfrak{p}_\infty$ any divisor of $\mathfrak{M}$ that divides p_∞. Since the product formula, when applied to elements of R, must reduce to the formula of Theorem 2, our set $\mathfrak{M}$ always contains at least one $\mathfrak{p}_\infty$. The other primes of $\mathfrak{M}$ shall be called finite. For elements a of R the valuations $|a|_{\mathfrak{p}_\infty}$ and $\|a\|_{p_\infty}$ are equivalent. We define the real numbers $\lambda(\mathfrak{p}_\infty)>0$ by

$$|a|_{\mathfrak{p}_\infty} = \|a\|_{p_\infty}^{\lambda(\mathfrak{p}_\infty)} \text{ for all } a \in R.$$

Lemma 7. *Let $\mathfrak{q}$ be one of the primes satisfying Axiom 2 and $\mathfrak{S}$ be a set of elements of k of an order $M>1$. Let x be an upper bound for $|\alpha|_{\mathfrak{q}}$ for all α of $\mathfrak{S}$: $|\alpha|_{\mathfrak{q}}\leqq x$. Then there is an element θ of k with the following properties:*

(1) *θ is either a difference of two elements of $\mathfrak{S}$ or, in case there is a field of constants, a linear combination of elements of $\mathfrak{S}$ with coefficients in k_0.*

(2) *$\theta\neq 0$.*

(3) *$|\theta|_{\mathfrak{q}}\leqq A_{\mathfrak{q}}x/M^{\rho(\mathfrak{q})}$ where $A_{\mathfrak{q}}$ is a constant depending only on $\mathfrak{q}$.*

Proof. 1. $\mathfrak{q}$ archimedean, $k_{\mathfrak{q}}$ real. In this case we may treat k as a subfield of the real field. We have

$$\|\alpha\|_{\mathfrak{q}} \leqq x^{1/\rho(\mathfrak{q})}$$

for each of the M elements α of $\mathfrak{S}$. Divide the interval from $-x^{1/\rho(\mathfrak{q})}$ to $x^{1/\rho(\mathfrak{q})}$ into $M-1$ equal parts. Two of the α's must be in the same compartment so their difference θ satisfies

$$\|\theta\|_{\mathfrak{q}} \leqq \frac{2x^{1/\rho(\mathfrak{q})}}{M-1} \leqq \frac{4x^{1/\rho(\mathfrak{q})}}{M},$$

hence

$$|\theta|_{\mathfrak{q}} \leqq \frac{4^{\rho(\mathfrak{q})}x}{M^{\rho(\mathfrak{q})}}.$$

2. $\mathfrak{q}$ archimedean, $k_{\mathfrak{q}}$ complex. Treating k as a subfield of the complex field, $\|\alpha\|_{\mathfrak{q}}^{1/2} = |\alpha|_{\mathfrak{q}}^{1/2\rho(\mathfrak{q})}$ is the ordinary distance from the origin to the point α and is less than $x^{1/2\rho(\mathfrak{q})}$. Writing $\alpha = \xi + i\eta$ for each $\alpha \in \mathfrak{S}$ we know that $\mathfrak{S}$ is in the square $|\xi| \leqq x^{1/2\rho(\mathfrak{q})}$ $|\eta| \leqq x^{1/2\rho(\mathfrak{q})}$. Divide this square into N^2 small squares by dividing each side into N equal parts, where $N < M^{1/2} \leqq N+1$. Then some two α's are in the same subdivision so their difference θ satisfies:

$$\|\theta\|_{\mathfrak{q}}^{1/2} \leqq \frac{2^{3/2}x^{1/2\rho(\mathfrak{q})}}{N} \leqq \frac{2^{5/2}x^{1/2\rho(\mathfrak{q})}}{M^{1/2}}$$

so

$$|\theta|_{\mathfrak{q}} \leqq \frac{(2^{5/2})^{2\rho(\mathfrak{q})}x}{M^{\rho(\mathfrak{q})}}.$$

3. $\mathfrak{q}$ discrete. Let α_1 be an α for which $|\alpha|_{\mathfrak{q}}$ is maximum. This exists since $\mathfrak{q}$ is discrete and $|\alpha|_{\mathfrak{q}} \leqq x$. Then for each $\alpha \in \mathfrak{S}$, $|\alpha/\alpha_1|_{\mathfrak{q}} \leqq 1$.

Choose r so that $N\mathfrak{q}^r < M \leqq N\mathfrak{q}^{r+1}$. If the number of elements in the residue class field is finite then the local theory shows easily that $\mathfrak{o}_{\mathfrak{q}}$ contains at most $N\mathfrak{q}^r$ residue classes mod $\mathfrak{q}^r$. Hence two of the $M > N\mathfrak{q}^r$ elements of $(1/\alpha_1)\mathfrak{S}$ are in the same residue class and their difference θ/α_1 has at least the ordinal number r. Should there be a field k_0, let f be the degree of $\mathfrak{o}_{\mathfrak{q}}/\mathfrak{q}$ over k_0, so that $N\mathfrak{q} = q^f$. Then there are at most rf elements of $\mathfrak{o}_{\mathfrak{q}}$ that are linearly independent mod $\mathfrak{q}^r$. Taking more than rf of our elements α/α_1 that are independent considered as elements of k (possible since $M > N\mathfrak{q}^r$) we can find a linear combination $\theta/\alpha_1 \neq 0$ of them that is congruent to 0 (mod $\mathfrak{q}^r$) and hence has at least the ordinal number r. In both cases we get

$$\left\|\frac{\theta}{\alpha_1}\right\|_{\mathfrak{q}} \leqq \frac{1}{N\mathfrak{q}^r} = \frac{N\mathfrak{q}}{N\mathfrak{q}^{r+1}} \leqq \frac{N\mathfrak{q}}{M}, \qquad \|\theta\|_{\mathfrak{q}} \leqq \frac{N\mathfrak{q}\cdot\|\alpha_1\|_{\mathfrak{q}}}{M},$$

or

$$|\theta|_{\mathfrak{q}} \leqq \frac{N\mathfrak{q}^{\rho(\mathfrak{q})}|\alpha_1|_{\mathfrak{q}}}{M^{\rho(\mathfrak{q})}} \leqq \frac{N\mathfrak{q}^{\rho(\mathfrak{q})}\cdot x}{M^{\rho(\mathfrak{q})}}.$$

LEMMA 8. *Let M be the order of the set of elements $\alpha \in k$ that is contained in a parallelotope of dimensions $x_{\mathfrak{p}}$. If $\mathfrak{q}$ is a prime satisfying Axiom 2 we can find a constant $B_{\mathfrak{q}}$ depending only on $\mathfrak{q}$ such that either $M = 1$ (if our set contains only $\alpha = 0$) or*

$$M \leqq B_{\mathfrak{q}}\Big(\prod_{\mathfrak{p}} x_{\mathfrak{p}}\Big)^{1/\rho(\mathfrak{q})}.$$

PROOF. Assume $M>1$. By Lemma 7 there is a $\theta\neq 0$ satisfying

$$|\theta|_{\mathfrak{q}} \leqq \frac{A_{\mathfrak{q}}x_{\mathfrak{q}}}{M^{\rho(\mathfrak{q})}}.$$

For the other $\mathfrak{p}$ of $\mathfrak{M}$ we estimate θ directly and get

$$|\theta|_{\mathfrak{p}} \leqq \begin{cases} x_{\mathfrak{p}} \text{ at any nonarchimedean } \mathfrak{p}, \\ 4^{\rho(\mathfrak{p})}\cdot x_{\mathfrak{p}} \text{ at any archimedean } \mathfrak{p}. \end{cases}$$

Substituting in the product formula $\prod_{\mathfrak{p}}|\theta|_{\mathfrak{p}}=1$ we get (if $D_{\mathfrak{q}}$ is a certain constant):

$$1 \leqq \frac{D_{\mathfrak{q}}\cdot \prod_{\mathfrak{p}} x_{\mathfrak{p}}}{M^{\rho(\mathfrak{q})}},$$

hence the lemma.

LEMMA 9. *If $\alpha_1, \alpha_2, \cdots, \alpha_l$ are linearly independent with respect to the subfield R and if y is a given nonzero integer of R, we can construct a certain set $\mathfrak{S}$ of elements α with the following properties:*

1. $|\alpha|_{\mathfrak{p}} \leqq a_{\mathfrak{p}} = \max_{i=1,\ldots,l}(|\alpha_i|_{\mathfrak{p}})$ *for every finite* $\mathfrak{p}$.

2. $|\alpha|_{\mathfrak{p}_\infty} \leqq B\cdot|y|_{\mathfrak{p}_\infty}$ *with a certain constant B that can be easily estimated.*

3. *If there is a field of constants k_0 then $\mathfrak{S}$ is closed under addition and under multiplication by elements of k_0, so $\mathfrak{S}$ may be considered as a vector space over k_0.*

4. *The order of $\mathfrak{S}$ is greater than $\|y\|^l_{p_\infty}$.*

PROOF. Let $\mathfrak{S}$ consist of all α of the form

$$\alpha = \nu_1\alpha_1 + \nu_2\alpha_2 + \cdots + \nu_l\alpha_l$$

where the ν_i range over all integers of R that satisfy

$$\|\nu_i\|_{p_\infty} \leqq \|y\|_{p_\infty}.$$

This settles at once property 3 and implies $|\nu_i|_{\mathfrak{p}_\infty} \leqq |y|_{\mathfrak{p}_\infty}$ for each $\mathfrak{p}_\infty$ and consequently property 2. Property 1 holds since $|\nu_i|_{\mathfrak{p}}\leqq 1$ for all finite $\mathfrak{p}$. Property 4 is clear if p_∞ is archimedean; if not then assume $\|y\|_{p_\infty}=q^d$ so that y is a polynomial of degree d. Each ν_i ranges over all polynomials of degree not greater than d. This gives for $\mathfrak{S}$ a vector space of $(d+1)l$ dimensions and our statement is obvious.

LEMMA 10. *The degree n of k over R is finite; every $\mathfrak{p}$ of M satisfies Axiom 2; and the inequality*

$$n \leqq \frac{1}{\rho(\mathfrak{p})} \cdot \sum_{\mathfrak{p}_\infty} \lambda(\mathfrak{p}_\infty)$$

holds for each p.

PROOF. Apply Lemma 8 to the set of Lemma 9. We get the inequality

$$\|y\|^{l}_{p_\infty} \leqq E \cdot \prod_{\mathfrak{p}_\infty} |y|^{1/\rho(\mathfrak{q})}_{\mathfrak{p}_\infty} = E \cdot \|y\|_{p_\infty}^{(1/\rho(\mathfrak{q}))\Sigma_{\mathfrak{p}_\infty}\lambda(\mathfrak{p}_\infty)},$$

where E is a certain constant that depends on the constants in the previous lemmas. Since $\|y\|_{p_\infty}$ takes on arbitrarily large values we get

$$l \leqq \frac{1}{\rho(\mathfrak{q})} \sum_{\mathfrak{p}_\infty} \lambda(\mathfrak{p}_\infty).$$

This proves that n is finite. None of our valuations $\mathfrak{p}$ is trivial on R or else it would be trivial on the finite extension k. Let p be the divisor of R that is divisible by $\mathfrak{p}$. The local theory shows now (since p is nontrivial) that $\mathfrak{p}$ satisfies Axiom 2. In our previous inequality we can therefore assume $l = n$ and take for $\mathfrak{q}$ each prime $\mathfrak{p}$ of $\mathfrak{M}$.

Let r be a positive real number and let us replace each valuation $|\alpha|_\mathfrak{p}$ by its rth power $|\alpha|^r_\mathfrak{p}$. This would be a new set of valuations for which Axioms 1 and 2 would hold again. The numbers $\lambda(\mathfrak{p}_\infty)$ would then be replaced by $r\lambda(\mathfrak{p}_\infty)$. This shows that it is no restriction of generality to assume that

$$\sum_{\mathfrak{p}_\infty} \lambda(\mathfrak{p}_\infty) = n.$$

Then Lemma 10 gives

$$\rho(\mathfrak{p}) \leqq 1$$

for every $\mathfrak{p}$.

Assume now that $\mathfrak{p} | p$, where p is a nonarchimedean divisor of R, and let us compare $\|a\|_\mathfrak{p}$ and $\|a\|_p$ for elements a of R. The ordinal number of a in k is $e(\mathfrak{p})$ times the ordinal number of a measured in R; we also have $N\mathfrak{p} = (Np)^{f(\mathfrak{p})}$. $e(\mathfrak{p})$ is the ramification number and $f(\mathfrak{p})$ the degree of the residue class fields. So

$$\|a\|_\mathfrak{p} = \|a\|_p^{e(\mathfrak{p})f(\mathfrak{p})} = \|a\|_p^{n(\mathfrak{p})}.$$

For an archimedean $\mathfrak{p}$ this equality follows directly from the definitions. Hence we have

$$|a|_{\mathfrak{p}} = \|a\|_{p}^{n(\mathfrak{p})\rho(\mathfrak{p})} \text{for all } \mathfrak{p} \text{ and all } a \in R.$$

We note in particular $\lambda(\mathfrak{p}_\infty) = n(\mathfrak{p}_\infty)\rho(\mathfrak{p}_\infty)$ so that

$$\sum_{\mathfrak{p}_\infty} n(\mathfrak{p}_\infty)\rho(\mathfrak{p}_\infty) = n.$$

Now we apply the product formula to an $a \in R$:

$$1 = \prod_{\mathfrak{p}} |a|_{\mathfrak{p}} = \prod_{p}\left(\prod_{\mathfrak{p}|p,\,\mathfrak{p}\in\mathfrak{M}} \|a\|_{p}^{n(\mathfrak{p})\rho(\mathfrak{p})}\right) = \prod_{p} \|a\|_{p}^{\nu(p)},$$

where

$$\nu(p) = \sum_{\mathfrak{p}|p,\,\mathfrak{p}\in\mathfrak{M}} n(\mathfrak{p})\rho(\mathfrak{p}).$$

But Lemma 6 shows that all $\nu(p)$ are equal and the special case $p = p_\infty$ shows finally

$$n = \sum_{\mathfrak{p}|p,\,\mathfrak{p}\in\mathfrak{M}} n(\mathfrak{p})\rho(\mathfrak{p}).$$

If we compare this with $\rho(\mathfrak{p}) \leqq 1$ and $\sum_{\mathfrak{p}|p} n(\mathfrak{p}) \leqq n$ we find:
(1) all $\mathfrak{p}|p$ are in $\mathfrak{M}$,
(2) all $\rho(\mathfrak{p}) = 1$,
(3) $\sum_{\mathfrak{p}|p} n(\mathfrak{p}) = n$.
Thus we have proved:

THEOREM 3. *If k is a field that satisfies the Axioms 1 and 2 it is an extension of a finite degree n either of the rational field R or of the field $R = k_0(z)$ of rational functions over its field of constants k_0. All valuations satisfy Axiom 2. $\mathfrak{M}$ consists of all extensions of the well known valuations of R. Replacing if necessary all valuations in the product formula by the same power we can assume that they are all normed (that is, $|\alpha|_{\mathfrak{p}} = \|\alpha\|_{\mathfrak{p}}$). We have $\sum_{\mathfrak{p}|p} n_{\mathfrak{p}} = n$ for all p of R.*

Let now α be an element of k and $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r$ those among the finite primes for which $\|\alpha\|_{\mathfrak{p}_\nu} > 1$. Let $p_1, p_2, \cdots, p_l$ be the primes of R that have the $\mathfrak{p}_\nu$ as divisors. Construct an integer in R whose absolute value at each p_ν is sufficiently small and $a\alpha$ will now be less than or equal to 1 at all finite primes. This shows that any set $\alpha_1, \alpha_2, \cdots, \alpha_l$ of elements that are linearly independent with respect to R can be replaced by a set of elements which are integral at all finite primes. This will be useful in the next section.

4. **Parallelotopes.** We still make the same assumptions as in the previous chapter so that we can assume Theorem 3. Thus we will take $|\alpha|_{\mathfrak{p}} = \|\alpha\|_{\mathfrak{p}}$ and $\rho(\mathfrak{p}) = 1$. In Lemma 9 we may assume $l = n$.

THEOREM 4. *There are two positive constants C and D such that for all idèles $\mathfrak{a}$ we have*

$$CV(\mathfrak{a}) < M(\mathfrak{a}) \leqq \max\,(1, DV(\mathfrak{a})).$$

PROOF. If we apply Lemma 8 for one particular $\mathfrak{q}$ we get the right half of the inequality.

If we replace $\mathfrak{a}$ by $\alpha\mathfrak{a}$ then $V(\mathfrak{a})$ and $M(\mathfrak{a})$ remain unchanged. Select $\alpha = \alpha_1 y$ where α_1 and y are selected as follows:

Theorem 1 shows that there is an α_1 such that

$$4B \leqq \|\alpha_1\mathfrak{a}\|_{\mathfrak{p}_\infty} \leqq 5B \quad \text{for all} \quad \mathfrak{p}_\infty,$$

where B is the constant of Lemma 9. We choose such an α_1 and then select an integer y of R in such a way that $\|\alpha_1 y\mathfrak{a}\|_{\mathfrak{p}} \leqq 1$ at all finite $\mathfrak{p}$. This shows that it is sufficient to prove our theorem for all idèles $\mathfrak{a}$ satisfying

$$4B\|y\|_{\mathfrak{p}_\infty} \leqq \|\mathfrak{a}\|_{\mathfrak{p}_\infty} \leqq 5B\|y\|_{\mathfrak{p}_\infty} \quad \text{for all} \quad \mathfrak{p}_\infty$$

and $\|\mathfrak{a}\|_{\mathfrak{p}} \leqq 1$ at all finite $\mathfrak{p}$, where y is an integer of R. Using this integer y, we now apply Lemma 9, taking $a_{\mathfrak{p}} = 1$ at each finite $\mathfrak{p}$ and constructing a set $\mathfrak{S}$ of elements α of k with the following properties:

1. $\|\alpha\|_{\mathfrak{p}} \leqq 1$ for all finite $\mathfrak{p}$.
2. $\|\alpha\|_{\mathfrak{p}_\infty} \leqq B\|y\|_{\mathfrak{p}_\infty}$.
3. In case there is a field k_0, $\mathfrak{S}$ is a vectorspace over k_0.
4. The order of $\mathfrak{S}$ is greater than $\|y\|^n_{p_\infty}$ and we have

$$\|y\|^n_{p_\infty} = \|y\|^{\Sigma_{\mathfrak{p}_\infty}\lambda(\mathfrak{p}_\infty)}_{p_\infty} = \prod_{\mathfrak{p}_\infty}\|y\|_{\mathfrak{p}_\infty} \geqq \prod_{\mathfrak{p}_\infty}\frac{1}{5B}\|\mathfrak{a}\|_{\mathfrak{p}_\infty}.$$

So the order is greater than $C\prod_{\mathfrak{p}_\infty}\|a\|_{\mathfrak{p}_\infty}$ where C is a certain constant not equal to 0.

We distinguish two cases:

1. Order of a set means number. Consider the set $\mathfrak{o}$ of all integers of k (that is, all elements that are integers for every finite $\mathfrak{p}$) and the subset $\{\mathfrak{a}\}$ of all integers β satisfying $\|\beta\|_{\mathfrak{p}} \leqq \|\mathfrak{a}\|_{\mathfrak{p}}$ for all finite $\mathfrak{p}$. This subset $\{\mathfrak{a}\}$ forms an additive group which is the intersection of all the groups $\{\mathfrak{a}\}_{\mathfrak{p}}$ of integers satisfying $\|\beta\|_{\mathfrak{p}} \leqq \|\mathfrak{a}\|_{\mathfrak{p}}$ for only this particular $\mathfrak{p}$. The local theory shows that the index of $\mathfrak{o}$ mod $\{\mathfrak{a}\}_{\mathfrak{p}}$ is at most $1/\|\mathfrak{a}\|_{\mathfrak{p}}$. So the index of $\{\mathfrak{a}\}$ in the group of all integers is at most $N = \prod_{\mathfrak{p}\,\text{fin}}(1/\|\mathfrak{a}\|_{\mathfrak{p}})$. If we consider now our set $\mathfrak{S}$ modulo $\{\mathfrak{a}\}$ we get at most N residue classes. So one residue class contains more than

$$\frac{C\cdot\prod\|\mathfrak{a}\|_{\mathfrak{p}_\infty}}{N} = CV(\mathfrak{a})$$

elements. If we select one special element of this residue class and subtract it from each of the others, we get more than $CV(\mathfrak{a})$ elements γ of $\{\mathfrak{a}\}$. As such they satisfy $\|\gamma\|_{\mathfrak{p}} \leqq \|\mathfrak{a}\|_{\mathfrak{p}}$ for all finite p. At a $\mathfrak{p}_\infty$ we get $\|\gamma\|_{\mathfrak{p}_\infty} \leqq 4B\|y\|_{\mathfrak{p}_\infty} \leqq \|\mathfrak{a}\|_{\mathfrak{p}_\infty}$. (The factor 4 instead of the expected 2 must be used since one of the valuations may be the square of a true valuation.) So we have found more than $CV(\mathfrak{a})$ elements in our parallelotope of size $\mathfrak{a}$.

2. There is a constant field k_0. We define $\{\mathfrak{a}\}$ and $\{\mathfrak{a}\}_{\mathfrak{p}}$ as before. Assume that m is the dimension of the vector space $\mathfrak{S}$ over k_0. The residue classes of the integers mod $\{\mathfrak{a}\}_{\mathfrak{p}}$ also form a vector space over k_0; let $d(\mathfrak{p})$ denote its dimension. Then $q^{d(\mathfrak{p})} \leqq 1/\|\mathfrak{a}\|_{\mathfrak{p}}$. Let $\mathfrak{S}_1$ be the intersection of $\mathfrak{S}$ and $\{\mathfrak{a}\}_{\mathfrak{p}_1}$. Starting with a basis for the space $\mathfrak{S}_1$, we need at most $d(\mathfrak{p}_1)$ vectors to complete it to a basis of $\mathfrak{S}$. So the dimension of $\mathfrak{S}_1$ is at least $m - d(\mathfrak{p}_1)$. Repeating this process for all finite $\mathfrak{p}$ and calling $\mathfrak{T}$ the intersection of $\mathfrak{S}$ and $\{\mathfrak{a}\}$, we see that its dimension is at least $m - \sum_{\mathfrak{p}\text{ fin}} d(\mathfrak{p})$. So the order of $\mathfrak{T}$ is at least

$$q^m \cdot \prod_{\mathfrak{p}\text{ fin}} \frac{1}{q^{d(\mathfrak{p})}} \geqq q^m \cdot \prod_{\mathfrak{p}\text{ fin}} \|\mathfrak{a}\|_{\mathfrak{p}}.$$

Since the order q^m of $\mathfrak{S}$ is $C\prod_{\mathfrak{p}_\infty} \|\mathfrak{a}\|_{\mathfrak{p}_\infty}$ we find that the order of $\mathfrak{T}$ is greater than $CV(\mathfrak{a})$. That the elements γ of $\mathfrak{T}$ satisfy $\|\gamma\|_{\mathfrak{p}} \leqq \|\mathfrak{a}\|_{\mathfrak{p}}$ follows for a finite $\mathfrak{p}$ from the fact that they are in $\{\mathfrak{a}\}$. For $\mathfrak{p}_\infty$, since they are in $\mathfrak{S}$,

$$\|\gamma\|_{\mathfrak{p}_\infty} \leqq B\|y\|_{\mathfrak{p}_\infty} \leqq \|\mathfrak{a}\|_{\mathfrak{p}_\infty}.$$

COROLLARY. *If* $V(\mathfrak{a}) \geqq 1/C$ *then there is a* β *in* k *such that*

$$1 \leqq \|\beta\mathfrak{a}\|_{\mathfrak{p}} \leqq V(\mathfrak{a}) \quad \textit{for all} \quad \mathfrak{p}.$$

PROOF. The field elements in the parallelotope of size $\mathfrak{a}$ form a set of order greater than 1 so there is an $\alpha \neq 0$ such that $\|\alpha\|_{\mathfrak{p}} \leqq \|\mathfrak{a}\|_{\mathfrak{p}}$ for all $\mathfrak{p}$. Put $\beta = 1/\alpha$; then $1 \leqq \|\beta\mathfrak{a}\|_{\mathfrak{p}}$. Now for each $\mathfrak{q}$

$$\|\beta\mathfrak{a}\|_{\mathfrak{q}} = \frac{V(\beta\mathfrak{a})}{\prod_{\mathfrak{p}\neq\mathfrak{q}} \|\beta\mathfrak{a}\|_{\mathfrak{p}}} \leqq V(\beta\mathfrak{a}) = V(\mathfrak{a}).$$

LEMMA 11. *Let* $\mathfrak{a}$ *be any idèle and* $\mathfrak{q}$ *a fixed prime; then there is a* β *in* k *such that*

$$1 \leqq \|\beta\mathfrak{a}\|_{\mathfrak{p}} \leqq N\mathfrak{q}/C \quad \textit{for} \quad \mathfrak{p} \neq \mathfrak{q};$$

$$(C/N\mathfrak{q})V(\mathfrak{a}) \leqq \|\beta\mathfrak{a}\|_{\mathfrak{q}} \leqq V(\mathfrak{a}).$$

For an archimedean prime $\mathfrak{q}$ we mean by $N\mathfrak{q}$ the number 1. C is the constant of the preceding corollary.

PROOF. If we replace in $\mathfrak{a}$ the component $\alpha_{\mathfrak{q}}$ by a suitable $\alpha_{\mathfrak{q}}'$ and leave all other components unchanged we can achieve that the new idèle $\mathfrak{a}'$ satisfies

$$1/C \leqq V(\mathfrak{a}') \leqq N\mathfrak{q}/C.$$

Then we determine the β of our corollary and get

$$1 \leqq \|\beta\mathfrak{a}\|_{\mathfrak{p}} \leqq V(\mathfrak{a}') \leqq N\mathfrak{q}/C \quad \text{for} \quad \mathfrak{p} \neq \mathfrak{q}$$

and

$$1 \leqq \|\beta\mathfrak{a}'\|_{\mathfrak{q}} \leqq V(\mathfrak{a}').$$

Now

$$\|\beta\mathfrak{a}\|_{\mathfrak{q}} = \frac{V(\beta\mathfrak{a})}{V(\beta\mathfrak{a}')}\cdot\|\beta\mathfrak{a}'\|_{\mathfrak{q}} = \frac{V(\mathfrak{a})}{V(\mathfrak{a}')}\|\beta\mathfrak{a}'\|_{\mathfrak{q}}.$$

Hence

$$(C/N\mathfrak{q})V(\mathfrak{a}) \leqq V(\mathfrak{a})/V(\mathfrak{a}') \leqq \|\beta\mathfrak{a}\|_{\mathfrak{q}} \leqq V(\mathfrak{a}).$$

Let now U be the multiplicative group of all absolute units, that is, the set of all ζ of k satisfying $\|\zeta\|_{\mathfrak{p}}=1$ for all $\mathfrak{p}$. In case there is a constant field k_0, our group consists of the elements not equal to 0 of k_0. In case order means number of elements, U must be a finite group since it is contained in the parallelotope of size 1; so U consists in this case of all roots of unity of k and is a finite cyclic group.

We select a finite non-empty set S of primes $\mathfrak{p}$ that contains at least all archimedean primes. By $\mathfrak{a}_S$ we mean the idèles satisfying $\|\mathfrak{a}_S\|_{\mathfrak{p}}=1$ for all $\mathfrak{p}$ not in S. An element ϵ_S of k that belongs to $\mathfrak{a}_S$ is called an S-unit.

Let $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s$ be the primes of S. If ϵ_S is an S-unit and we know the s positive numbers $\|\epsilon_S\|_{\mathfrak{p}_1}, \|\epsilon_S\|_{\mathfrak{p}_2}, \cdots, \|\epsilon_S\|_{\mathfrak{p}_s}$, then we know $\|\epsilon_S\|_{\mathfrak{p}}$ for all $\mathfrak{p}$, so $\|\epsilon_S\|$ is known except for a factor in U. Let us call two S-units equivalent if they differ only by a factor in U. The product formula gives

$$\prod_{\nu=1}^{s}\|\epsilon_S\|_{\mathfrak{p}_\nu} = 1$$

and shows that it suffices to know the $s-1$ numbers $\|\epsilon_S\|_{\mathfrak{p}_1}, \|\epsilon_S\|_{\mathfrak{p}_2}, \cdots, \|\epsilon_S\|_{\mathfrak{p}_{s-1}}$. (Should $s=1$ then ϵ_S is already in U as the product formula shows.)

It is more convenient to take the logarithms of our numbers so we map the unit ϵ_S onto the following vector $v(\epsilon_S)$ of an ordinary space R_{s-1} of $s-1$ dimensions:

$$v(\epsilon_S) = (\log \|\epsilon_S\|_{\mathfrak{p}_1}, \log \|\epsilon_S\|_{\mathfrak{p}_2}, \cdots, \log \|\epsilon_S\|_{\mathfrak{p}_{s-1}}).$$

We have then for two units ϵ_S and η_S the relation

$$v(\epsilon_S\eta_S) = v(\epsilon_S) + v(\eta_S).$$

So the maps $v(\epsilon_S)$ form an additive group of vectors in R_{s-1}.

The product formula gives

$$\log \|\epsilon_S\|_{\mathfrak{p}_s} = -\sum_{\nu=1}^{s-1} \log \|\epsilon_S\|_{\mathfrak{p}_\nu}.$$

Let us consider a bounded region in R_{s-1} that gives bounds for $\log \|\epsilon_S\|_{\mathfrak{p}_\nu}$ $(\nu=1, 2, \cdots, s-1)$, say

$$-K \leqq \log \|\epsilon_S\|_{\mathfrak{p}_\nu} \leqq K \qquad (\nu = 1, 2, \cdots, s-1).$$

Then we get for $\log \|\epsilon_S\|_{\mathfrak{p}_s}$ the bounds $-(s-1)K \leqq \log \|\epsilon_S\|_{\mathfrak{p}_s} \leqq (s-1)K$.

In case all the $\mathfrak{p}_\nu$ of S are discrete this gives only a finite number of possibilities for the ordinal number at each $\mathfrak{p}_\nu$; hence only a finite number of units inequivalent mod U. If there are archimedean primes in S then all ϵ_S of our region are contained in a parallelotope, so their order is finite. But order means number in this case. So we have proved:

LEMMA 12. *There are only a finite number of vectors $v(\epsilon_S)$ in a bounded region of R_{s-1}.*

The following lemma is well known; we repeat its proof here for the convenience of the reader.

LEMMA 13. *Let G be an additive group of vectors in an ordinary euclidean n-space R_n, such that no bounded region of R_n contains an infinite number of vectors of G. Assume that we can find m but not more vectors of G that are linearly independent with respect to real numbers. Then these m vectors may be selected in such a fashion that any vector of G is a linear combination of them with integral coefficients. In other words: G is a lattice of dimension m.*

PROOF. The proof is by induction according to m.

Let $v_1, v_2, \cdots, v_m$ be a maximal set of independent vectors and G_0 be the subgroup of G contained in the subspace spanned by the vectors $v_1, v_2, \cdots, v_{m-1}$. Because of induction we may already assume that any vector in G_0 is a linear integral combination of $v_1, v_2, \cdots, v_{m-1}$.

Consider the subset $\mathfrak{S}$ of all v of G of the form

$$v = x_1v_1 + x_2v_2 + \cdots + x_{m-1}v_{m-1} + x_mv_m$$

with real coefficients $x_1, x_2, \cdots, x_m$ that satisfy

$$0 \leqq x_i < 1 \qquad \text{for } i = 1, 2, \cdots, m-1$$

and

$$0 \leqq x_m \leqq 1.$$

It is a bounded set. Let v'_m be a vector of S with the smallest possible $x_m \neq 0$, say

$$v'_m = \xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_m v_m.$$

Starting now with any vector v of G we can select integral coefficients $y_1, y_2, \cdots, y_m$ in such a way that

$$v' = v - y_m v'_m - y_1 v_1 - y_2 v_2 - \cdots - y_{m-1} v_{m-1}$$

is in $\mathfrak{S}$ and the coefficient of v_m is even less than ξ_m. So this coefficient of v_m is 0, that is, v' is in G_0. So v' is an integral linear combination of $v_1, v_2, \cdots, v_{m-1}$ and therefore v is an integral linear combination of $v_1, v_2, \cdots, v_{m-1}$ and v'_m.

THEOREM 5. *The vectors $v(\epsilon_S)$ form a lattice of at most $s-1$ dimensions. The ϵ_S themselves form* mod U *a free abelian group with at most $s-1$ generators.*

5. **A more restrictive axiom.** If we wish to derive stronger theorems as for instance that of the existence of enough units, we must replace Axiom 2 by a stronger axiom. So we assume from now on that we have in k besides Axiom 1 also

AXIOM 2a. *There is at least one prime in $\mathfrak{M}$ that is either archimedean, with the real field or the complex field as its completed field, or else discrete with residue class field having only a finite number of elements.*

Since Axiom 2 is a consequence of Axiom 2a we can assume all the results we derived thus far and thus we see that k is either a number field or else a function-field where k_0 has only a finite number of elements. We see immediately that Axiom 2a holds for all primes of $\mathfrak{M}$.

LEMMA 14. *To any integer M there are only a finite number of primes $\mathfrak{p}$ with $N\mathfrak{p} \leqq M$.*

PROOF. Since there are only a finite number of archimedean primes we are concerned only with the nonarchimedean ones. Consider $M+1$ integers α_ν of R and let $\mathfrak{p}$ be a prime with $N\mathfrak{p} \leqq M$. Two α_i are in the same residue class, say α_1 and α_2; hence $|\alpha_1 - \alpha_2|_{\mathfrak{p}} < 1$. So our $\mathfrak{p}$'s are contained among the primes for which one of the differences $\alpha_i - \alpha_k$ $(i \neq k)$ has an absolute value $|\alpha_i - \alpha_k|_{\mathfrak{p}} < 1$. Because of Axiom 1 our lemma holds.

Now let S be again a finite and non-empty set of primes.

LEMMA 15. *There is a constant E such that to any idèle $\mathfrak{a}_S$ and any prime $\mathfrak{q}$ of S we can find an S-unit ϵ_S such that*

$$\|\epsilon_S\mathfrak{a}_S\|_\mathfrak{p} \leqq E \quad \textit{for all } \mathfrak{p} \neq \mathfrak{q} \textit{ of } S.$$

PROOF. Select β according to Lemma 11. Then $1\leqq\|\beta\mathfrak{a}_S\|_\mathfrak{p}\leqq N\mathfrak{q}/C$ for $\mathfrak{p}\neq\mathfrak{q}$. So

$$1 \leqq \|\beta\|_\mathfrak{p} \leqq N\mathfrak{q}/C \quad \text{for all } \mathfrak{p} \text{ not in } S.$$

If $\|\beta\|_\mathfrak{p}\neq 1$, then $\|\beta\|_\mathfrak{p}\geqq N\mathfrak{p}$. Since there are only a finite number $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_l$ of primes with $N\mathfrak{p}_i\leqq N\mathfrak{q}/C$ we get $\|\beta\|_\mathfrak{p}=1$ for all $\mathfrak{p}\neq\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_l$ and $\mathfrak{p}$ not in S. Since $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_l$ are discrete we get only a finite number of possibilities for each $\|\beta\|_{\mathfrak{p}_i}$.

Assume that $\beta_1, \beta_2, \cdots, \beta_r$ already realize any possible distribution of values for $\|\beta\|_{\mathfrak{p}_i}$. Then to any of our β there is a β_k with $\|\beta\|_\mathfrak{p}=\|\beta_k\|_\mathfrak{p}$ for all $\mathfrak{p}$ not in S, or $\beta=\beta_k\epsilon_S$. Substituting back we get

$$\|\beta_k\epsilon_S\mathfrak{a}_S\|_\mathfrak{p} \leqq N\mathfrak{q}/C \quad \text{for} \quad \mathfrak{p}\neq\mathfrak{q}.$$

So $\|\epsilon_S\mathfrak{a}_S\|_\mathfrak{p}\leqq E$ for all $\mathfrak{p}\neq\mathfrak{q}$ of S where

$$E = \max_{\nu=1,2,\cdots,r;\, \mathfrak{p}\in S}\left(\frac{N\mathfrak{q}}{C\|\beta_\nu\|_\mathfrak{p}}\right).$$

Now select an $\mathfrak{a}_S$ so that $\|\mathfrak{a}_S\|_\mathfrak{p}>E$ for all $\mathfrak{p}$ of S. If ϵ_S is the corresponding unit then $\|\epsilon_S\|_\mathfrak{p}<1$ for all $\mathfrak{p}\neq\mathfrak{q}$ of S.

Assume now that $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s$ are all the primes in S. Then $\mathfrak{q}$ could be any of the primes $\mathfrak{p}_i$. We get in this fashion s S-units $\epsilon_1, \epsilon_2, \cdots, \epsilon_s$, where ϵ_i satisfies $\|\epsilon_i\|_{\mathfrak{p}_k}<1$ for $k\neq i$. Because of the product formula we also get

$$\|\epsilon_i\|_{\mathfrak{p}_i} > 1.$$

The first $s-1$ of these S-units are mapped onto vectors

$$v_i = (a_{i1}, a_{i2}, \cdots, a_{i,s-1}), \qquad i = 1, 2, \cdots, s-1,$$

where $a_{ik}=\log\|\epsilon_i\|_{\mathfrak{p}_k}$. Then $a_{ii}>0$ and $a_{ik}<0$ for $i\neq k$, but $\sum_{\nu=1}^{s-1}a_{i\nu}=\sum_{\nu=1}^{s-1}\log\|\epsilon_i\|_{\mathfrak{p}_\nu}=-\log\|\epsilon_i\|_{\mathfrak{p}_s}>0$.

We prove now that the vectors v_i are linearly independent, that is, that the homogeneous equations

$$\sum_{\nu=1}^{s-1} x_\nu a_{\nu k} = 0, \qquad k = 1, 2, \cdots, s-1,$$

have only the trivial solution. To that effect it suffices to show that the homogeneous equations

$$\sum_{\nu=1}^{s-1} a_{i\nu}y_\nu = 0, \qquad i = 1, 2, \cdots, s-1,$$

have only the trivial solution.

Assume indeed that $y_1, y_2, \cdots, y_{s-1}$ is a non-trivial solution and that y_i has the greatest absolute value. It is no restriction to assume $y_i>0$ so that $y_i \geqq y_j$ for all $j \neq i$. Since $a_{ij}<0$ we get $a_{ij}y_i \leqq a_{ij}y_j$. Now

$$\begin{aligned} a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n &\geqq a_{i1}y_i + a_{i2}y_i + \cdots + a_{in}y_i \\ &\geqq (a_{i1} + a_{i2} + \cdots + a_{in})y_i. \end{aligned}$$

The left side of the inequality should be 0 but on the right side both factors are positive.

This proves:

THEOREM 6 (UNIT THEOREM).[6] *If Axioms* 1 *and* 2a *hold then the dimension mentioned in Theorem* 5 *is precisely* $s-1$, *so the S-units form* mod U *a free abelian group with* $s-1$ *generators.*

Another consequence of Axiom 2a is the following: If we go back to Lemma 11 and select in it for $\mathfrak{q}$ one of the primes of S then the inequalities show just as in the proof of Lemma 15 that $\|\beta\mathfrak{a}\|_\mathfrak{p}=1$ for all $\mathfrak{p}$ with $N\mathfrak{p}>N\mathfrak{q}/C$ and that outside of S there are only a finite number of possibilities for the value distribution of $\|\beta\mathfrak{a}\|_\mathfrak{p}$. Assume that the idèles $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_m$ realize any possible case; then there is always an i such that $\|\beta\mathfrak{a}\|_\mathfrak{p}=\|\mathfrak{a}_i\|_\mathfrak{p}$ for all p not in S or $\beta\mathfrak{a}=\mathfrak{a}_i\cdot\mathfrak{a}_S$. This proves:

THEOREM 7 (FINITENESS OF CLASS NUMBER). *There is a finite set of idèles* $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_m$ *such that any idèle* $\mathfrak{a}$ *is of the form*

$$\mathfrak{a} = \alpha\mathfrak{a}_i\mathfrak{a}_S$$

for a suitable i, $\alpha \in k$ *and* $\mathfrak{a}_S$.

We mention the special case, important for class field theory:

THEOREM 8. *If the set* S *is big enough then* $\mathfrak{a}=\alpha\mathfrak{a}_S$ *for all idèles* $\mathfrak{a}$.

PROOF. Add to the previous set S also the primes $\mathfrak{p}$ where any $\|\mathfrak{a}_i\|_\mathfrak{p} \neq 1$.

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[6] The unit theorem in this form is due to Hasse. It is proved by Chevalley in [4].

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INDIANA UNIVERSITY AND
UNIVERSITY OF PENNSYLVANIA

THE APRIL MEETING IN STANFORD UNIVERSITY

The four hundred tenth meeting of the American Mathematical Society was held at Stanford University on Saturday, April 28, 1945. Approximately forty people attended, including the following twenty-eight members of the Society:

H. M. Bacon, B. A. Bernstein, H. F. Blichfeldt, E. A. Davis, T. C. Doyle, Evelyn Fix, S. M. Hallam, O. H. Hamilton, J. G. Herriot, Glenn James, D. H. Lehmer, J. C. C. McKinsey, F. J. Massey, E. D. Miller, K. A. Morgan, Jerzy Neyman, C. D. Olds, George Polya, W. C. Randels, J. B. Robinson, R. M. Robinson, S. A. Schaaf, A. C. Schaeffer, T. W. Simpson, Gabor Szegö, Alfred Tarski, A. R. Williams, František Wolf.

There was a section for contributed papers in the morning at which Professor B. A. Bernstein presided. The afternoon session was devoted to a symposium on *Contagious probability distributions* at which Professor Jerzy Neyman and Professor George Polya spoke. Professor Polya spoke about *Interpretation or interpolation? Remarks on certain applications of, and certain views about, contagious probability distributions*. This address discussed some of the known dependent events ("influence of the crowd," "influence of the predecessor," and so on), their most intuitive applications, and the notable fact that a probability distribution resulting from such a structure of mutual dependence can be interpreted also as generated by the concurrence of independent events, and even in various ways (inhomogeneity, delivery by "packages"). Professor Neyman's address, which will be printed in greater detail, was about *Multivariate Poisson and contagious distributions*. Professor H. F. Blichfeldt presided at the afternoon session.

Titles and cross references to papers read at the meeting follow below. Paper 7, whose abstract number is followed by the letter *t*, was read by title. Mr. Jonsson was introduced by Dr. Tarski.

1. D. H. Lehmer: *On the Graeffe process for power series*. (Abstract 51-5-87.)

2. Kathryn A. Morgan: *Representation of a positive binary form by a positive quaternary form*. (Abstract 51-5-81.)

3. R. M. Robinson: *Univalent majorants*. (Abstract 51-5-89.)

4. Glenn James: *Certain general polynomial expansions*. (Abstract 51-5-86.)

5. George Polya and Gabor Szegö: *On finite Fourier integrals*. (Abstract 51-7-119.)

6. Bjarni Jonsson: *On unique factorization problem for torsionfree Abelian groups*. (Abstract 51-5-75.)

7. E. R. Lorch: *The Cauchy-Schwarz inequality and self-adjoint spaces*. (Abstract 51-3-64-*t*.)

A. C. SCHAEFFER,
Associate Secretary

REPORT OF THE MEETING OF THE COUNCIL OF THE SOCIETY IN NEW YORK CITY ON APRIL 28, 1945

A meeting of the Council of the Society was held in the Men's Faculty Club at Columbia University at 1:30 P.M. on Saturday, April 28, 1945. The following members of the Council were present: A. A. Albert, Garrett Birkhoff, S. S. Cairns, Richard Courant, Arnold Dresden, B. P. Gill, M. R. Hestenes, T. R. Hollcroft, Harold Hotelling, Saunders MacLane, J. F. Ritt, P. A. Smith, I. S. Sokolnikoff, J. D. Tamarkin, Hassler Whitney, and the Secretary.

Certain papers submitted for presentation at various meetings of the Society were read by title at the meeting of the Council; titles and cross references to the abstracts of these papers appear at the end of this report.

The Secretary announced the election of the following ten persons to ordinary membership in the Society:

Mr. Silvio Aurora, Columbia University;
Professor Harold Frederick Bright, University of Rochester;
Miss Helen M. Clark, Northwestern University;
Dr. Vincent Frederick Cowling, Rice Institute;
Mrs. Anne Carples Davis, Seaman Second Class, U.S.N.R.;
Mr. Richard Latter, Lieutenant, U.S.N.R.;
Mr. Douglas Blakeshaw Netherwood, Captain, A.U.S.;
Mr. Joseph Thomas Olsztyn, Private, U.S.A.;
Mr. Clarence A. Phillips, University of Illinois;
Mr. Marlow Canon Sholander, Brown University.

The Department of Mathematics of the University of Oregon was elected to institutional contributing membership.

The following appointment by the President was reported: as a Committee to Select Gibbs Lecturers for 1946 and 1947, Professors L. M. Graves (Chairman), G. A. Hedlund, and S. S. Wilks.

It was reported that the Board of Trustees had approved the recommendation of the Council that a catalogue of the Society library be published during 1945 and that the Board had appropriated $2500 for this purpose.

The Secretary called the attention of the Council to the following policy which has been followed in recent years: if both husband and wife are members of the Society, one is allowed to receive the Transactions, instead of the Bulletin, as one of the privileges of membership.

Professor Oswald Veblen was appointed representative of the Society in the Division of Physical Sciences of the National Research Council for a period of three years, beginning July, 1945.

The Council authorized the appointment by the President of a committee to study the problem of aid to devastated libraries.

Secretary Kline reported for the War Policy Committee as follows:

1. A subcommittee of the War Policy Committee has been appointed to study the relation of mathematics to the compulsory military training program being considered by the government; this subcommittee consists of Professors W. L. Hart (Chairman), Saunders MacLane, and C. B. Morrey, Jr. The final report of this subcommittee will probably be offered for publication in *Science* as well as in the mathematical journals.

2. The Rockefeller Foundation has extended its second grant for the work of the War Policy Committee from December 31, 1944, to September 30, 1945, and has made a grant of $1,000 for the period January 1–September 30, 1945.

3. The Subcommittee to Advise the Examinations Staff of the Armed Forces Institute (Professor W. T. Reid, Chairman) has been discharged with thanks. A general report on the activities of this subcommittee appeared in the March, 1945, issue of *The American Mathematical Monthly*. Examinations have been prepared in algebra, plane trigonometry, analytic geometry, differential and integral calculus; these are to be standardized under a project which is being undertaken jointly by the War Department and the Cooperative Test Service of the American Council on Education.

4. Since the work of the Subcommittee on War Training Programs (Professor W. L. Hart, Chairman) has been finished, this subcommittee has been discharged with thanks.

5. The Committee recommended that the Presidents and Secretaries of the Society and Association make application to the War Committee on Conventions for permission to hold the 1945 Summer Meeting in Montreal, Canada.

6. The Committee approved the bill calling for the assignment of students to a national scientific education program.

7. The Committee has studied the problem of the place of the National Roster of Scientific and Specialized Personnel in the postwar

period and has made recommendations to a national committee appointed to study this problem. In this connection the Secretary called attention to the lists of publications maintained in his office. These are complete lists of publications for all persons who have received the doctorate in mathematics in America. The lists for those who received the doctorate abroad are complete for the period subsequent to their appointments at American institutions.

The Council approved with certain changes a proposed bill to authorize the assignment of persons to a national scientific program instead of to service in the armed forces, in order to make possible the education and training of scientists to meet essential needs.

The Secretary reported that application had been made to the War Committee on Conventions for permission to hold a meeting of the Society in Montreal at the conclusion of the Canadian Mathematical Congress. (The regulations of the Office of Defense Transportation require that permission be secured, even though meetings are scheduled outside continental United States.) The President and Secretary were empowered to take action regarding the 1945 Summer Meeting in light of the ruling of the Office of Defense Transportation.

Titles and cross references to the papers read at the Council meeting follow below. Mr. DeBaggis was introduced by Professor Menger.

1. R. P. Agnew: *Spans of translations of peak functions.* (Abstract 51-5-84-*t*.)

2. R. P. Boas: *Fundamental sets of analytic functions.* (Abstract 51-5-85-*t*.)

3. H. F. DeBaggis: *A simplified projective theory of order and parallelism in the hyperbolic plane.* (Abstract 51-5-97-*t*.)

4. Herbert Federer: *Coincidence functions and their integrals.* (Abstract 51-7-113-*t*.)

5. Evelyn Frank: *Corresponding type continued fractions.* (Abstract 51-7-114-*t*.)

6. A. P. Hillman: *On identities for differential polynomials.* (Abstract 51-5-73-*t*.)

7. A. P. Hillman: *Theorems obtained from the Newton polygon process for differential polynomials.* (Abstract 51-5-74-*t*.)

8. L. K. Hua: *Geometries of matrices.* I. *Generalizations of von Staudt's theorem.* (Abstract 51-5-95-*t*.)

9. L. K. Hua: *Geometries of matrices.* I_1. *Arithmetical construction.* (Abstract 51-5-96-*t*.)

10. S. B. Jackson: *The four-vertex theorem for surfaces of constant curvature.* (Abstract 51-3-68-*t*.)

11. R. E. Johnson: *Systems of finite linear equations.* (Abstract 51-3-55-*t*.)

12. G. K. Kalisch: *A spectral theory for symmetric operators over generalized Hilbert spaces.* (Abstract 51-5-76-*t*.)

13. Irving Kaplansky: *A note on groups without isomorphic subgroups.* (Abstract 51-5-78-*t*.)

14. Irving Kaplansky: *Maximal fields with valuations.* II. (Abstract 51-5-79-*t*.)

15. Irving Kaplansky: *The asymptotic distribution of runs of consecutive elements.* (Abstract 51-5-101-*t*.)

16. J. N. B. Livingood: *A partition function with the prime modulus* $p>3$. (Abstract 51-3-56-*t*.)

17. N. H. McCoy: *Subdirectly irreducible commutative rings.* (Abstract 51-5-80-*t*.)

18. Morris Marden: *A note on the zeros of the sections of a partial fraction.* (Abstract 51-5-88-*t*.)

19. Karl Menger: *A projective definition of hyperbolic congruency independent of parallelism.* (Abstract 51-5-99-*t*.)

20. Karl Menger: *A projective definition of hyperbolic perpendicularity without reference to parallelism.* (Abstract 51-5-98-*t*.)

21. L. J. Mordell: *Further contributions to the geometry of numbers for non-convex regions.* (Abstract 51-7-108-*t*.)

22. V. C. Poor: *On the Hamilton differential.* (Abstract 51-7-121-*t*.)

23. A. R. Schweitzer: *Functional relations valid in the domains of abstract groups and Grassmann's space analyais.* III. (Abstract 51-5-90-*t*.)

24. Otto Szász: *On some summability methods with triangular matrix.* (Abstract 51-5-91-*t*.)

25. H. S. Wall: *Polynomials with real coefficients whose zeros have negative real parts.* (Abstract 51-3-60-*t*.)

26. F. T. Wang: *Strong summability of Fourier series.* (Abstract 51-5-92-*t*.)

27. F. T. Wang: *Tauberian theorem of oscillating series.* (Abstract 51-5-93-*t*.)

28. J. E. Wilkins: *A generalization of the Euler ϕ-function.* (Abstract 51-5-83-*t*.)

J. R. KLINE,
Secretary

BOOK REVIEWS

Theory of games and economic behavior. By John von Neumann and Oskar Morgenstern. Princeton University Press, 1944. 18+625 pp. $10.00.

Posterity may regard this book as one of the major scientific achievements of the first half of the twentieth century. This will undoubtedly be the case if the authors have succeeded in establishing a new exact science—the science of economics. The foundation which they have laid is extremely promising. Since both mathematicians and economists will be needed for the further development of the theory it is in order to comment on the background necessary for reading the book. The mathematics required beyond algebra and analytic geometry is developed in the book. On the other hand the non-mathematically trained reader will be called upon to exercise a high degree of patience if he is to comprehend the theory. The mathematically trained reader will find the reasoning stimulating and challenging. As to economics, a limited background is sufficient.

The authors observe that the give-and-take of business has many of the aspects of a game and they make an extensive study of the strategy of games with this similarity in mind (hence the title of this book). In the game of life the stakes are not necessarily monetary; they may be merely *utilities.* In discussing utilities the authors find it advisable to replace the questionable marginal utility theory by a new theory which is more suitable to their analysis. They note that in the game of life as well as in social games the players are frequently called upon to choose between alternatives to which probabilities rather than certainties are attached. The authors show that *if a player can always arrange such fortuitous alternatives in the order of his preferences, then it is possible to assign to each alternative a number or numerical utility expressing the degree of the player's preference for that alternative. The assignment is not unique but two such assignments must be related by a linear transformation.*

The concept of a game is formalized by a set of postulates. Even the status of information of each player on each move is accounted for and is characterized by a partition of a certain set. *The amount which player k receives at the conclusion of the play is a function* $\mathfrak{F}_k(\sigma_1, \sigma_2, \cdots, \sigma_\nu)$ *of the moves* $\sigma_1, \sigma_2, \cdots, \sigma_\nu$ *where some of the* σ's *may be the moves of chance* (dealing cards, throwing dice, and so on).

The concept of a game admits of a rather drastic simplification which practically relieves the players of the necessity of playing.

Imagine that all possible strategies of all players have been catalogued. Then player k can tell his secretary that he wishes to play strategy τ_k. When she looks up this strategy she finds a complete prescription determining every move for every possible eventuality. Thus the secretaries could get together and determine the outcome of the game if they could only find an equitable method of accounting for the moves of chance. But chance enters into the game very much as one of the players. Thus we can imagine a cataloguing of the possible strategies of chance. Suppose for the moment that the strategy τ_0 of chance has been decided upon and that the players have chosen respectively the strategies $\tau_1, \tau_2, \cdots, \tau_n$. Then the strategies determine the moves. Hence $\mathfrak{F}_k(\sigma_1, \sigma_2, \cdots, \sigma_\nu)$ *is a function* $\mathfrak{G}_k(\tau_0, \tau_1, \cdots, \tau_n)$ *of the strategies* and the outcome of the game is determined. But how should τ_0 be selected? Instead of selecting τ_0 the secretaries could *assign to each player* k *the amount* $\mathfrak{H}_k(\tau_1, \tau_2, \cdots, \tau_n)$ *which he would receive on the average if strategies* $\tau_1, \tau_2, \cdots, \tau_n$ *of the players were chosen.* The amount $\mathfrak{H}_k$ is the mathematical expectation of $\mathfrak{G}_k$. It is computed in terms of the probabilities of the various strategies τ_0 and these probabilities are in turn computed in terms of the probabilities of the moves of chance.

The game has now been reduced to one in which each player makes just one move—the selection of a strategy. Each player makes his move in complete ignorance of the moves of the other players. The authors have accomplished this simplification of the game with complete rigor and with complete adherence to the rules laid down by the postulates.

A 1-*player game corresponds to the economy of a man on a desert island.* It is the Robinson Crusoe economy or a strictly regimented communism. If the player is wise, he will choose his strategy τ_1 so that $\mathfrak{H}_1(\tau_1)$ is a maximum. This is the only case where a game is settled by simple maximum considerations.

An n-player zero-sum game is one for which the sum of the $\mathfrak{F}_k$'s is zero for all choices of $\sigma_1, \sigma_2, \cdots, \sigma_\nu$ and hence one for which the sum of the $\mathfrak{H}_k$'s is zero for all choices of $\tau_1, \tau_2, \cdots, \tau_n$. Social games are zero-sum but the game of economics is decidedly not zero-sum since society as a whole can improve its status if all members behave properly. *However an arbitrary n-player game can be reduced to a zero-sum* $(n+1)$-*player game* by introducing a fictitious player $n+1$ who receives the amount $\mathfrak{H}_{n+1}(\tau_1, \tau_2, \cdots, \tau_n)$ which is the negative of the total received by the remaining n players. Note that the functions $\mathfrak{H}_k$ do not contain the variable τ_{n+1}, that is, the fictitious player is not permitted to choose a strategy. It will appear later that further restrictions on the activities of this player are necessary to prevent his influencing the outcome of the game.

A zero-sum 2-player game Γ can be characterized by a single function $\mathfrak{H}(\tau_1, \tau_2)=\mathfrak{H}_1(\tau_1, \tau_2)$ since $\mathfrak{H}_2(\tau_1, \tau_2)=-\mathfrak{H}(\tau_1, \tau_2)$ by virtue of the relation $\mathfrak{H}_1+\mathfrak{H}_2=0$. In this game player 1 will attempt to maximize $\mathfrak{H}$ (or $\mathfrak{H}_1$) whereas player 2 will attempt to minimize $\mathfrak{H}$ (or maximize $\mathfrak{H}_2$). Since these are diametrically opposed tendencies, it looks as though nothing could be decided. However we can gain insight into the problem by considering a modified game Γ_1 which is the same as Γ except that player 1 moves first and player 2 knows 1's move. In Γ_1 after player 1 chooses τ_1, player 2 will choose τ_2 so as to minimize $\mathfrak{H}$. It is therefore advisable for 1 to choose τ_1 so as to maximize $\min_{\tau_2} \mathfrak{H}(\tau_1, \tau_2)$ where $\min_{\tau_2} \mathfrak{H}(\tau_1, \tau_2)$ is the minimum with respect to τ_2 of $\mathfrak{H}(\tau_1, \tau_2)$. Player 1 will then receive.

$$v_1 = \max_{\tau_1} \min_{\tau_2} \mathfrak{H}(\tau_1, \tau_2)$$

and 2 will receive $-v_1$. Next consider a third game Γ_2 which is the same as Γ except that 2 moves first and 1 knows 2's move. If both players of Γ_2 are skillful, then 1 will receive the amount

$$v_2 = \min_{\tau_2} \max_{\tau_1} \mathfrak{H}(\tau_1, \tau_2)$$

and 2 will receive $-v_2$. In the original game Γ if both players are skillful, 1 will receive at least v_1 and at most v_2 whereas 2 will receive at least $-v_2$ and at most $-v_1$. *Hence* $v_1 \leqq v_2$ *and these quantities are bounds for the outcome of the game.* If $v_1=v_2$, the game is determined but in general this is not the case.

Note that Γ reduces to Γ_1 if 2 discovers 1's strategy whereas Γ becomes Γ_2 if 1 discovers 2's strategy. Hence it is advisable for the players to conceal their strategies. The concealment is accomplished by using probabilities. Thus 1 chooses τ_1 with probability ξ_{τ_1} and 2 chooses τ_2 with probability η_{τ_2}. *The average outcome* $K(\xi, \eta)$ *for player 1 is the mathematical expectation of* $\mathfrak{H}(\tau_1, \tau_2)$ *with respect to the probabilities* ξ_{τ_1} *and* η_{τ_2} where ξ is the vector with components $\xi_1, \xi_2, \cdots$ and η is the vector with components $\eta_1, \eta_2, \cdots$. The introduction of these probabilities modifies Γ and consequently modifies Γ_1, Γ_2 and the bounds v_1, v_2. The new bounds become

$$v_1' = \min_{\xi} \max_{\eta} K(\xi, \eta)$$

and

$$v_2' = \min_{\eta} \max_{\xi} K(\xi, \eta).$$

It is easily shown that $v_1 \leqq v_1' \leqq v_2' \leqq v_2$, that is, that each player is at least as well off as before the probabilities were introduced. Moreover it can be shown that

$$v_1' = v_2' = v$$

and hence that the game is determined. The proof of the latter result depends on the fact that the numbers $x_{\tau_2} = \sum_{\tau_1} \mathfrak{H}(\tau_1, \tau_2)\xi_{\tau_1}$ are components of a vector χ which depends on ξ and that the tips of the vectors χ for all possible ξ's constitute a convex set of points.

Next consider an n-player game in which the players divide into two hostile groups called S and $-S$. This can be interpreted as a 2-player game between the players S and $-S$. If probabilities are employed in the manner described above, then S will receive

$$v(S) = v_1' = v_2' = v$$

and $-S$ will receive

$$v(-S) = -v(S).$$

If I is the set of all players, then $v(I)=0$, that is, the game is zero-sum. Finally

$$v(S+T) \geqq v(S) + v(T)$$

if S and T are mutually exclusive groups. That is, the players of $S+T$ can obtain at least as much by cooperating as they can by splitting up into two groups S and T. The function $v(S)$ satisfying the above relations is called a characteristic function. Corresponding to any function satisfying these relations there exists a game having this $v(S)$ as its characteristic function. The construction of such a game involves partitions of I into subsets called rings and solo sets.

If the equality $v(S+T)=v(S)+v(T)$ always holds, that is, if $v(S)$ is additive, then the coalitions will be ineffective and the game will be determined. This is the case for $n=2$. Moreover two characteristic functions (whether additive or not) which differ by an additive function will produce the same strategies of coalitions. If $v(S)$ is not additive, it can be modified by a suitable additive function and a suitable scale factor so that $v(S)=-1$ for all 1-element sets. Thus for $n=3$, $v(S)$ is given by the following table

$$v(S) = \left\{\begin{array}{c} 0 \\ -1 \\ +1 \\ 0 \end{array}\right. \textit{ for the } \left\{\begin{array}{l} \textit{0-element set } (-I \textit{ or the complement of } I) \\ \textit{1-element sets} \\ \textit{2-element sets (complements of 1-element sets)} \\ \textit{3-element set } (I). \end{array}\right.$$

For $n \geqq 4$, $v(S)$ is no longer determined and the number of possibilities becomes almost bewildering. The reader will begin to realize that there is never a dull moment with these games. We have seen that for each of the cases $n=1, 2, 3, 4$ a new situation appears. For $n=5$ no new phenomenon has as yet been discovered but for $n \geqq 6$ we first meet the possibility of a game which splits into two or more games which are in some respects quite distinct but which nevertheless exert potent influences on one another. This phenomenon has the counterpart of nations whose economies are distinct yet interdependent.

It remains to consider what coalitions can be expected to form in a given game and how the stakes will be divided in the presence of such coalitions. A division of stakes is called an *imputation* and is represented by a vector α with components $\alpha_1, \alpha_2, \cdots, \alpha_n$ where α_k is the amount the kth player receives. One could imagine that if a group of novices were playing one of these games a certain chaos would result. Coalitions would be made and broken as each player sought to improve his own status. Finally as the players became more acquainted with the game certain imputations would come to be trusted because of the stability of the corresponding coalitions and because of the profitableness to an effective group of players. There would thus emerge *a set V of trusted imputations*. There would of course be players who were dissatisfied with any given trusted imputation but they would not be strong enough to force a change unless they could bribe some of the favored players to desert their coalitions. Nor would such bribery be effective since the potential recipient of the bribe would realize that the chaos produced by his desertion would eventually leave him in a less favorable position. Thus V corresponds to a group behavior pattern. It is an institution or a morality arising from enlightened self interest.

But how can V be described mathematically? We begin with a definition. *We say that an imputation α dominates an imputation β if there is an effective group of players each of which is better off under α than under β. The group is effective provided it can guarantee for its members the stakes prescribed by α against any opposition from without the group.* A set V of imputations is called a solution provided every imputation outside of V is dominated by some imputation of V and no imputation in V is dominated by any other imputation in V. Thus *V is a maximal set of mutually undominated imputations.* Unfortunately dominance does not produce even a partial ordering of the set of all imputations. It is not a transitive relation. This makes the discovery of solutions a difficult task. We shall however outline a method of finding solutions for the case $n=3$.

When $n=3$ we have

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) \quad \text{with} \quad \alpha_1 + \alpha_2 + \alpha_3 = 0,$$

that is, the game is zero-sum. Thus the tip of α lies in a plane which passes through the origin and is equally inclined to the coordinate axes. This plane is divided into six congruent sectors by the traces of the coordinate planes. Next $\alpha_k \geqq -1$ (for $k=1, 2, 3$) since each player can obtain at least -1 without the benefit of any coalition (see the above table). These inequalities require the tip of α to lie within an equilateral triangle whose center is at the common intersection of the traces of the coordinate planes and whose sides are parallel to these traces. An imputation α dominates those imputations which are represented by points interior to three parallelograms each of which has two sides in common with the above equilateral triangle and one vertex at the tip of α. On the basis of these geometrical considerations it is easy to find solutions V. We first look for a V whose imputations do not all lie on a line $\alpha_k=$a constant (that is, a line parallel to a trace). There is only one such solution, namely,

$$V: \quad (1/2, 1/2, 0), \qquad (1/2, 0, 1/2), \qquad (0, 1/2, 1/2).$$

We next look for a V whose imputations do lie on a line, say, $\alpha_3=c$. The corresponding solutions are

$$V_c: \quad (a, -a-c, c)$$

where a and c are required to satisfy certain inequalities. Thus V_c contains a continuum of solutions corresponding to values of the parameter a. This exhausts the possible solutions. The first solution V seems quite reasonable whereas V_c seems unnatural and difficult to interpret but let us return to this question later.

Let us consider the following non-zero-sum 2-player game. Each player (1 or 2) chooses either the number 1 or the number 2. If both players choose 1, then each receives the stake 1/2. Otherwise each receives -1. If we reduce this game to a zero-sum 3-player game by the introduction of a fictitious player 3, then the characteristic function becomes the one given in the above table. Now if we take the first solution V, we discover that the fictitious player may play an active part in the formation of coalitions. Hence if we wish to retain the 2-player character of the game, we must choose the solution V_c and it is reasonable to assign to c the value -1.

The authors apply this theory of games to the analysis of a market consisting of one buyer and one seller and also of a market consisting of two buyers and one seller.

The book leaves much to be done but this fact only enhances its interest. It should be productive of many extensions along the lines of economic interpretation as well as of mathematical research. In fact the authors suggest a number of directions in which research might profitably be pursued.

ARTHUR H. COPELAND

Principles of stellar dynamics. By S. Chandrasekhar. The University of Chicago Press, 1942. 10+251 pp.

The primary field of this book is astronomy and not mathematics, although the latter is used as an essential tool. The readers of this review, professional mathematicians almost exclusively, will have a normal human interest in the major astronomical aspects of the book, but their critical scrutiny is bound to be concentrated on how the astronomical problems are formulated mathematically and what sort of mathematics has been proposed for their solution. For this reason, and partly also in the interest of brevity, this review treats only of the mathematical aspects of the book.

In the first chapter is given a detailed discussion of the kinematical concepts appropriate to the study of stellar systems. Since these systems contain a large number of stars, it becomes necessary to introduce a method similar to that employed in hydrodynamics, where the motion of a fluid is described by a vector field, representing at each point and for each instant of time the velocity of the fluid. In hydrodynamics the velocity of the fluid at a point is conceived as the velocity of the "fluid particle" at the point in question. But this notion of a "particle" at the point in question is difficult to make precise, especially if one assumes the fluid to consist of a large number of small atoms with relatively large empty spaces between them. Nevertheless such a concept (in which the stars play the role of the atoms) is characteristic of stellar dynamics as distinguished from celestial (particle) mechanics, which considers systems containing but a relatively small number of bodies.

The components $U_0(x, y, z, t)$, $V_0(x, y, z, t)$, $W_0(x, y, z, t)$ of the vector field thus introduced do not, of course, necessarily represent the components of velocity of a star which might happen to be at the point (x, y, z), but rather the velocity of the centroid of stars in a "small volume" about the point (x, y, z). The components of velocity of an individual star are written in the form $U = U_0 + u$, $V = V_0 + v$, $W = W_0 + w$, where the vector (u, v, w) is called the *residual* velocity. The statistical consideration of these residual velocities is a characteristic of stellar dynamics and gas theory as distinguished from classical hydrodynamics. Associated with each point of space is a distribution density function $F(u, v, w)$, such that $Fdudvdw$ is regarded as approximately measuring the number of stars per unit volume, the components of whose residual velocities lie in the intervals $(u, u+du)$, $(v, v+dv)$, and $(w, w+dw)$ respectively. Considerable space is devoted to a review of observed astronomical phenomena to justify the assumption that such functions U_0, V_0, W_0, F can be introduced in the manner indicated. Still more space is devoted to the study of the exact form of the distribution density F. The fact that it can often be assumed to be Gaussian (but not necessarily spherically symmetric as in the Maxwellian special case) is known as Schwarzschild's law. In cases when the distribution is not Gaussian, there is a hint that the given stellar system can be considered as the superposition of two or more systems each one of which obeys Schwarzschild's law. An example of such a situation is that afforded by the phenomenon of the so-called high velocity stars in our galaxy.

The most general form for the distribution density function, $F(u, v, w) = F[x, y, z, t; u, v, w] = F[x, y, z, t; U-U_0, V-V_0, W-W_0]$, considered in this book is a generalization of the Schwarzschild form, namely

$$F(u, v, w) = \Psi(Q + \sigma),$$

where Q is a positive definite quadratic form in u, v, w, whose coefficients together with σ are functions of x, y, z, and t.

The motion of a single star is assumed to be governed to a high degree of approximation by equations of the familiar type,

$$\ddot{r} = \text{grad } \Phi \qquad (r = ix + jy + kz). \tag{1}$$

Here Φ, a function of x, y, z, and t, is a so-called "smoothed-out" potential function, depending upon the general distribution of the other stars of the system. This assumption is evidently justified as long as the star in question is relatively far from the other stars, but it loses its significance if the star has a near encounter with another star. To indicate that the approximation (1) is indeed likely to be valid over extraordinarily long periods of time is the fundamental object of the work on the "relaxation time" of Chapter II, a subject to which we shall presently return.

Since $\Psi(Q+\sigma)$, considered as a function of x, y, z, U, V, W, may be interpreted as a distribution density function of the star in six-dimensional phase space, it is at once obvious that

$$\int\int\int\int\int\int \Psi(Q + \sigma)dxdydzdUdVdW$$

is an integral invariant of (1). Hence, by Liouville's theorem, $\Psi(Q+\sigma)$, and, in fact, $Q+\sigma$ itself, is a first integral of (1), the condition for which is the familiar partial differential equation,

$$\frac{\partial\Psi}{\partial x}U+\frac{\partial\Psi}{\partial y}V+\frac{\partial\Psi}{\partial z}W+\frac{\partial\Psi}{\partial U}\frac{\partial\Phi}{\partial x}+\frac{\partial\Psi}{\partial V}\frac{\partial\Phi}{\partial y}+\frac{\partial\Psi}{\partial W}\frac{\partial\Phi}{\partial z}=0. \tag{2}$$

From this equation there are two alternative ways of proceeding. One method is to derive (with J. H. Jeans) the equations of hydrodynamics. But these equations can not carry the full implications of the kinematical hypotheses, especially the one regarding the existence of a distribution function for the residual velocities. Indeed, the four hydrodynamical equations (that is, the equation of continuity plus the three equations expressing Newton's second law of motion for a fluid) are only the first four of a series of equations obtained by multiplying equation (2) by $U^pV^qW^r$, integrating with respect to U, V, and W over all values, and reducing by appropriate partial integrations. If the distribution function has the proper behaviour at infinity, one thus obtains an infinite number of equations by letting p, q, and r assume independently of each other the values 0, 1, 2, 3, $\cdots$, whereas the four hydrodynamical equations are only those corresponding to $(p=q=r=0)$, $(p=1,\ q=r=0)$, $(q=1,\ p=r=0)$, $(r=1,\ p=q=0)$. Accordingly the author, although the hydrodynamical equations are mentioned and even briefly applied in Chapter IV, uses for the most part a more powerful method which will now be described.

Remembering that $\Psi=\Psi(Q+\sigma)$, where Q is a quadratic form in $U-U_0$, $V-V_0$, $W-W_0$, one sees that equation (2) after division by Ψ' has the form of a cubic polynomial in U, V, W, equated to zero. Since U, V, W are independent variables, we can equate to zero each one of the coefficients of this polynomial, thus obtaining twenty partial differential equations in the eleven unknown functions U_0, V_0, W_0, Φ, σ, and the six coefficients of Q. The study of these twenty equations in Chapter III appears to the reviewer to be the central theme of the entire work. The problem is evidently that of determining the most general conditions under which equations (1) admit an integral quadratic in the velocities. Although various special cases of this problem are well known in classical dynamics or have been considered in an astronomical context by various authors including Eddington and Jeans, the completely general problem seems to have been first formulated and studied by Chandrasekhar.

A very interesting and general result of the investigation of the twenty partial differential equations is to the effect that, for stellar systems in steady states and possessing some distribution function of the form $\psi(Q+\sigma)$, the potential Φ is necessarily characterized by helical symmetry. If, in addition, the stellar system is of finite extent, this helical symmetry reduces to axial symmetry. The case of nonsteady states has not been completely treated except for certain special cases.

In Chapter IV the discussion centers on the case when the residual velocities have a spherical distribution. In this way the author is led to a theory of the spiral structure of extragalactic nebulae. He is careful to emphasize, however, that the class of spiral orbits predicted is so wide that the theory is really too general to give any indication as to why certain forms of spiral orbits are preferred to the exclusion of others.

In the fifth and last chapter there seems to be a departure from the central theme of the work as described above. The subject here is the classical n body problem. After a standard discussion of the elementary first integrals and of Lagrange's identity, statistical considerations are introduced for the purpose of obtaining information on the dispersion of velocities in clusters, the rate of disintegration of clusters by the escape of stars, and the time of relaxation of a cluster.

Although a cluster is usually regarded as a much smaller and more compact group of stars than a stellar system in the sense of the preceding chapters, a formula from Chapter II for the relaxation time of a stellar system is applied in Chapter V in the determination of the relaxation time of a cluster. The method used in Chapter II is to estimate the effect of two body encounters in the long history of a representative star.

Since a cluster may be a small part of a much larger stellar system, a generalization of the n body problem is also considered, in which superimposed on the potential of the cluster itself is the general "tidal" potential of the rest of the stellar system; but the only instance of such a situation considered in any detail is a case in which the tidal potential has both a plane and axis of symmetry and in which the center of gravity of the cluster describes a circular orbit.

It is statistically impossible in a work of such scope and originality to avoid a certain number of blemishes. Some which have attracted the attention of the reviewer are the following:

In concluding Chapter I, the author states, as if it were an important principle, that due to the considerations of this chapter "we are able to express the distribution function $\Psi(x, y, z; U, V, W; t)$ in the form $\Psi(x, y, z; U-U_0, V-V_0, W-W_0; t)$, where U_0, V_0, W_0 are functions of position and time only." Clearly this is not what the author really means, since the principle stated is trivially true with $U_0 \equiv V_0 \equiv W_0 \equiv 0$.

In Chapter II it would be well to insert an explanation of the term "impact parameter," as this term is not used in any of a half dozen standard treatises on dynamics which a reader might consult. Although what may be taken as a strictly mathematical definition is given in Appendix I, equation 26, there is no explanation of the word "impact" nor is it clear without further discussion why the integrated average over the impact parameter must be weighted proportionally to the parameter (cf. the factor DdD in the formula (2.313) instead of simply dD).

The remarks inserted in the bibliography at the end of Chapter III (pp. 133, 134) involving the expansion of the potential function in a Taylor's series give a deceiving sense of generality. Evidently the results to be obtained by choosing the origin so that the linear terms disappear would be valid only in the neighborhoods of the presumably rather rare critical points, unless it were possible to prove that the Taylor's series had a large domain of convergence.

In Chapter IV (pp. 185, 186) it would be desirable to note that $\overline{U}$, $\overline{V}$, and $\overline{W}$ are identical respectively with U_0, V_0, and W_0; it would be equally desirable to indicate the well known relation existing between $\overline{U^2}$, $\overline{UV}$, and so on, U_0, U_0V_0, and so on, and the strain tensor.

In spite of these criticisms, the reviewer found the book to be extremely interesting, and he feels that it has reached the highest level of scientific merit.

Daniel C. Lewis

The mathematics of physics and chemistry. By H. Margenau and G. M. Murphy. New York, Van Nostrand, 1943. 12+581 pp. $6.50.

In this textbook Professor Margenau and Murphy have assembled a very useful collection of mathematical principles as applied in prewar fundamental research in physics and chemistry. Mathematicians may not, in general, be in sympathy with the authors' deliberate compromising of rigor of derivation to maintain an emphasis on applications. It is doubtful that the book will prove successful as a textbook without prerequisites including the conventional course in advanced calculus. On the other hand, it does fulfill a long standing need, particularly evident in smaller universities, for a textbook suitable as the basis for a mathematics course at this level for graduate students of physics and chemistry.

Chapters are devoted to thermodynamics, ordinary differential equations, special functions, vector analysis, coordinate systems, calculus of variations, partial differential equations of classical physics, eigenvalues and eigenfunctions, mechanics of molecules, matrices and matrix algebra, quantum mechanics, statistical mechanics, numerical calculations, linear integral equations and group theory.

An advanced course, based upon this text, for physicists and chemists, could afford to be supplemented by material on linear systems, transforms, approximate and asymptotic evaluating of integrals, and some elementary theory of acoustics and hydrodynamics.

Some illustrative problems are worked as examples, others, often supplementary to the text material, are included as student exercises. Typographical errors were fairly numerous in the first printing.

This book has been enthusiastically received by graduate students of physics, chemistry and mathematics, although not without criticism of certain sections. It has stimulated considerable interest in the topics covered.

Paul C. Cross

O sentido da nova lógica. By Willard Van Orman Quine. São Paulo, Livraria Martins Editora, 1944. 252 pp.

This is a well organized exposition of modern formal logic addressed to the general philosophical reader. There is at the end a list of the principles most often referred to, a list of all the formal definitions (in symbolic notation), an extensive bibliography, and an alphabetical index. All these will help the studious reader who wishes to master the technique, but the book is not equipped with exercise material, nor designed for classroom drill. The style, at least up to the last chapter, is reasonably simple, with well chosen local illustrations and appropriate cautionary remarks. Rarely, except on questions of ontology, does the author present at length, and then dispose of, views opposed to his own. Throughout the text, generous but brief remarks attribute credit to previous workers.

Following an illuminating introduction, the text falls into four chapters: I Theory of composition, II Theory of quantification, III Identity and existence, IV Class, relation and number. In I, truth value tables (called "quadros") are treated in detail. This treatment covers the translation from everyday language to the more precise, symbolic notation. As distinguished from the "if-then" rule of composition of propositions, the author here, as previously, reserves "implication" to state a relation between propositions. If "$\sim(p \cdot \sim q)$," then "'p' implies 'q'," where 'p', 'q' are substantives, naming the propositions concerned. "Compositional implication" and "compositional truth" are then explained. In II, the theory of quantification introduces logical pronouns (the "variable arguments" of some writers), and matrices (called by many "propositional forms"). Seven deductive operations are listed and used. In III is emphasized the contrast between object and symbol, and the nature of description. Only one kind of existence (material existence in the case of physical objects) is accepted; the question is reduced to whether a pronoun has an application. A brief argument is presented for the elimination, in logic, of all substantives other than pronouns. In IV, attributes are compared with and contrasted to the more readily handled classes. The *pure matrices* in the theory of classes employ only the notions of conjunction, negation, quantification and pertinence (using the "ϵ" of membership). Applications, even in the abstract field of mathematics, which bring in other signs, not themselves defined in logical terms, involve *impure* matrices. A brief survey of the construction of the number system out of pure logical concepts follows. The notion of pertinence adopted is such that "$z \epsilon y$" has the sense: "z is a number of y, if y is a class, and $z=y$, if y is not a class." A virtual theory of classes and relations, while using familiar phraseology, succeeds in eliminating the ontological presuppositions by offering acceptable defined equivalents. The chapter closes with a description of Gödel's Theorem.

Save possibly that this book may give the idea that all important problems in logical theory have been disposed of, the confident tone and clear, enthusiastic and well organized statements commend the book to all thoughtful readers—who read Portuguese.

ALBERT A. BENNETT

Rings with minimum condition. By Emil Artin, Cecil J. Nesbitt and Robert M. Thrall. University of Michigan Press, 1944. 7+123 pp. $1.50.

The major portion of this book is devoted to a new exposition of the classical theory of associative and distributive rings satisfying the minimum condition for left ideals. The authors envisage in their treatment further developments of the theory. They announce the basic observation that the theory of rings proper should not be separated from the corresponding theory of representations. In connection with this thesis it is worthwhile mentioning that the proof of Wedderburn's structure theorem for simple rings involves the theory of representations.

The plan of the authors thus required a careful study of vector spaces admitting rings for left (right) operators (Chapters I and II). The discussion of the implications of the minimum condition contains R. Brauer's result that the minimum condition for left ideals implies the maximality of the radical (Chapter III). The development of the theory of semi-simple and simple rings puts due emphasis on the existence of a nonzero idempotent element in a non-nilpotent left ideal (Chapters IV and V). To make the proof of Wedderburn's theorem on simple rings more lucid the connection between homomorphisms, matrices, and rings of homomorphisms is given emphasis. The following theorem (p. 38) foreshadows the important concept of analytic linear transformation. "Let R be a simple ring with the minimum right ideal eR. Suppose that $\alpha_1, \cdots, \alpha_m$ is a basis of eR over its division algebra of homomorphisms eRe. Then there exists for any elements $\beta_1, \cdots, \beta_m$ in eR a unique element ξ in R such that $\beta_i=\alpha_i\xi$, $1\leqq i\leqq m$."

A linear function $L(x)$, $x\in R$, of a simple ring R over its center $\mathfrak{k}$ is defined as a homomorphism of $R/\mathfrak{k}$ given by $L(x)=\sum a_\nu x a_\nu'$ where the a_ν and a_ν' lie in R. This concept furnishes an important tool for the discussion of the subrings of a simple ring and their commutator algebras (Chapter VII). In this connection the authors present anew the theory of the Kronecker or direct product of linear spaces (Chapter VI). The new definition of the Kronecker product is independent of the particular bases of the component spaces. The definition involves a careful study of the identifications which have to be made in set $\{\sum A_\nu B_\nu\}$, A_ν in a space V, B_ν in a space W, admitting a ring $\mathfrak{k}$ with unit element for right and left operators, respectively. The identifications leading to the one-sided Kronecker product $V\times_{\mathfrak{k}}W$ depend on the distributivity of the sums $\sum A_\nu B_\nu$ with respect to both the elements A_ν and B_ν. The symmetry of $V\times_{\mathfrak{k}}W$ in both factors is established readily. A simplification of the proof is achieved if only locally finite vector spaces are considered. Such spaces have the property that any finite set of vectors lies in a subspace generated by independent vectors. The new definition has the advantage of being invariant and thus useful in the theory of infinite algebras. (See, for example, Nathan Jacobson, *Structure theory of simple rings without finiteness assumptions*, Trans. Amer. Math. Soc. vol. 57 (1945) pp. 228–245.)

Another illustration of the benefits gained by the use of the representation theory combined with the new tools can be found in the theory of crossed products (Chapter VIII). The latter theory is a significant part of the theory of representations of groups and simple rings over fields. The authors present some new simpler proofs of classical results. Many of the bothersome arguments involving idempotent elements are avoided and thus the proofs on crossed products are less computational.

The book terminates with the formulation of the theory of "nonsemisimple rings with minimum condition" (Chapter VIII). The basic algebraic facts concerned with rings admitting composition series leading to the Cartan invariants are described with an indication of their importance for the theory of modular representations of finite groups. The authors have shown that the superficial sacrifice of purity, as far as methods are concerned, was worthwhile. The literature of algebra has been enriched by an inspiring book.

O. F. G. SCHILLING

NOTES

The Royal Society of Edinburgh has awarded the Gunning Victoria Jubilee Prize for the period 1940–1944 to Professor H. W. Turnbull of the University of St. Andrews for his distinguished contributions to mathematical science and the history of mathematics.

The Carty Gold Medal of the National Academy of Sciences with an honorarium of $2,500 has been awarded to Professor Emeritus W. F. Durand of Stanford University, a member of the National Advisory Committee for Aeronautics. The medal is awarded every other year for "noteworthy and distinguished contributions in any field of science."

The following have been elected fellows of the Royal Society: Mr. A. E. Ingham of the University of Cambridge, Professor P. C. Mahalanobis of Presidency College, Calcutta, and Professor R. E. Peierls of the University of Birmingham.

Dr. A. Erdelyi of the University of Edinburgh has been elected a fellow of the Royal Society of Edinburgh.

Dr. Harold Jeffreys of St. John's College, Cambridge, and Sir Geoffrey I. Taylor, Trinity College, Cambridge, have been elected foreign associates of the National Academy of Sciences.

Dr. Harald Bohr, formerly of the University of Copenhagen, has been elected a foreign member of the American Philosophical Society.

Reverend J. T. O'Callahan of the College of the Holy Cross, Lieutenant Commander in the Navy, has been recommended for the Congressional Medal of Honor for outstanding bravery while in service on the U.S.S. Franklin.

Professor Hassler Whitney of Harvard University and Professor E. P. Wigner of Princeton University have been elected members of the National Academy of Sciences.

Professor Garrett Birkhoff of Harvard University and Professor Norman Levinson of the Massachusetts Institute of Technology have been elected fellows of the American Academy of Arts and Sciences.

Professor R. P. Agnew of Cornell University has been awarded the degree of Doctor of Science by Allegheny College.

Professor Albert Einstein of the Institute for Advanced Study has retired with the title professor emeritus.

Professor L. M. Coffin of Coe College, Cedar Rapids, Iowa, has retired with the title professor emeritus.

Dr. Paul Erdös has been awarded a Guggenheim fellowship.

Professor Harold Davenport of the University College, Bangor, Wales, has been appointed to the Astor chair of mathematics at the University of London.

Professor L. J. Mordell of the University of Manchester has been appointed to the Sudlerian professorship of mathematics at the University of Cambridge, succeeding Professor G. H. Hardy.

Associate Professor Harriet W. Allen of Hollins College has accepted a position with the Air Reduction Company of Stamford, Connecticut.

Associate Professor I. A. Barnett of the University of Cincinnati has been promoted to a professorship.

Mr. A. F. Bartholomay of Keuka College has been promoted to an assistant professorship.

Professor J. A. Benner of Lafayette College has been appointed director of the summer session.

Assistant Professor Z. W. Birnbaum of the University of Washington has been promoted to an associate professorship.

Assistant Professor I. W. Burr of Purdue University has been promoted to an associate professorship.

Dr. Mary D. Clement of Wells College has been promoted to an assistant professorship.

Dr. Helen W. Dodson has been appointed to an associate professorship at Goucher College.

Associate Professor Will Feller of Brown University has been appointed to a professorship at Cornell University.

Dean Tomlinson Fort of Lehigh University has been appointed to a professorship at the University of Georgia.

Mr. C. F. Hall of Rensselaer Polytechnic Institute has been promoted to an assistant professorship.

Dr. Margaret M. Hansman of Colorado College has been promoted to an assistant professorship.

Assistant Professor M. L. Hartung of the University of Chicago has been promoted to an associate professorship of teaching of mathematics.

Dr. F. E. Hohn of the University of Arizona has been appointed to an assistant professorship at Guilford College, Guilford College, North Carolina.

Professor P. L. Hsu of Kunming, China, has been appointed to a visiting professorship of statistics at Columbia University, beginning January 1946.

Assistant Professor H. F. S. Jonah of Purdue University has been promoted to an associate professorship.

Assistant Professor H. T. Karner of Louisiana State University has been promoted to an associate professorship.

Dr. Anne L. Lewis of the University of Chicago has been appointed to an assistant professorship at the Woman's College of the University of North Carolina.

Assistant Professor Charles Loewner of the University of Louisville has been appointed to an associate professorship at Syracuse University.

Dr. Murray Mannos of the University of Notre Dame has been promoted to an assistant professorship.

Professor O. H. Rechard of the University of Wyoming has been appointed Dean of the Liberal Arts College.

Dr. Helene Reschovsky of Russell Sage College has been promoted to an assistant professorship.

Assistant Professor C. E. Rhodes of Union College has been appointed to a professorship at Washington College, Chestertown, Maryland.

Dr. J. A. Sharpe has been made Vice President of C. H. Frost Gravimetric Surveys, Inc., of Tulsa, Oklahoma.

Associate Professor P. A. Smith of Columbia University has been promoted to a professorship.

Associate Professor Marion E. Stark of Wellesley College has been promoted to a professorship.

Professor R. P. Stephens of the University of Georgia will retire on September 1, 1945.

Assistant Professor S. M. Ulam of the University of Wisconsin has been appointed to an associate professorship at the University of Southern California.

Assistant Professor W. A. Vezeau of the University of Detroit has been appointed to an assistant professorship at St. Louis University.

Associate Professor Abraham Wald of Columbia University has been promoted to a professorship.

Dr. J. M. Wolfe of Brooklyn College has been promoted to an assistant professorship.

The following appointments to instructorships are announced: Cedar Crest College, Allentown, Pennsylvania: Miss Minerva E. Butz; East Carolina Teachers College, Greenville, North Carolina: Dr. Ethel Sutherland; University of Illinois: Mr. M. E. Munroe; North Carolina State College: Mr. H. J. Zimmerberg; Rutgers University: Mr. A. G. Makarov, Dr. L. M. Rauch; St. Xavier College for Women, Chicago, Illinois: Miss Betty E. Alexander.

The death of Professor Stanislaw Zaremba of the University of Cracow has been reported.

Dr. Elizabeth B. Cowley died on April 13, 1945. She had been a member of the Society since 1903.

Professor Emeritus T. F. Holgate of Northwestern University died on April 11, 1945, at the age of eighty-six years. He had been a member of the Society since 1895.

Professor S. T. Sanders, Jr., of Southwestern Louisiana Institute died on April 10, 1945, at the age of thirty-seven years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

Algebra and Theory of Numbers

105. Richard Brauer and H. F. Tuan: *On simple groups of finite order.* I.

Using the theory of representations of groups the authors have obtained a number of results for groups of certain types of orders. The following result will be proved: If G is a non-cyclic simple group of order $g=pq^bh$, where p and q are primes and where b and h are positive integers with $h<p-1$, then either $G\cong LF(2, p)$ with $p=2^m\pm 1>3$ or $G\cong LF(2, 2^m)$ with $p=2^m+1>3$; conversely, these groups satisfy the assumptions. As an application, the authors determine all simple groups of order prq^b, where p, r, q are primes and where b is a positive integer. The only simple groups of this type are the well known groups of orders 60 and 168. (Received May 23, 1945.)

106. E. T. Browne: *Concerning a certain ring of homographies.*

In an earlier paper (Ann. of Math. (2) vol. 29 (1928) pp. 483–492) the author considered pairs of n-square matrices A, B which possess the property that not only $A^2=B^2=I$, but also that if $D=(1-\theta)A+\theta B$, then $D^2=I$ for every value of the scalar θ. Such a pair A, B may be said to determine a *linear class* of involutions to which D belongs. In this paper for n even he considers certain triples of matrices A, B, C over the field of complex numbers such that not only $A^2=B^2=C^2=I$, but, in addition, if $D=xA+yB+zC$, then $D^2=k^2I$, where k^2 depends only on x, y and z. Call such a triple a symmetrical triple. The four matrices A, B, C, I are shown to be linearly independent and to form the basis of an algebra of order four over the field of complex numbers. Methods are devised for obtaining from a given symmetrical triple other such triples, and in particular triples which are harmonic to the given triple. Geometric interpretations are given for the cases $n=2$ and $n=4$. (Received May 25, 1945.)

107. A. P. Hillman: *A note on differential polynomials.* II.

Let $F\neq 0$ and G be differential polynomials in the unknowns $y_1, \cdots, y_n$. Let G be of order q_i in y_i. It is known that if G has all the solutions of F, then $G^d=\sum_{j=0}^{s}C_jF_j$ where d and s are positive integers, the C's are differential polynomials, and F_j is the jth derivative of $F_0=F$. The author shows that for j greater than the maximum of the q_i, C_j is in the perfect ideal generated by F (that is, C_j has all the solutions of F). An analogous result holds for partial differential polynomials. (Received May 17, 1945.)

108. L. J. Mordell: *Further contributions to the geometry of numbers for non-convex regions.*

This gives an application of the methods recently discovered by the author to lattice points in the region $|x^n+y^n| \leqq 1$ where $m=p/q>4$ and p, q are odd integers. (Received April 9, 1945.)

ANALYSIS

109. R. P. Agnew: *Tauberian theorems for Nörlund summability.*

It is shown that a Tauberian theorem for Nörlund summability given by R. P. Cesco (Universidad Nacional de La Plata, Publicaciones de la Facultad de Ciencias Fisicomatematicas (2) vol. 4 (1944) pp. 443–445) and more general theorems are implied by familiar Tauberian theorems for Abel summability. (Received May 14, 1945.)

110. R. H. Cameron: *Some examples of Fourier-Wiener transforms of analytic functionals.*

Let $F[x]=F[x(\cdot)]$ be a functional which is defined throughout the space K of complex continuous functions $x(t)$ defined on $0 \leqq t \leqq 1$ and vanishing at $t=0$, and let F possess the property that $F[x+iy]$ is Wiener summable in x over C for each fixed $y(\cdot)$ in K. (C is the subspace of all real functions in K.) Then the functional $G[y]=\int_C^W F[x+iy]d_W x$ is called the Fourier-Wiener transform of $F(x)$. Examples are given of various functionals $F(x)$ which have the property that the transforms of their transforms exist and equal $F(-x)$. The fact that there exist extensive classes of functionals having this property is proved in a paper by W. T. Martin and the author. (Received May 9, 1945.)

111. R. H. Cameron and W. T. Martin: *Fourier-Wiener transforms of analytic functionals.*

Let K be the space of all continuous complex functions $x(t)$ defined on $0 \leqq t \leqq 1$ which vanish at $t=0$, let C be the subset of all real functions of K, and let $F[x]$ be a functional defined on K. Then, as defined by one of the authors in another paper, $G(y)=\int_C^W F(x+iy)d_W x$ is called the Fourier-Wiener transform of $F(x)$ if it exists for all y in K. The purpose of the present paper is to exhibit three extensive classes of functionals which are taken into themselves in a one-to-one manner by the Fourier-Wiener transformation in such a way that the inverse transformation is $F(x)=\int_C^W G(y-ix)d_W y$. In particular, this property is enjoyed by the class E_m of functionals $F(x)$ which are mean continuous, of mean exponential type, and "entire," that is, continuous in the topology defined by root mean square distance between functions, bounded by $A \exp [B\int_0^1 x(t)^2 dt]^{1/2}$, and "entire" in the sense that $F(x+\lambda y)$ is entire in λ for all x and y in K. (Received May 9, 1945.)

112. Nelson Dunford and Robert Schatten: *On the associate and conjugate space for the direct product of Banach spaces.*

The direct product $E_1 \otimes_N E_2$ of two Banach spaces E_1, E_2 (Trans. Amer. Math. Soc. vol. 53 (1943) pp. 195–217), which naturally depends on the norm N, determines uniquely an "associate space" $\bar{E}_1 \otimes_{\bar{N}} \bar{E}_2$ ($\bar{N}$ is the norm "associate" with N) and a conjugate space $\overline{E_1 \otimes_N E_2}$. It is shown that for a "natural crossnorm," $\bar{L} \otimes_{\bar{N}} \bar{L}$ is a proper subset of $\overline{L \otimes_N L}$. Similarly, for a "natural crossnorm," $\bar{l} \otimes_{\bar{N}} \bar{l}$ is a proper subset of $\overline{l \otimes_N l}$. A related example of a non-reflexive crossspace $E_1 \otimes_N E_2$ is constructed, for which E_1, E_2, and N are all reflexive. (Received April 23, 1945.)

113. Herbert Federer: *Coincidence functions and their integrals.*

Two functions f and g are said to coincide at (x, y) if and only if $f(x)=g(y)$. If f parametrizes a k-dimensional surface in Euclidean n-space, and g an $(n-k)$-dimensional surface, the number of coincidences of f and g is the number of intersections of these surfaces. Now keep the first surface fixed and move the second rigidly; or, otherwise said, superimpose upon g an arbitrary isometric transformation S of n-space. Count the number of coincidences of f with the superposition $(S:g)$, that is, the number of intersections of the fixed and the movable surface. Integrate this count over the group of all isometric transformations of n-space with respect to its Haar measure, properly and explicitly normalized. If f and g are sufficiently regular, say Lipschitzian, the value of the integral is the product of the areas of the two surfaces times a number $\beta(n, k)$, which depends only on n and k. This result is established, and $\beta(n, k)$ is evaluated. Further it is proved that the area of a sufficiently regular k-dimensional surface in n-space equals $\beta(n, k)^{-1}$ times the average area of its projection into a k-dimensional subspace of n-space. This, in turn, suggests the definition of a lower semicontinuous area for all continuous k-dimensional surfaces in terms of the stable values of their projections into k-dimensional subspaces. (Received April 11, 1945.)

114. Evelyn Frank: *Corresponding type continued fractions.*

This paper is concerned with a development of properties of *corresponding type continued fractions* ("C-fractions"), $1+a_1z^{\alpha_1}/1+a_2z^{\alpha_2}/1+\cdots$, first studied by Leighton and Scott (Bull. Amer. Math. Soc. vol. 45 (1939) pp. 596–605) and Scott and Wall (Ann. of Math. vol. 41 (1940) pp. 328–349). (i) There is an algorithm analogous to that given by Wall for J-fractions (Bull. Amer. Math. Soc. vol. 51 (1945) pp. 97–105) for expanding an arbitrary power series $P(z)=1+c_1z+c_2z^2+\cdots$ into a C-fraction. (ii) If (1) $\alpha_1\geqq\omega+1, \sum_0^p\alpha_{2k+1}-1\geqq\sum_1^p\alpha_{2k}+\omega\geqq\sum_1^p\alpha_{2k-1}, p=1, 2, 3, \cdots$; $\omega\geqq 0$, then the approximants of the C-fraction are all Padé approximants for $P(z)$. Incidentally, an extension is made of a known theorem on the Padé table. (iii) Necessary and sufficient conditions on the c_p for the α_p to satisfy (1) are given, which reduce to well known conditions in case $\alpha_p=1$. (iv) Certain transformations of Stieltjes (*Œuvres*, vol. 2) are extended to C-fractions for which (1) holds. (Received April 12, 1945.)

115. Fritz Herzog and J. D. Hill: *The Bernstein polynomials for discontinuous functions.*

The well known theorem of Weierstrass on the uniform approximation of continuous functions by polynomials was proved by S. Bernstein as follows: If $f(x)$ is continuous in $[0, 1]$ then the polynomials $B_n(f; x)=\sum_{k=0}^n f(k/n)C_{n,k}x^k(1-x)^{n-k}$ converge to $f(x)$ uniformly in $[0, 1]$. These polynomials, called the *Bernstein polynomials* of f, are determined for a function $f(r)$, defined only for rational numbers r in $[0, 1]$. Such an $f(r)$ will be called a *skeleton*. This paper proposes to investigate the Bernstein polynomials for arbitrary skeletons. For bounded skeletons upper and lower bounds for $\liminf_n B_n(f; x)$ and $\limsup_n B_n(f; x)$ are obtained for each x. Let $\mathfrak{S}$ be the class of skeletons $f(r)$ for which $\lim_n f(r_n)$ exists whenever $r_n\rightarrow x-$ or $r_n\rightarrow x+$, x being any point in $[0, 1]$. For skeletons of the class $\mathfrak{S}$, $\{B_n(f; x)\}$ converges in $[0, 1]$, and this convergence is uniform in any closed subinterval in which the limit function is continuous. The problem of when two skeletons in $\mathfrak{S}$ lead to the same limit function is also completely answered. Convergence conditions of a more special character are given as well as examples which illustrate the theory developed in the paper. (Received April 25, 1945.)

116. J. D. Hill: *Summability of sequences of* 0's *and* 1's.

Let T denote a regular matrix method of summability and $\mathfrak{X}$ the class of all sequences $x \equiv \{\alpha_k\}$, $\alpha_k = 0$ or 1, with infinitely many $\alpha_k = 1$. A result of Steinhaus states that corresponding to each T there is at least one x in $\mathfrak{X}$ which is not summable T. Let $\mathfrak{X}_0(T)$ be the set of all such x and $\mathfrak{X}_1(T)$ its complement with respect to $\mathfrak{X}$. Let $\mathfrak{X}_2(T)$ be the subset of $\mathfrak{X}_1(T)$ composed of all x which are summable T to 1/2. A one-to-one mapping of $\mathfrak{X}$ into the interval $\mathfrak{Y} \equiv (0 < y \leqq 1)$ is obtained by defining y as the dyadic fraction $0.\alpha_0\alpha_1\alpha_2 \cdots$ if $x = (\alpha_0, \alpha_1, \alpha_2, \cdots)$ is a point of $\mathfrak{X}$. The map of $\mathfrak{X}_i(T)$ is denoted by $\mathfrak{Y}_i(T)$ for $i = 0, 1, 2$. A result of Borel is equivalent to the statement that the Lebesgue measure of $\mathfrak{Y}_2(C_1)$ is one, where C_1 is the Cesàro method of order one. The author investigates further questions suggested by the results of Steinhaus and Borel. The principal results obtained are: (1) For each T the sets $\mathfrak{Y}_1(T)$ and $\mathfrak{Y}_2(T)$ have the same measure which is either zero or one. (2) For each T the set $\mathfrak{Y}_1(T)$ is of the first category and hence $\mathfrak{Y}_0(T)$ is of the second category. (3) A sufficient condition is given that $\mathfrak{Y}_2(T)$ have measure one. (4) By (3) it is shown that $\mathfrak{Y}_2(T)$ has measure one if T is either Cesàro, Euler, or Borel summability. (Received May 15, 1945.)

117. A. P. Hillman: *Complex zeros of the Bessel-Weber functions* Y_n.

It is shown that for integral n the Bessel-Weber functions Y_n (for the definition see G. N. Watson, *A treatise on the theory of Bessel functions*, p. 64) have no negative real zeros and that each branch of Y_n has complex zeros whose real parts tend to minus infinity. Numerical values of the first few zeros of Y_0, Y_1, and Y_1' as well as asymptotic expansions for the zeros of Y_n and Y_n' are also given. Nothing seems to have been known previously about zeros of Y_n in the left half-plane, not even for Y_0 or Y_1. (Received May 17, 1945.)

118. C. N. Moore: *Convergence factors in general analysis.* II.

In a paper presented at the summer meeting of 1944 (cf. Bull. Amer. Math. Soc. Abstract 50-9-221) the author developed certain theorems in the field of general analysis which include as particular cases some of the theorems concerning convergence factors in infinite series which may be found in vol. 22 of the American Mathematical Society's Colloquium Publications and analogous theorems concerning infinite integrals. In the present paper the theory is further developed so as to include additional special results in the field of infinite series and infinite integrals. (Received May 14, 1945.)

119. George Polya and Gabor Szegö: *On finite Fourier integrals.*

This paper deals with three independent problems connected by their relation to finite Fourier integrals. 1. As an analogue of a theorem of Fejér-F. Riesz on non-negative trigonometric polynomials the following is proved. Let $f(u)$ be L-integrable in $-a$, a, $F(x) = \int_{-a}^{a} f(u)e^{iux}du$. Let $F(x)$ be L-integrable in $-\infty$, ∞, and $F(x) \geqq 0$ for all real x. Then an L-integrable function $\phi(u)$ exists such that, putting $\Phi(x) = \int_0^a \phi(u)e^{iux}du$, we have $F(x) = |\Phi(x)|^2$. Various applications are given, for instance the inequality $F(x) \leqq af(0)$. 2. This part deals with the characterization of entire functions defined by a finite Hankel integral $x^{-\nu}\int_0^a f(u)J_\nu(xu)du$. The results are generalizations of those due to Paley-Wiener and Plancherel-Polya corresponding to the cases $\nu = \pm 1/2$. Applications are given to the evaluation of various integrals involving Bessel functions. 3. As an analogue of a well known minimum property of the partial sums of a Fourier series, the following is proved. Let $(f(u))^2$ be L-integrable in $-\infty$, ∞, $F(x) \sim \int_{-\infty}^{\infty} f(u)e^{iux}du$. Let $g(u)$ be an arbitrary L-integrable function in $-a$, a,

$G(x) = \int_{-a}^{a} g(u)e^{iux}du$. For all functions $g(u)$ the integral $\int_{-\infty}^{\infty}(F(x)-G(x))^2dx$ is a minimum if $g(u)=f(u)$ (except on a zero set). (Received April 12, 1945.)

120. V. C. Poor: *Complex functions possessing differentials.*

The purpose of this paper is to study the differentials of complex functions, in particular the restricted Hamilton differential which was shown elsewhere to be equivalent to the Rainich differential and also the Young differential and their relations to each other. The Young definition is also given by Mrs. Young in discussing functions possessing differentials. In her paper necessary and sufficient conditions for the existence of a Young differential of a complex function are given. However, some of these conditions are superfluous, and the theorem, properly formulated, is not proved. The necessary and sufficient conditions for the existence of a Young differential will be set up here and proved. Also certain results of the Fundamenta paper will be generalized to the complex plane. Finally, a power series expansion of a polygenic function will be given. (Received April 2, 1945.)

121. V. C. Poor: *On the Hamilton differential.*

This note concerns a modified form of the Hamilton differential, due to G. Y. Rainich, which is the Hamilton differential modified by a linearity condition. Necessary and sufficient conditions for the existence of this differential of a function on a vector n-space are set up and proved. It is shown that the modified definition possesses the linearity property. (Received April 2, 1945.)

122. Raphael Salem and Antoni Zygmund: *Capacity of sets and Fourier series.*

Beurling showed that, if $\sum n(a_n^2+b_n^2)$ is finite, the set of the points of divergence of the trigonometric series (*) $a_0/2+\sum(a_n \cos nx+b_n \sin nx)$ is of logarithmic capacity zero. He also stated without proof that, if $\sum n^{\alpha-\epsilon}(a_n^2+b_n^2)$ converges for some $0<\alpha<1$ and for every $\epsilon>0$, the capacitary dimension of the set of points where the harmonic function $\sum(a_n \cos nx+b_n \sin nx)r^n$ has no finite radial limit is greater than or equal to $1-\alpha$. The latter result may be generalized as follows. If $\sum n^{\alpha}(a_n^2+b_n^2)<\infty$, the set of the points of divergence of (*) is of $(1-\alpha)$-capacity zero. (Received May 8, 1945.)

123. Raphael Salem and Antoni Zygmund: *The approximation by partial sums of Fourier series.*

Let $f(x)$ be a function of period 2π and of the class Lip α, $0<\alpha<1$. It is familiar that the partial sums $s_n(x)$ of the Fourier series of f satisfy the condition $s_n-f=O(n^{-\alpha}\log n)$, and that the logarithm here cannot be removed. (a) The estimate cannot be improved even if in addition to the previous hypotheses it is assumed that f is of bounded variation. (b) If, however, not only the function f but also its absolute variation is of the class Lip α, $0<\alpha<1$, then $s_n(x)-f(x)=O(n^{-\alpha})$. (Received May 8, 1945.)

124. I. M. Sheffer: *Note on Appell polynomials.*

Thorne has recently given (Amer. Math. Monthly vol. 52 (1945) pp. 191–193) an interesting characterization of Appell polynomials by means of a Stieltjes integral. In this note is given a companion representation of Appell sets in terms of a Stieltjes integral. The result is extended to the case of polynomial sets of type zero. For the Appell set $\{P_n(x)\}$ the representation is $P_n(x)=\int_0^\infty \{(x+t)^n/n!\}d\beta(t)$, where $\beta(t)$ is a function of bounded variation on $(0, \infty)$, for which the moment constants $b_n = \int_0^\infty t^n d\beta(t)$ exist, with $b_0 \neq 0$. (Received May 10, 1945.)

125. Otto Szász: *On the absolute convergence of trigonometric series.*

Generalizing a theorem of Fatou on trigonometric series with monotonically decreasing coefficients, the author proves the following theorem: If $\rho_n > 0$, $\rho_{n+1} \leqq c\rho_n$, c a constant, and if the trigonometric series $\sum \rho_n \cos nx$ is absolutely convergent at one point x_0, then $\sum \rho_n < \infty$. The same is true for the sine series if in addition $x_0 \neq 0 \pmod{\pi}$. The proof is quite short and elementary. The author extends this result to series of the type $\sum \rho_n \cos \lambda_n x$, $\sum \rho_n \sin \lambda_n x$, where $0 < \lambda_1 < \lambda_2 < \cdots$. (Received May 10, 1945.)

126. W. J. Trjitzinsky: *Integral equations in problems of representation of functions of a complex variable.*

This work relates to the representation of functions of a complex variable, more general than analytic, in terms of "Cauchy double integrals." (Received May 16, 1945.)

APPLIED MATHEMATICS

127. H. E. Salzer: *Formulas for direct and inverse interpolation of a complex function tabulated along equidistant circular arcs.*

When an analytic function $f(z)$ may be approximated by a complex polynomial of degree $n-1$ passing through the values of the function at n points, according to the Lagrange-Hermite interpolation formula, it often happens that those n points are situated along the arc of a circle (equally spaced) and it is required to obtain $f(z)$ for z off the circle but near the arguments. An important case is when $f(z)$ is tabulated in polar form (including tabulation along the vertices of any regular polygon). The formulas that were obtained will facilitate direct interpolation when $f(z)$ is known at three, four, or five points. They furnish the real and imaginary parts of $L_k^{(n)}(P)$, where $f(z) \sim \sum L_k^{(n)}(P) f(z_k)$, as functions of $P^m = p_m + iq_m$ and θ. Here $P = (z - z_0)/h$, h being the distance between successive points z_k, and θ denotes the angle between successive chords joining the points z_k. For extensive use for a fixed θ, one can readily obtain $L_k^{(n)}(P)$ in the form $\sum C_k^{(n)}(p_m + iq_m)$. A method for inverse interpolation is given, employing the coefficients of the polynomials $L_k^{(n)}(P)$ in an expansion derived in the author's *A new formula for inverse interpolation*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 513–516. (Received May 18, 1945.)

128. H. E. Salzer: *Table of coefficients for double quadrature without differences, for integrating second order differential equations.*

On the basis of a double quadrature of the Lagrange interpolation formula, a table of coefficients has been computed to determine a function at equally spaced points (to within an arbitrary $Ax+B$), when its second derivative is known at those points. The coefficients cover the cases where the second derivative may be approximated by a polynomial ranging from the second to tenth degrees (that is, from three-point through eleven-point formulas), and are given exactly. Their chief value will occur in the numerical solution of ordinary linear differential equations of the second order, which can always be reduced to the form $y'' + g(x)y = h(x)$. They can also be employed to integrate the more general equation $y'' + \phi(x, y) = 0$. In every case it is necessary to begin with a few values of y'' which can always be found by the usual methods.

There are indications that these coefficients can be used to extend the solution of a second order partial differential equation of the form $u_{xx}=u_t+\psi(u, x, t)$, provided that it is known at a rectangular array of points in the x, t-plane and at two other points in the next row of values of t. (Received April 30, 1945.)

GEOMETRY

129. P. O. Bell: *Metric properties of a class of quadratic differential forms.*

In the present paper a new invariant quadratic differential form Ω is geometrically defined for a general pair of surfaces S, S' whose corresponding points x, x' determine the metric normal to S at x. The ratio of the form Ω to the first fundamental form ds^2 of S, in which Ω and ds^2 are defined for a common arc element of S at x, is found to be independent of the direction of the element if and only if the surface S' is the locus of the center of mean curvature of S; the ratio thus determined is the Gaussian curvature K of S at x. It is proved that the form Ω for an arbitrary arc element is identical with the form Kds^2 for either "conjugate" element if and only if the surface S' is the plane net at infinity. The principal directions at x of the tensor whose components are the coefficients of the form Ω are the classical principal directions of S at x for an arbitrary choice of S'. Finally, the net of lines of mean-curvature of S and the mean-conjugate net of S are characterized as integral nets of equations of the form $\Omega=0$, for suitable selections of S'. The author employs dual systems of linear equations of the first order with the use of a tensor notation. (Received May 19, 1945.)

130. L. M. Blumenthal: *Characterization of ϕ-spherical subsets and pseudo sets.*

The class of ϕ-spherical spaces is defined by four metric postulates involving an arbitrary function ϕ. The class contains, for example, those spaces derived from the surface of the ordinary sphere in euclidean $(n+1)$-space by making it metric with respect to geodesic (shorter arc) distance and with respect to euclidean (chord) distance. The paper develops the metric geometry of this class of spaces and obtains the metric characterizations of the subsets and pseudo sets. (The paper is to appear (in Spanish) in Revista de Universidad Nacional de Tucumán, Ser. A. vol. 5 under the title *La caracterización métrica de espacios ϕ-esféricos*.) (Received May 18, 1945.)

131. L. M. Blumenthal: *Metric study of elliptic spaces.*

Among the properties of elliptic spaces which make inapplicable the procedures usually employed in a metric study are (1) the "unusual" character of the locus of points equidistant from two points, (2) the lack of free movability in the large, (3) the necessity of distinguishing between "contained in" and "congruently contained in," (4) the notions of dependence and independence of subsets, if defined in the ordinary way, are not metrically invariant, and (5) the abnormal behavior with respect to equilateral subsets. The writer presents a new approach in which the metrically invariant notions of *relative independence* (*dependence*) and *class independence* (*dependence*) play fundamental roles. In terms of these notions the elliptic line is metrically defined. The extension to plane, and so on, is then possible in conventional manner. Necessary and sufficient conditions (of a *quasi-metric* nature) in order that congruent subsets of an elliptic space be superposable are obtained. (Received May 19, 1945.)

132. S. S. Chern: *Characteristic classes of Hermitian manifolds.* I.

With a manifold having a complex analytic structure is associated in an intrinsic way the fibre bundle of its complex tangent vectors. By retraction it can be assumed in most problems that the fibres are spheres of dimension $2n-1$ on which operates the complex unitary group in n variables. The following theorems are proved: (1) This fibre bundle can be imbedded in the sense of Whitney-Steenrod in the Grassmann manifold of all n-dimensional (complex dimension) linear spaces through the origin of a complex Euclidean space of dimension $n+N$, provided that $n \leq N$. (2) The imbedding is defined up to a homotopy in the sense that two such fibre bundles are equivalent when and only when the maps in the Grassmann manifold are homotopic. (3) The characteristic cohomology classes of the manifold, being the inverse images under this mapping of the cohomology classes of dimension less than or equal to $2n$ of the Grassmann manifold, can be obtained from n basic characteristic classes under operations of the cohomology ring. These n basic classes are the ones analogous to the Stiefel-Whitney classes in the case of real vector fields over real differentiable manifolds, the difference being that they are always classes with integral coefficients. (Received May 25, 1945.)

133. S. S. Chern: *Characteristic classes of Hermitian manifolds.* II.

In the previous paper the author proved that all characteristic classes of a complex analytic manifold of dimension n can be derived from n basic ones by operations of the cohomology ring. In case the manifold carries an Hermitian metric which is everywhere regular, these n classes can be expressed in the sense of de Rham in terms of simple invariant differential forms constructed from the Hermitian metric. One of these relations (namely, that corresponding to the highest dimension) is the formula of Gauss-Bonnet as generalized by Allendoerfer-Weil. If the manifold is the complex projective space with the elliptic Hermitian metric, certain formulas of Cartan and Wirtinger can be derived from these formulas as simple consequences. (Received May 25, 1945.)

134. H. F. DeBaggis: *A reduction of the postulational basis of non-euclidean geometry.*

In order to prove the law of Pasch in Bolyai-Lobachevsky geometry Jenks (Reports of a Mathematical Colloquium vol. 2, p. 10) introduced a postulate VIII which Landin (ibid. vols. 5–6, p. 57) reformulated as follows: If B is between A and C, and if a and c are non-intersecting lines on A and C, respectively, and d is a line intersecting a and c, then d intersects every line on B which does not intersect a and c. This statement can be derived from Jenks' postulates I, V–VII (ibid. vol. 1, p. 46) and the author's postulates II′, III′, IV′ (Bull. Amer. Math. Soc., abstract 51-5-97) which are weaker than Jenks' postulates II, III, IV. From I, II′–IV′, V–VII one can deduce the entire theory of linear and planar order except the theorem that for each two points P and Q there exists a point R such that Q is between P and R. This theorem can be obtained if IV′ is replaced by Jenks' original IV. (Received April 3, 1945.)

135. A. R. Schweitzer: *A theory of congruence in the foundations of geometry.* II.

Supplementing his descriptive axioms in terms of a relation $\alpha\beta K\lambda\mu$ (Trans. Amer. Math. Soc. vol. 10 (1909) p. 309) the author constructs axioms for metrical geometry in terms of a relation $\alpha\beta E\lambda\mu$ as follows: 1. External transitiveness: $\alpha\beta E\lambda\mu$ and $\lambda\mu E\nu\omega$ imply $\alpha\beta E\nu\omega$. 2. Internal transitiveness: $\alpha\beta K\beta\gamma$, $\lambda\mu K\mu\nu$, $\alpha\beta E\lambda\mu$ and $\beta\gamma E\mu\nu$ imply

$\alpha\gamma E\lambda\nu$. 3. Construction: $\alpha\beta\gamma K$, $\lambda\mu\nu K$ imply the existence of ξ, η such that $\lambda\mu K\lambda\xi$, $\lambda\mu\nu K\lambda\xi\eta$, $(\alpha\beta, \beta\gamma, \gamma\alpha)E(\xi\lambda, \lambda\eta, \eta\xi)$, that is, $\alpha\beta E\xi\lambda$, and so on. 4. Linear uniqueness: $\alpha\beta K\alpha\xi$ and $\alpha\beta E\xi\alpha$ imply $\xi=\beta$. 5. Planar uniqueness: $\alpha\beta\gamma K\alpha\beta\xi$, $\alpha\gamma E\alpha\xi$, $\beta\gamma E\beta\xi$ imply $\xi=\gamma$. 6. Equality of angles: If $\alpha\beta\gamma K$, $\lambda\mu\nu K$, $(\alpha\beta, \beta\gamma, \gamma\alpha)$ E $(\lambda\mu, \mu\nu, \nu\lambda)$ then $\alpha\xi\eta I\alpha\beta\gamma$ (that is, $\alpha\xi K\alpha\beta$, $\alpha\eta K\alpha\gamma$) implies the existence of ζ, τ such that $\lambda\zeta\tau I\lambda\mu\nu$ and $(\alpha\xi, \xi\eta, \eta\alpha)$ E $(\lambda\zeta; \zeta\tau, \tau\lambda)$. Definitions: 1. The angle of the triangle $\alpha\beta\gamma$ at α is the class of triads $\alpha\xi\eta$ such that $\alpha\xi\eta I\alpha\beta\gamma$ and $\alpha\beta\gamma K$. 2. The angles of the triangles $\alpha\beta\gamma$, $\lambda\mu\nu$ at α and λ are equal, $\Lambda\beta\gamma=\Lambda\mu\nu$, if and only if, $\alpha\beta\gamma K$, $\lambda\mu\nu K$ and $\alpha\xi\eta I\alpha\beta\gamma$ implies the existence of ζ, τ as indicated under axiom 6. Reference is made to a paper previously reported in Bull. Amer. Math. Soc. vol. 43 (1937) p. 475. (Received May 18, 1945.)

136. A. R. Schweitzer: *A theory of congruence in the foundations of geometry.* III.

Relatively to the set of axioms in the preceding abstract, equality of dyads, $\alpha\beta=\lambda\mu$, "modulo K" and "modulo E" is defined to be $\alpha\beta K\lambda\mu$ and $\alpha\beta E\lambda\mu$ respectively. If E is replaced by K in the preceding axioms then axiom 3 is contradicted, axiom 4 is ineffective ("vacuously satisfied") and the remaining axioms are satisfied. If to the hypothesis of axiom 3 is added "$E\neq K$" then for $E=K$ axiom 3 is ineffective; thus a descriptive or metrical system results according as $E=K$ or $E\neq K$. Correspondingly, equality of angles is defined modulo E $(E\neq K)$ and modulo K $(E=K)$. In the latter case angles are equal if and only if they coincide. Finally, an alternative set of axioms $(E\neq K)$ results from replacing the symbol $(\xi\lambda, \lambda\eta, \eta\xi)$ by its conjugate $(\lambda\xi, \xi\eta, \eta\lambda)$ in axiom 3, assuming that $\alpha\beta K\alpha\beta$ implies $\alpha\beta E\beta\alpha$, and replacing the dyad $\xi\alpha$ by its conjugate $\alpha\xi$ in axiom 4. (Received May 18, 1945.)

TOPOLOGY

137. R. F. Arens: *The linear homogeneous continua of G. D. Birkhoff.*

A linear homogeneous continuum (LHC), in the sense of Birkhoff, is a linearly ordered set L in which every increasing (or decreasing) sequence of elements converges, and which can be placed in one-to-one, order preserving correspondence with any of its closed subintervals. Vasquez and Subieta (*Sopre los continuous homogeneos linealis de George D. Birkhoff*, Boletin de la Sociedad Matematica Mexicana vol. 1 (1944)) have given the first example of an LHC which is not an ordinary real closed interval. The present paper proves (1) if L is an LHC, then L^{ω}, the class of all sequences in L, lexicographically ordered, is also an LHC, (2) if each well ordered subset of L has only countably many distinct elements, the same is true of L^{ω}, and (3) if L is a real closed interval, L^{ω} is not isomorphic to L. (Received May 10, 1945.)

138. R. H. Bing: *Concerning simple plane webs.*

It is shown that a necessary and sufficient condition that a compact plane continuous curve be a simple plane web is that it remain connected and locally connected on the omission of any countable set of points. Using this characterization of a simple plane web, the author considers some of its properties. (Received May 11, 1945.)

139. Salomon Bochner and Deane Montgomery: *Groups of differentiable and real or complex analytic transformations.*

The authors prove the following results: (1) If a Lie group acts on a manifold in such a way that the transforming functions $f(g; x)$ are continuous in g and x simultaneously and if for fixed g the functions $f(g; x)$ are of class C^k (analytic) in x, then the functions $f(g; x)$ are of class C^k (analytic) in the variables g and x simultaneously. (2) If a complex analytic group is compact it is abelian. From (1) and a known theorem is also obtained (3). If a compact group acts effectively on a connected manifold and if for each g, $f(g; x)$ is of class C^k $(k \geqq 1)$ or analytic, then G is a Lie group and the functions $f(g; x)$ are of class C^k or analytic in the variables (g, x) simultaneously. In (1) and (3) the manifold of course must be taken as of class C^k or analytic. The result (2) has been familiar to some mathematicians. (Received May 28, 1945.)

140. R. H. Fox: *Knots in 3-dimensional manifolds.*

Let M be a compact connected 3-dimensional manifold, let γ denote an element of $\pi_1(M)$ and $\bar{\gamma}$ the class of elements of $\pi_1(M)$ conjugate to γ. Let K be any polygonal simple closed curve in M which represents $\bar{\gamma}$ and does not intersect the boundary of M. Let R denote the nucleus of the injection $\pi_1(M-K) \rightarrow \pi_1(M)$ and let $[R]$ denote the commutator subgroup of R. It is proved that the group $\omega(M, \bar{\gamma}) = \pi_1(M-K)/[R]$ is independent of the choice of representative K. The groups $\omega(M, \bar{\gamma})$, determined by the various classes $\bar{\gamma}$ of $\pi_1(M)$, are invariants of the 3-dimensional manifold M. These groups are generally non-abelian and are independent of the homology groups. (Received May 16, 1945.)

141. Dean Montgomery: *Topological groups of differentiable transformations.*

If a locally compact group G acts on a manifold of class C^k in such a way that for a fixed g the transforming functions $f(g; x)$ are of class C^k in x, then all partial derivatives with respect to the x's of order k or less are simultaneously continuous in g and x. It follows that if G is compact and $k \geqq 1$ then G is a Lie group (considered in itself). The author had previously demonstrated this result in the analytic case by another method. (Received May 28, 1945.)

142. A. D. Wallace: *Extension sets.* I.

Although the results to be presented are valid for more general spaces, for simplicity the space H is assumed to be compact metric. By a subspace is meant a closed subset of H. A subspace M will be termed an *extension set* in dimension n (briefly, a J_n) if for each subspace Y any mapping of $Y \cdot M$ into S_n may be extended to Y. (1) Each J_n is a J_{n+1}. (2) The intersection of any collection of J_n's is a J_n. A subset [subspace] Z is said to be n-connected [an n-continuum or a C_n] if every mapping of Z into S_n is inessential on any compact subset contained in Z. (3) The intersection of an ordered (by inclusion) collection of C_n's is a C_n. (4) The intersection of a C_n and a J_n is a C_n. If P is an admissible property for subspaces then a subspace is a P-endelement if it is contained in arbitrarily small neighborhoods with P-boundaries. (5) The intersection of an ordered collection of P-endelements is a P-endelement. (Received April 30, 1945.)

143. A. D. Wallace: *Extension sets.* II.

A point x is an n-cutpoint of a set X if $X-x$ is not n-connected. A subspace will be termed a T_n if each of its subspaces is a C_n. (1) An n-cutpoint of a J_n is an n-cutpoint of H. (2) In order that a subspace be a J_n it is sufficient that its complement be the union of a collection of pairwise disjoint open sets whose boundaries are of type T_n. If f maps the subspace X into S_n then the subspace Y will be termed an essential membrane for f provided that f can be extended to every proper subspace of Y but not to Y. (3) If M is a J_n containing the subspace X, and Y is an essential membrane for the mapping f of X into S_n [a C_n irreducible about X] then Y is contained in M. (4) If X is a subspace not cut by any T_n and maximal relative to this property then X is a J_n. (Received April 30, 1945.)

144. A. D. Wallace: *Extension sets.* III.

Referring to definitions given in previous abstracts, the author proves five theorems. (1) Any T_n is a T_{n+1}. (2) If H is a C_n and for each subspace X each mapping of X into S_n has a unique essential membrane then H is a T_{n+1}. (3) If X is a subspace and Y is a C_n irreducible about X [an essential membrane for some mapping of X into S_n] then no point of $Y-X$ is a T_n-endelement. (4) If H is a C_n and Z is the set of all points not T_n-endelements then Z is n-connected. (5) If H is a C_n and X is a C_n-endelement then X is a C_n. Many of the results presented are valid if H is a compact Hausdorff space. In some cases S may be replaced by a space having the neighborhood extension property. In the paper reference will be made to the work of Ayres, Borsuk, Eilenberg, Hurewicz, and G. T. Whyburn. (Received April 30, 1945.)

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A NOTE ON GROUPS WITHOUT ISOMORPHIC SUBGROUPS

IRVING KAPLANSKY

1. Introduction. One of the theorems obtained by R. A. Beaumont in a recent paper[1] states that if G is an abelian group of finite rank all of whose elements have finite order, then G has no proper isomorphic subgroups. If we interpret G as a vector space over the ring of integers I, it is natural to raise the question: what properties of I are needed for this result? In this note we find two conditions which are sufficient, given as (1) and (2) in our main theorem. Whether these conditions are also necessary remains to be determined.

2. A preliminary lemma. We shall require the following lemma, which is closely related to various known results.[2]

LEMMA. *Let G be a vector space over a commutative principal ideal ring R. Suppose that G can be spanned by r elements, and that H is a subspace that can be spanned by a finite number of elements of H. Then H can be spanned by r elements of H.*

PROOF. Let G be spanned by $g_1, \cdots, g_r$; H by $h_1, \cdots, h_s$. Then $h_i = \sum_j \alpha_{ij} g_j$ $(\alpha_{ij} \in R)$. Let β be the H.C.F. of $\alpha_{11}, \cdots, \alpha_{s1}$, so that $\beta = \sum_i \lambda_i \alpha_{i1}$. Define $k_1 = \sum_i \lambda_i h_i$. Then $k_1 \in H$, and since

$$k_1 = \beta g_1 + \text{a linear combination of } g_2, \cdots, g_r,$$

$k_1, g_2, \cdots, g_r$ span H. After r such steps we shall obtain elements $k_1, \cdots, k_r$ in H which span H.

3. The theorem. Let V be a vector space over a ring R.

DEFINITION. V has rank r over R if any finite subset of V can be spanned over R by r elements of V, and if r is the smallest integer with this property.

THEOREM. *Let V have finite rank r over R and suppose that:*

(1) *R is a commutative principal ideal ring.*

(2) *Every proper residue class ring of R is finite.*

(3) *For every $v \in V$, there is an $\alpha \neq 0$ in R such that $\alpha v = 0$.*

Then V has no isomorphic proper subspaces.

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[1] *Groups with isomorphic proper subgroups*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 381–387.

[2] For example, C. J. Everett, *Vector spaces over rings*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 312–316.

PROOF. Take any $v \in V$ and an $\alpha \neq 0$ with $\alpha v=0$. Let W be the subspace consisting of all w with $\alpha w=0$, and let A be the ideal in R consisting of all β with $\beta W=0$. By hypothesis R/A is finite; say it has n elements. Then W is finite; in fact, it has at most n^r elements. For take any k elements of W and let H be the subspace they span. By our hypothesis H can be embedded in a subspace G spanned by r elements. Then by our lemma, H can also be spanned by r elements $h_1, \cdots, h_r$ of H. Since H is annihilated by A, there are at most n^r different elements obtainable by taking linear combinations of $h_1, \cdots, h_r$. Hence $k \leqq n^r$, as desired.

Now suppose there is an isomorphism mapping V into a proper subspace V'. In this mapping W must be sent into part of itself, for W consists of all elements annihilated by α. But W is finite; hence W must be mapped into all of itself. In particular, the (arbitrary) element v is part of the image space V', whence $V'=V$.

NEW YORK CITY

THE BASIS THEOREM FOR VECTOR SPACES OVER RINGS

C. J. EVERETT

It is the purpose of this note to establish the following theorem:

THEOREM. *A vector space* $M = u_1K + \cdots + u_mK$ *of* m *basis elements over a ring* $K = \{0, a, b, \cdots, 1\}$ *with unit* 1 *has the property that every subspace* $N > 0$ *possesses a basis of* $n \leqq m$ *elements if and only if* K *is a right principal-ideal-ring without zero-divisors.*

That such a ring insures the basis condition for subspaces is well known [3, p. 121].[1]

Suppose now that every subspace $N > 0$ has a basis of $n \leqq m$ elements. It has been shown [2, Theorem (F)] that every right ideal $R > 0$ of K must then have a single generator: $R = r_0K$, where $r_0k = 0$ implies $k = 0$. Moreover, since every right ideal has a finite set of generators, the ascending chain condition must hold for right ideals of K [3, p. 26]. It therefore suffices to prove the following two lemmas.

LEMMA 1. *In a ring* K *with unit* 1 *and ascending chain condition for right ideals, equations* $ab = 1$, $ac = 0$ *imply* $c = 0$.

If $c \neq 0$, the linear transformation $k \rightarrow ak$, $k \in K$, would be of type (iv) [2, p. 313], that is, $K/K_0 \cong K$, and $0 < K_0 < K_1 < K_2 < \cdots$, where K_i is defined inductively as the set of all elements of K mapped into elements of K_{i-1}. This contradicts the chain condition.

LEMMA 2. *A ring* K *with unit in which every right ideal* $R > 0$ *is of the form* r_0K, *where* $r_0k = 0$ *implies* $k = 0$, *has no zero divisors.*

Let $sc = 0$, $s \neq 0$, and $sK = r_0K \neq 0$, where $r_0k = 0$ implies $k = 0$. We have $s = r_0a$, $r_0 = sb = r_0 \cdot ab$, $r_0(ab - 1) = 0$, and hence $ab = 1$. Also, $sc = 0 = r_0ac$, and $ac = 0$. Since Lemma 1 applies to K, $c = 0$.

It should be noted that the result follows also from a result of Baer's [1, Theorem 5 or Lemma 4] which states that in a ring with unit and weak maximal condition, $ab = 1$ implies $ba = 1$.

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[1] Numbers in brackets refer to the bibliography.

2. C. J. Everett, *Vector spaces over rings*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 312–316.
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ON A CONSTRUCTION FOR DIVISION ALGEBRAS OF ORDER 16

R. D. SCHAFER

It is not known whether there exist division algebras of order 16 (or greater) over the real number field $\mathfrak{R}$. In discussing the implications of this question in algebra and topology, A. A. Albert told the author that the well known Cayley-Dickson process[1] does not yield a division algebra of order 16 over $\mathfrak{R}$ and suggested a modification of that process which might. It is the purpose of this note to show that, while Albert's construction can in no instance yield such an algebra over $\mathfrak{R}$, it does yield division algebras of order 16 over other fields, in particular the rational number field R.

Initially consider an arbitrary field F. Let C be a Cayley-Dickson division algebra of order 8 over F. Define[2] an algebra of order 16 over F with elements $c=a+vb$, $z=x+vy$ (a, b, x, y in C) and with multiplication given by

$$cz = (a + vb)(x + vy) = (ax + g\cdot ybS) + v(aS\cdot y + xb) \tag{1}$$

where S is the involution $x \rightleftarrows xS = t(x) - x$ of C and g is some fixed element of C. The Cayley-Dickson process is of course the instance $g=\gamma$ in F.

For A to be a division algebra over F the right multiplication[1] R_z must be nonsingular for all $z \neq 0$ in A. Now

$$R_z = \begin{pmatrix} R_x & SR_y \\ SL_yL_g & L_x \end{pmatrix}$$

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[1] See [1] and [2] for background and notations. Numbers in brackets refer to the references cited at the end of the paper.

[2] We should remark that this modification of the Cayley-Dickson process does yield non-alternative division algebras of orders 4 and 8 over $\mathfrak{R}$ when applied to the algebras of complex numbers and real quaternions instead of to C. See R. H. Bruck, *Some results in the theory of linear non-associative algebras*, Trans. Amer. Math. Soc. vol. 56 (1944) pp. 141–199, Theorem 16C, Corollary 1, for a generalization.

and, if $g\neq 0$, R_z is nonsingular in case either $x=0$, $y\neq 0$ or $x\neq 0$, $y=0$. Therefore let $x\neq 0$, $y\neq 0$. Then

$$\begin{aligned} |R_z| &= \begin{vmatrix} R_x & SR_y \\ SL_yL_g & L_x \end{vmatrix} = \begin{vmatrix} R_x & 0 \\ SL_yL_g & L_x - SL_yL_gR_x^{-1}SR_y \end{vmatrix} \\ &= |R_x|\cdot\left| L_x - \frac{1}{n(x)} R_ySR_gSL_xR_y \right| \\ &= |R_x|\cdot\left| L_x - \frac{1}{n(x)} R_ySR_gSR_x^{-1}R_{xy}L_x \right| \end{aligned}$$

by a lemma of Moufang.[3] Hence $|R_z| = |L_x|\cdot|n(x)R_yR_g|^{-1}\cdot|n(x)R_xR_gR_y - n(g)n(y)R_{xy}|$. That is, A is a division algebra over F if and only if the transformation

$$n(x)R_xR_gR_y - n(g)n(y)R_{xy} \tag{2}$$

is nonsingular for all $x, y\neq 0$ in C.

Now let $F=\mathfrak{R}$, the field of real numbers. The non-scalar[4] element g generates $B\subset Q=B+uB$, Q a real quaternion algebra, $u^2=-n(u)$, $gu=u\cdot gS$. Multiplication in the Cayley algebra $C=Q+wQ$ is defined by the right multiplication

$$R_{q+wr} = R_{(q,r)} = \begin{pmatrix} R_q & SR_r \\ -SL_r & L_q \end{pmatrix}$$

for q, r in Q and S the involution $q\rightleftarrows qS=t(q)-q$ of Q. Specialize two elements x, y of C in the following manner. Let $y=u\in Q$; then $y^2=-n(y)$, $gy=y\cdot gS$. Let $x\in wQ$ and $n(x)=\zeta n(y)$ where $\zeta>0$, $\zeta^2=n(g)$. Then $xy=(0, x)R_{(y,0)}=(0, yx)$ and

$$\begin{aligned} &|n(x)R_xR_gR_y - n(g)n(y)R_{xy}| \\ &= \left| n(x)\begin{pmatrix} 0 & SR_x \\ -SL_x & 0 \end{pmatrix}\begin{pmatrix} R_g & 0 \\ 0 & L_g \end{pmatrix}\begin{pmatrix} R_y & 0 \\ 0 & L_y \end{pmatrix} - n(g)n(y)\begin{pmatrix} 0 & SR_{yx} \\ -SL_{yx} & 0 \end{pmatrix} \right| \\ &= \begin{vmatrix} 0 & n(x)SR_xL_{yg} - n(g)n(y)SR_{yx} \\ n(g)n(y)SL_{yx} - n(x)SL_xR_{gy} & 0 \end{vmatrix} \\ &= |R_xL_x|\cdot|n(x)L_{yg} - n(g)n(y)R_y|\cdot|n(g)n(y)L_y - n(x)R_{gy}| \end{aligned}$$

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[3] [2, Lemma 1].

[4] The Cayley-Dickson process (the case $g=\gamma$, a scalar) may be eliminated by this argument too. If $\gamma\geqq 0$, let $y=\beta$ in $\mathfrak{R}$, $n(x)=\gamma\beta^2$; if $\gamma<0$, let $y=i$, $f=j$, $n(x)=-\gamma$ in what follows.

since x, y, g are quaternions. That is, $|n(x)L_{yg}-n(g)n(y)R_y|=0$ would imply that transformation (2) is singular.

Choose $f=\zeta y+yg$. Then $f\{n(x)L_{yg}-n(g)n(y)R_y\}=n(x)\zeta ygy+n(x)ygyg-n(g)n(y)\zeta y^2-n(g)n(y)ygy=n(x)y^2gS\cdot g-n(g)n(x)y^2=0$. Hence (2) is singular and A is not a division algebra over $\mathfrak{R}$.

The easy generalization that there is no choice of g to make A a division algebra of order 16 over any field F should not be made. For the singularity of transformation (2) implies that there exists an element $h\neq 0$ in C such that $n(x)\{(hx)g\}y=n(g)n(y)h(xy)$. Since the norm of a product is the product of the norms in an alternative division algebra,[5]

$$\overline{n(x)}^3n(h)n(g)n(y)=\overline{n(g)}^2\overline{n(y)}^3n(h)n(x)\quad\text{or}\quad\overline{n(x)}^2=n(g)\overline{n(y)}^2$$

in case $g\neq 0$. That is, the transformation (2) cannot be singular (and A is therefore a division algebra) for any choice of g in C such that $n(g)$ is not the square of an element in F.

For example, let F be in particular the field R of rational numbers, and $g=1+i$ so that $n(g)=2$. Then the algebra A with multiplication defined by (1) is a division algebra of order 16 over R.

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[5] [2, Lemma 2].

A PROOF OF A THEOREM ON COMMUTATIVE MATRICES

PACO LAGERSTROM

The following theorem is well known (see, for example, Wedderburn, *Lectures on matrices*, p. 106):

"*If the matrix B commutes with every matrix that commutes with A, then B is a scalar polynomial of A.*"

It is thought, however, that the proof given below is simple enough to be of interest. The proof is based on the main theorem for abelian groups with a finite number of generators. The version of this theorem given in van der Waerden, Moderne Algebra, vol. 2, pp. 114 and 122, is especially well suited for our purpose. Let $\mathfrak{M}$ be a finite-dimensional vector space over a commutative field K. Let A be a fixed linear endomorphism of $\mathfrak{M}$. All endomorphisms of the form $P(A)$, where $P(x)$ is a polynomial with coefficients in K, form a euclidean ring of operators on $\mathfrak{M}$. "Admissible subgroups" (van der Waerden, Moderne Algebra, vol. 1, p. 145) with respect to this set of operators are those subspaces of $\mathfrak{M}$ which are invariant under A. The main theorem about the decomposition of abelian groups, as applied to $\mathfrak{M}$, then reads: There exist a finite number of subspaces $\mathfrak{M}_i$ and polynomials over K, $P_i(x)$, such that:

(1a) $\mathfrak{M}$ is a direct sum of the $\mathfrak{M}_i$.

(1b) $\mathfrak{M}_i$ is invariant under A.

(1c) Each $\mathfrak{M}_i$ is cyclic. This means that there exist elements e_i such that each element of $\mathfrak{M}_i$ is of the form $P(A)e_i$.

(1d) $P_i(x)$ generates the annihilating ideal of $\mathfrak{M}_i$.

(1e) $P_{i+1}(x)$ divides $P_i(x)$.

It follows that $P_1(A)=0$ and that $P(A)=0$ implies that $P_1(x)$ divides $P(x)$. (In a terminology sometimes used $P_i(x)$ is the order of e_i with respect to A and $P_1(x)$ is the minimal polynomial of A. Thus the order of e_1 is the minimal polynomial of A. Conversely, once the existence of an element with this property has been demonstrated, the decomposition theorem is easily proved.)

We denote by E_i the projection on $\mathfrak{M}_i$, that is, the linear endomorphism uniquely defined by: $E_if=f$ if f is in $\mathfrak{M}_i$ and $E_if=0$ if f is in $\mathfrak{M}_j$, $j\neq i$. It follows that f is in $\mathfrak{M}_i$ if and only if $E_if=f$. An endomorphism C which commutes with E_i leaves $\mathfrak{M}_i$ invariant because if f is in $\mathfrak{M}_i$, then $E_iCf=CE_if=Cf$. Conversely, if all $\mathfrak{M}_i$ are invariant un-

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der C, then $CE_i = E_iC$. Also, if E is the identity mapping, then $E = \Sigma E_i$.

After these preliminaries we are ready to give a concise formulation and a proof of the theorem stated at the very beginning of this note:

THEOREM 1. *Let $\mathfrak{M}$ be a finite-dimensional vector space over a commutative field K and A and B linear endomorphisms of $\mathfrak{M}$ such that B commutes with every endomorphism that commutes with A. Then there exists a polynomial $\overline{Q}(x)$ with coefficients in K such that $B = \overline{Q}(A)$.*

Using the notation explained above we first prove a lemma:

LEMMA. *A polynomial $Q(x)$ may be found satisfying the relation $Be_1 = Q(A)e_1$ (where as remarked above any element whose order is the minimal polynomial of A may be taken as e_1). $Q(x)$ also satisfies the relation $BE_1 = Q(A)E_1$.*

PROOF. Since each $\mathfrak{M}_i$ is invariant under A, each E_i commutes with A and hence with B. Thus each $\mathfrak{M}_i$ is also invariant under B, in particular Be_1 is in $\mathfrak{M}_1$ from which the existence of $Q(x)$ follows by (1c). Now let f be in $\mathfrak{M}_1$. Then for some polynomial $P(x)$, $f = P(A)e_1$ and $BE_1f = Bf = BP(A)e_1 = P(A)Be_1 = P(A)Q(A)e_1 = Q(A)f = Q(A)E_1f$. If f is in $\mathfrak{M}_i$, $i \neq 1$, then $BE_1f = Q(A)E_1f = 0$ which concludes the proof of the lemma.

In order to prove that the $Q(x)$ defined in the lemma may be taken as the $\overline{Q}(x)$ of Theorem 1, we have to show that $BE_i = Q(A)E_i$ for any i. For this purpose we define a mapping M_i by:

(2a) $M_iR(A)e_1 = R(A)e_i$ for any polynomial $R(x)$.

(2b) $M_if = 0$ for f in $\mathfrak{M}_j$, $j \neq 1$.

(2c) M is a linear endomorphism of $\mathfrak{M}$.

Condition 2a defines a one-valued mapping because if $R(A)e_1 = S(A)e_1$, then $R(x) - S(x)$ is divisible by $P_1(x)$ (by 1d) and by $P_i(x)$ (by 1e) and hence $R(A)e_i = S(A)e_i$. Evidently M_i commutes with A and hence with B. From this follows: $Be_i = BM_ie_1 = M_iBe_1 = M_iQ(A)e_1 = Q(A)e_i$. Since $Be_i = Q(A)e_i$, we may prove as above that $BE_i = Q(A)E_i$.

The proof is now complete since $B = \Sigma BE_i = \Sigma Q(A)E_i = Q(A)$.

The above proof suggests a generalization. Namely, in Theorem 1 the vector space $\mathfrak{M}$ and the ring of polynomials of A may be replaced by any abelian group with a commutative ring of operators for which the decomposition theorem is valid. It can then be seen easily that the proof given in this note remains valid without essential changes.

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A REMARK ON A RESULT DUE TO BLICHFELDT

LOO-KENG HUA

Let $\sigma \geqq 1$ and $\xi_1, \cdots, \xi_n$ be $n \geqq 3$ linear forms of the real variables $x_1, \cdots, x_n$ of nonvanishing determinant Δ. For simplicity's sake we assume $|\Delta| = 1$. Let $2s$ of the forms be pairwise conjugate complex and the remaining $n-2s$ be real. Then

$$|\xi_1|^\sigma + \cdots + |\xi_n|^\sigma \leqq 1$$

defines a symmetric convex body in the x-space, the volume $V(\sigma)$ of which equals

$$2^n \cdot \frac{\{\Gamma(1+\alpha)\}^{n-2s}\{\pi\Gamma(1+2\alpha)/2^{1+2\alpha}\}^s}{\Gamma(1+n\alpha)} \qquad (\alpha = 1/\sigma).$$

Minkowski's principle states that there is a lattice point $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$ satisfying the inequality

$$|\xi_1|^\sigma + \cdots + |\xi_n|^\sigma \leqq r^\sigma$$

provided

$$r^n \geqq 2^n V^{-1}(\sigma). \tag{1}$$

By means of Blichfeldt's method, van der Corput and Schaake[1] obtained a sharpening of this result for $\sigma \geqq 2$. Decisive in this procedure is an inequality of the following form

$$\sum_{p,q=1}^{k} |z_p - z_q|^\sigma \leqq \epsilon(\sigma)k \cdot \sum_{p=1}^{k} |z_p|^\sigma, \tag{2}$$

where the factor $\epsilon(\sigma)$ depends neither on the arbitrary complex numbers z_p nor on k. Once such an inequality is known, (1) may be replaced by

$$r^n \geqq (\epsilon(\sigma))^{n/\sigma} \cdot \frac{n+\sigma}{\sigma} \cdot V^{-1}(\sigma). \tag{3}$$

The elementary relation

$$|u - v|^\sigma \leqq 2^{\sigma-1}(|u|^\sigma + |v|^\sigma)$$

(following from the fact that x^σ is a convex function of $x>0$) implies (2) with $\epsilon(\sigma) = 2^\sigma$. Substituted in (3) this does not improve, but on

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[1] Acta Arithmetica vol. 2 (1936) pp. 152–160.

the contrary worsens, Minkowski's inequality. However, van der Corput and Schaake obtained the better value $2^{\sigma-1}$ for $\sigma \geqq 2$. I shall show here that $\epsilon(\sigma)=2$ is a legitimate choice for $1 \leqq \sigma \leqq 2$ and that both facts follow almost immediately from Marcel Riesz's convexity theorem.

Indeed, specialize this theorem (Theorem 296 on p. 219 of Hardy, Littlewood and Pólya's *Inequalities*) by taking $\gamma=\alpha$ and the X as the linear forms $X_{pq}=z_p-z_q$. It then turns out that the logarithm of the maximum $M_k(\alpha)$ of

$$\left\{\sum_{p,q=1}^{k} | z_p - z_q |^{1/\alpha} / k \sum_{p=1}^{k} | z_p |^{1/\alpha}\right\}^{\alpha}$$

for fixed k and variable $z_1, \cdots, z_k$ is a convex function of α in the interval $0 \leqq \alpha \leqq 1$. One readily verifies that

$$M_k(0) = 2, \qquad M_k(1/2) = 2^{1/2}, \qquad M_k(1) = 2(1 - 1/k) \leqq 2.$$

As

$$\left\{\sum_{p} | z_p |^{1/\alpha}\right\}^{\alpha} \to \max | z_p | \quad \text{for} \quad \alpha \to 0,$$

the first equation follows from $\max |z_p - z_q| \leqq 2 \cdot \max |z_p|$ together with the observation that the upper bound 2 is attained for $z_1=1$, $z_2=-1$, $z_3= \cdots =z_k=0$. Similarly the two other equations are immediate consequences of the elementary inequalities

$$\sum_{p,q} | z_p - z_q |^2 = 2k \sum_{p} | z_p |^2 - 2 \left| \sum_{p} z_p \right|^2 \leqq 2k \sum_{p} | z_p |^2,$$

$$\sum_{p \neq q} | z_p - z_q | \leqq \sum_{p \neq q} (| z_p | + | z_q |) = 2(k - 1) \sum_{p} | z_p |,$$

and the corresponding obvious observations about the z_p for which the upper bound is reached.

Let us use 2 as the basis of our logarithms. Then the values of $\log_2 M_k(\alpha)$ are 1, 1/2 and less than or equal to 1 for $\alpha=0, 1/2, 1$ respectively, and hence the broken line consisting of $1-\alpha$ for $0 \leqq \alpha \leqq 1/2$ and α for $1/2 \leqq \alpha \leqq 1$ gives an upper bound for the convex function $\log_2 M_k(\alpha)$. We thus obtain the promised result that (2) holds with

$$\epsilon(\sigma) = 2^{\sigma-1} \quad \text{for} \quad \sigma \geqq 2 \qquad \text{and} \qquad \epsilon(\sigma) = 2 \quad \text{for} \quad 1 \leqq \sigma \leqq 2. \tag{4}$$

Both choices are the best possible of their kinds, as, for $0 \leqq \alpha \leqq 1/2$, is shown by the example $k=2$, $z_1=-z_2=1$, and, for $1/2 \leqq \alpha \leqq 1$, by the example $z_1=-z_2=1$, $z_3= \cdots =z_k=0$, with large k.

Consider the case $1 \leqq \sigma \leqq 2$. If we substitute the value $\epsilon(\sigma) = 2$ in (3), we shall find that it does not always improve Blichfeldt's known inequality, in particular not for the most interesting case $\sigma = 1$. We observe that

$$\left(\frac{|\xi_1|^\sigma + \cdots + |\xi_n|^\sigma}{n}\right)^{1/\sigma}$$

is an increasing function of the exponent σ, while the upper bound for its lattice minimum as derived from (3), namely,

$$\left(\frac{2}{n}\right)^{1/\sigma}\left(\frac{n+\sigma}{\sigma}\right)^{1/n}(V(\sigma))^{-1/n}, \tag{5}$$

is not. For $s = 0$ the expression (5) tends to a limit with $n \to \infty$, namely

$$\frac{1}{2}\left(\frac{2}{\sigma e}\right)^{1/\sigma}\Big/\Gamma\left(1+\frac{1}{\sigma}\right) = 2^{\alpha-1}\left(\frac{\alpha}{e}\right)^{\alpha}\Big/\Gamma(1+\alpha).$$

The logarithmic derivative of this function with respect to α is negative for $\alpha = 1/2$ and positive for $\alpha = 2/3$, and hence this function has a minimum between $\sigma = 2$ and $\sigma = 1.5$; numerical computation gives as its location $\sigma = \sigma_0 = 1.8653 \cdots$.[2] At this point the value of the function is

$$\leqq 1/(3.146e)^{1/2}$$

which is slightly better than the constant

$$1/(\pi e)^{1/2}$$

due to Blichfeldt.[3]

In conclusion, for $2 \geqq \sigma \geqq \sigma_0$, (1) may be replaced by

$$r^n \geqq 2^{n/\sigma}\left(\frac{n+\sigma}{\sigma}\right)V^{-1}(\sigma),$$

and, for $1 \leqq \sigma \leqq \sigma_0$, (1) may be replaced by

$$r^n \geqq 2^{n/\sigma_0}\left(\frac{n+\sigma_0}{\sigma_0}\right)V^{-1}(\sigma_0).$$

This would be true however σ_0 were chosen within the limits $1 \leqq \sigma_0 \leqq 2$; our special choice approaches the best possible for $n \to \infty$ (and $s = 0$) and is sharp enough to beat Blichfeldt's record by a slight margin, even for small n.

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[2] The author is indebted to Mr. Sze for this numerical value.

[3] Trans. Amer. Math. Soc. vol. 15 (1914) pp. 227–235.

SOME REMARKS ON EULER'S ϕ FUNCTION AND SOME RELATED PROBLEMS

PAUL ERDÖS

The function $\phi(n)$ is defined to be the number of integers relatively prime to n, and $\phi(n)=n\cdot\prod_{p|n}(1-p^{-1})$.

In a previous paper[1] I proved the following results:

(1) The number of integers $m\leqq n$ for which $\phi(x)=m$ has a solution is $o(n[\log n]^{\epsilon-1})$ for every $\epsilon>0$.

(2) There exist infinitely many integers $m\leqq n$ such that the equation $\phi(x)=m$ has more than m^c solutions for some $c>0$.

In the present note we are going to prove that the number of integers $m\leqq n$ for which $\phi(x)=m$ has a solution is greater than $cn(\log n)^{-1}\log\log n$.

By the same method we could prove that the number of integers $m\leqq n$ for which $\phi(x)=m$ has a solution is greater than $n(\log n)^{-1}(\log\log n)^k$ for every k. The proof of the sharper result follows the same lines, but is much more complicated. If we denote by $f(n)$ the number of integers $m\leqq n$ for which $\phi(x)=m$ has a solution we have the inequalities

$$n(\log n)^{-1}(\log\log n)^k < f(n) < n(\log n)^{\epsilon-1}.$$

By more complicated arguments the upper and lower limits could be improved, but to determine the exact order of $f(n)$ seems difficult.

Also Turán and I proved some time ago that the number of integers $m\leqq n$ for which $\phi(m)\leqq n$ is $cn+o(n)$. We shall give this proof, and also discuss some related questions:

LEMMA 1. *Let* $a<\epsilon$, $b<n$, $a\neq b$, $\epsilon=(\log\log n)^{-100}$. *Then the number of solutions* $N_n(a,b)$ *of*

$$(1)\qquad (p-1)a=(q-1)b,\qquad p\leqq na^{-1},\qquad q\leqq nb^{-1},$$

p, q *primes, does not exceed*

$$(2)\qquad \frac{(a,b)}{ab}\,\frac{n}{(\log n)^2}(\log\log n)^{30}.$$

PROOF. Put $(a,b)=d$. Then we have $p\equiv 1 \bmod bd^{-1}$. Also $(p-1)ab^{-1}+1=q$ is a prime. We can assume that both p and q in (1) are greater

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[1] *On the normal number of prime factors of* $p-1$, Quart. J. Math. Oxford Ser. vol. 6 (1935) pp. 205–213.

than $n^{1/2}$, for the exceptional values of p and q give only $2n^{1/2}$ solutions of (1). Let $r<n^{\delta}$, where $\delta=(\log\log n)^{-10}$, be a prime. If p is a solution of (1) it must satisfy the following conditions

$$\begin{aligned} &p \equiv 1 \bmod bd^{-1}, && p < na^{-1}, \\ &p \not\equiv 0 \bmod r, && p \not\equiv (-ba^{-1}+1) \bmod r. \end{aligned}$$

If r is not a divisor of $a(a-b)$ the excluded two residues are different. Thus we obtain by Brun's argument[2]

$$N_n(a, b) < 2n^{1/2} + c_1 nd(ab)^{-1} \prod_{r \nmid a(a-b)} (1 - 2r^{-1}),$$

where r runs through the primes less than n^{δ}.

Now it is well known that[3]

$$\prod_{r \leqq x} (1 - 2r^{-1}) < c_2(\log x)^{-2}, \qquad \prod_{r \mid x} (1 - 2r^{-1}) > c_3(\log\log x)^{-2}.$$

Hence

$$\begin{aligned} N_n(a, b) &< 2n^{1/2} + c_4 nd(ab)^{-1}(\log\log n)^{22}(\log n)^{-2} \\ &< nd(ab)^{-1}(\log\log n)^{30}(\log n)^{-2}, \end{aligned}$$

which completes the proof.

LEMMA 2. $\sum(p-1)^{-1}<(\log\log n)^{20}d^{-1}$ *if this sum is extended over all* $p<n^{\epsilon}$ *for which* $p\equiv 1 \bmod d$.

Clearly (summing over the indicated p)

$$\sum p^{-1} \leqq d^{-1} \sum{}' x^{-1},$$

where the dash indicates that the summation is extended over the x for which $x<nd^{-1}$ and $xd+1$ is a prime. Let $y<nd^{-1}$; first we estimate the number of these $x\leqq y\leqq n$. Let $r<y^{\delta}$ ($\delta=(\log\log n)^{-10}$) be a prime; if $(r, d)=1$ then $x\not\equiv -d^{-1} \bmod r$. Brun's method[4] gives that the number of these $x\leqq y$ is less than

$$cy \prod (1 - r^{-1}) < cy(\log y)^{-1}(\log\log y)^{10} \log\log d,$$

where the product is extended over the r which satisfy $r<y^{\delta}$, $(r, d)=1$. Thus a simple argument gives

$$\sum{}' x^{-1} < c \sum_{z<n} (\log\log z)^{10}(\log\log d)(z \log z)^{-1} < (\log\log n)^{20},$$

which proves the lemma.

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[2] Landau, *Vorlesungen über Zahlentheorie*, vol. 1, p. 71.

[3] Hardy-Wright, *Theory of numbers*.

[4] Landau, ibid.

Lemma 3. *The number $A(n)$ of integers m of the form $m=pq$, where*

$$pq \leqq n, \tag{3}$$

p, q primes, $p>q$, $q<n^{\epsilon}$, equals

$$n(\log\log n)(\log n)^{-1} + o([n(\log\log n)(\log n)^{-1}]) = \pi_2(n) + o(\pi_2(n)).$$

Remark. Thus the number of integers satisfying (3) is asymptotically equal to the number $\pi_2(n)$ of integers which are less than n and have 2 prime factors.[5]

The number of integers satisfying (3) is clearly not less than

$$\begin{aligned}\sum (\pi(nq^{-1}) - n^{\epsilon}) &= \sum nq^{-1}(\log(nq^{-1}))^{-1} - n^{2\epsilon} \\ &\quad + \sum o(nq^{-1}[\log(nq^{-1})]^{-1}) \\ &= n(\log\log n)(\log n)^{-1} + o(n(\log\log n)(\log n)^{-1})\end{aligned}$$

(here $\pi(n)$ denotes the number of primes, and the sums are taken over $q<n^{\epsilon}$), since $\sum q^{-1} = \log_2 n + \log \epsilon + o(1)$ and $\log(nq^{-1})$ is asymptotic to $\log n$ for $q<n^{\epsilon}$. (The sum $\sum q^{-1}$ is for $q<n^{\epsilon}$.)

Theorem. *The number $f(n)$ of different integers m of the form $m=\phi(pr)$ where p, r are primes and $pr \leqq n$ equals*

$$n(\log\log n)(\log n)^{-1} + o(n(\log\log n)(\log n)^{-1}) = \pi_2(n) + o(\pi_2(n)).$$

Denote by $B(n)$ the number of solutions of $(p-1)(r-1) = (q-1)(s-1)$, where p, q, r, s are primes, with pq, $rs<n$ and s, $r<n^{\epsilon}$. Clearly

$$f(n) \geqq A(n) - B(n).$$

We have by Lemma 1 (the following sum being for r, $s<n^{\epsilon}$)

$$\begin{aligned}B(n) &= \sum N_n(r-1, s-1) \\ &< n(\log\log n)^{30}(\log n)^{-1} \sum (r-1, s-1)(r-1)^{-1}(s-1)^{-1}.\end{aligned}$$

Put $(r-1, s-1)=d$. Then

$$B_n < n(\log n)^{-2}(\log\log n)^{30} \sum\sum d(q-1)^{-1}(s-1)^{-1},$$

where the first sum is for $d<n^{\epsilon}$ and the second for $r\equiv s\equiv 1 \bmod d$, with r, $s<n^{\epsilon}$. By Lemma 2 we have, summing over the same r and s,

$$\sum (r-1)^{-1}(s-1)^{-1} < (\log\log n)^{40} d^{-2}.$$

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[5] Denote by $\pi_k(n)$ the number of integers having k different prime factors. Landau proves (*Verteilung der Primzahlen*, vol. 1, pp. 208–213) that $\pi_k(n) \sim (n/\log n)(\log\log n)^{k-1}/(k-1)!$. The same asymptotic formula holds if $\pi_k(n)$ denotes the number of integers having k prime factors, multiple factors counted multiply. (Landau, ibid.)

Hence

$$B(n) = c\epsilon n(\log n)^{-1}(\log\log n)^{70} = o(n(\log n)^{-1}).$$

Hence by Lemma 3

$$f(n) \geqq n(\log\log n)(\log n)^{-1} - o(n(\log n)^{-1}),$$

which completes the proof. (Clearly $f(n) < \pi_2(n) < (1+\epsilon)n(\log\log n)\cdot(\log n)^{-1}$.) Our result shows that the number of different integers not greater than n of the form $(p-1)(q-1)$ is asymptotic to the total number of integers not greater than n of the form $(p-1)(q-1)$. Nevertheless there exist integers m such that $(p-1)(q-1)=m$ has arbitrarily many solutions.[6]

By similar but more complicated methods we can prove:

The number of integers not greater than n of the form

$$\prod_{i=1}^{k} (p_i - 1) = \phi(p_1, \cdots, p_k) \qquad (p_i \text{ primes})$$

is greater than

$$cn(\log\log n)^{k-1}[(k-1)!\log n]^{-1} = c\pi_k(n) + o(\pi_k(n))$$

($\pi_k(n)$ denotes the number of integers not greater than n having exactly k prime factors). The constant c depends on k and tends to 0 as $k\to\infty$. For $k\geqq 3$, $c<1$. We omit the proof of these results.

THEOREM. *The number $M(n)$ of integers for which $\phi(m)\leqq n$ equals $cn+o(n)$.*

Denote by $f(x)$ the density of integers for which $m/\phi(m)\geqq x$. It is well known that this density exists.[7] We are going to prove that

$$c = 1 + \int_1^\infty f(x)dx.$$

First we have to show that $\int_1^\infty f(x)dx$ exists. Since $f(x)$ is nondecreasing it will suffice to show that for large r, $f(r)<cr^{-2}$. We have

$$\sum_{m=1}^{n} (m/\phi(m))^2 = \sum_{m=1}^{n}\prod_{p|m}(1+p^{-1}+\cdots)^2 < \sum_{m=1}^{n}\prod_{p|m}(1+5p^{-1})$$
$$= \sum_{m=1}^{n}\sum_{d|m}\mu(d)d^{-1}5^{\nu(d)} < n\sum_{d=1}^{\infty}5d^{-2} < cn.$$

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[6] P. Erdös, *On the totient of the product of two primes*, Quart. J. Math. Oxford Ser. vol. 7 (1936) pp. 227–229.

[7] Schönberg, Math. Zeit. vol. 28 (1928) pp. 171–199.

Hence

$$\lim n^{-1} \sum_{m=1}^{n} (m/\phi(m))^2 < c$$

and this shows $f(r) < cr^{-2}$.

Let k be a large number. Consider the integers m satisfying $nuk^{-1} \leqq m < n(u+1)k^{-1}$, $u \geqq k$. We clearly have

$$\limsup M(n)/n < 1 + k^{-1} \sum_{u=k}^{\infty} f(uk^{-1}),$$

$$\liminf M(n)/n > 1 + k^{-1} \sum_{u=k}^{\infty} f((u+1)k^{-1}).$$

(If $uk^{-1} \leqq m \leqq (u+1)k^{-1}$ and $m/\phi(m) \geqq (u+1)k^{-1}$, $\phi(m) < n$ and if $m/\phi(m) < uk^{-1}$, $\phi(m) > n$.) If $k \rightarrow \infty$ both sums tend to $\int_1^{\infty} f(x)dx$, thus

$$\lim M(n)/n = 1 + \int_1^{\infty} f(x)dx$$

which completes the proof.

Let $\sigma(m)$ be the sum of the divisors of m. By the same methods as used before we can prove the following results:

(1) The number of integers m for which $\sigma(m) \leqq n$ is $cn + o(n)$.

(2) Denote by $g(m)$ the number of integers $m \leqq n$ for which $\sigma(x) = m$ is solvable. Then $n(\log n)^{-1}(\log \log n)^k < g(n) < n(\log n)^{-1}(\log n)^{\epsilon}$.

It seems likely that there exist integers m such that the equation $\phi(x) = m$ has more than $m^{1-\epsilon}$ solutions, and also that there exist, for every k, consecutive integers $n, n+1, \cdots, n+k-1$ such that $\phi(n) = \phi(n+1) \cdots \phi(n+k-1)$.[8] We can make analogous conjectures for $\sigma(n)$. It also would seem likely that there are infinitely many pairs of integers x and y with $\sigma(x) = \sigma(y) = x + y$, that is, there are infinitely many friendly numbers, but these conjectures seem intractable at present.

One final remark: Let $\psi(n) \geqq 0$ be a multiplicative function which has a distribution function.[9] $f(x)$ denotes the density of integers with $\psi(n) \geqq x$. Denote by $M(n)$ the number of integers for which $n\psi(n) \leqq n$. Then $\lim M(n)/n$ always exists since it can be shown that $\int_0^{\infty} f(x)dx$ always exists. The proof is the same as in the case of $\phi(n)$.

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[8] It is known that there exists a number $n < 10000$ such that $\phi(n) = \phi(n+1) = \phi(n+2)$, but I do not remember n and cannot trace the reference.

[9] The necessary and sufficient condition for the existence of the distribution function is given by Erdös-Wintner, Amer. J. Math. vol. 61 (1939) pp. 713–721.

A CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS AND THEIR PROPERTIES

STEFAN BERGMAN

1. The class $\mathcal{C}$ of complex solutions of a linear partial differential equation. Any two real harmonic functions $U(x, y)$, $V(x, y)$—that is, solutions of the Laplace equation—can be combined to form a complex harmonic function $U+iV$. The class of *all* complex harmonic functions is of no interest, because in effect it possesses no special properties not already possessed by real harmonic functions. However, the theory of analytic functions of a complex variable, which is the subclass of complex harmonic functions where U and V satisfy the Cauchy-Riemann equations, has proved to be a powerful means for the study of *real* harmonic functions.

There is an analogous situation in the case of real solutions of the general linear equation of elliptic type,

$$U_{xx} + U_{yy} + A(x, y)U_x + B(x, y)U_y + C(x, y)U = 0,$$
$$U_x = \partial U/\partial x, \cdots .$$

As in the case of the Laplace equation, any two real solutions of this equation can be combined to form a complex solution, and it has been shown ([1a, 1b, 1c])[1] that there exists a certain subclass $\mathcal{C}$ of complex solutions which frequently aids in the study of real solutions in a manner which bears close analogies to the relationships between analytic functions and real harmonic functions in the case of the Laplace equation. Many properties of functions of class $\mathcal{C}$ are closely related to those of ordinary analytic functions: there exists a set of functions in class $\mathcal{C}$ which behave like powers of z; any function of class $\mathcal{C}$ can be expanded in a uniformly convergent series of these analogs of powers; any function of this class which is regular in a simply-connected domain can be approximated in this domain by the analog of a polynomial (see §4); singularities of functions of class $\mathcal{C}$ have properties analogous to those of analytic functions [1c, §§5–6]; and so on. Speaking generally, there exists a method of translating properties of analytic functions of a complex variable into properties of complex solutions of the elliptic type equation which belong to class $\mathcal{C}$ [1c, §1].

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

Before we characterize class $\mathcal{C}$ more precisely, it is convenient to introduce the complex notations $z=x+iy$, $\bar z=x-iy$, $\partial/\partial z=2^{-1}(\partial/\partial x-i\partial/\partial y)$, $\partial/\partial\bar z=2^{-1}(\partial/\partial x+i\partial/\partial y)$. Every function of the real variables x. y can be expressed as a function of the conjugate complex variables z, $\bar z$. It will be necessary, from time to time, to extend x and y to complex values; z and $\bar z$ will then no longer be conjugate, but may then be considered independent complex variables; and our functions will then have been extended to a four-dimensional complex domain. To return to the two-dimensional real domain, it will be sufficient to let z and $\bar z$ be complex conjugates. In this notation the Laplace equation assumes the form $U_{z\bar z}=0$, and the general elliptic equation assumes the form

$$(1.1)\qquad \mathrm{L}(u) = U_{z\bar z} + aU_z + \bar a U_{\bar z} + cU = 0.$$

The class $\mathcal{C}$ may now be defined. To every equation (1.1) whose coefficients a, c are entire functions in the real variables x, y, their corresponds an entire function $e(z, \bar z, t)$ (so-called *generating function of the first kind*) depending only on the coefficients of L and such that, if f is an *arbitrary* analytic function of a complex variable, then

$$(1.2)\qquad \phi(z, \bar z) = \int_{t=-1}^{1} e(z, \bar z, t) f\left[\frac{z}{2}(1-t^2)\right] dt/(1-t^2)^{1/2}$$

is a *complex solution of* (1.1) (cf. [1a, §1]). $\mathcal{C}$ *consists of the totality of functions* $\phi(z, \bar z)$ *defined by* (1.2) *as* f *ranges over the field of analytic functions of a complex variable which are regular at the origin.*

If $\Phi(z, \bar z)$ is any *real* solution of (1.1) which is regular at the origin, then we can determine a complex solution $\phi(z, \bar z)$ whose real part is Φ in the following manner. Let Φ be given as a power series

$$(1.3)\qquad \Phi(z, \bar z) = \sum_{m,n=1}^{\infty} D_{mn} z^m \bar z^n,$$

where $D_{mn}=\bar D_{nm}$ because Φ is real, and let ϕ be represented by the series

$$(1.4)\qquad \phi(z, \bar z) = \sum_{m,n=1}^{\infty} A_{mn} z^m \bar z^n, \qquad A_{00} \text{ real.}$$

If we extend x, y to complex values, then it is known that $\phi(z, 0)=2\Phi(z, 0)-D_{00}\exp[-\int_0^z \bar a(0, z)dz]$ and $\phi(0, \bar z)=D_{00}\exp[-\int_0^{\bar z} a(0, \bar z)d\bar z]$ [1c, §2]. The coefficients A_{m0} $(m=0, 1, 2, \cdots)$ and A_{0n} $(n=1, 2, \cdots)$ can be determined from these relations and (1.3); then, because ϕ satisfies (1.1), the remaining coefficients A_{mn} can be determined from the A_{m0}, A_{0n}. Thus, a complex solution ϕ may be determined uniquely[2] from a real solution Φ in such a way that $\mathrm{Re}\phi=\Phi$; and it can be shown that when Φ ranges over the totality of real solutions of (1.1) which are regular at the origin then the class of complex solutions ϕ obtained by the above procedure is the class $\mathcal{C}$ [1c, §3]. (Note that this is precisely analogous to the manner in which it is possible to determine an analytic function whose real part is a given real harmonic function.)

We may write $\phi(z, \bar{z})=\Phi(z, \bar{z})+i\Psi(z, \bar{z})$, the "conjugate" Ψ being uniquely obtained from the relation $\Psi=-i(\phi-\Phi)$; and it can be shown that Ψ has the same domain of regularity as Φ [1c]. Since the complex solution $\phi(z, \bar{z})$ which is obtained from a given real solution $\Phi(z, \bar{z})$ by the above method depends on the choice of origin, the conjugate also depends on the choice of origin. If Φ is single-valued but has a singular point in its domain, then Ψ will not necessarily be single-valued [1c]. It is evident that Ψ is also a real solution of the differential equation (1.1).

Here again we may note a certain analogy to ordinary analytic functions in that the real and imaginary parts of a complex solution ϕ belonging to class $\mathcal{C}$ are connected by a definite relationship, so that if one of them is known the other is uniquely determined. The analogy between class $\mathcal{C}$ and the class of analytic functions of a single complex variable is investigated in more detail in [1b, 1c]. However, the analogy between these two classes breaks down in an important respect: whereas any rational combination of analytic functions is again an analytic function, the product of two functions of class $\mathcal{C}$ in general does not belong to class $\mathcal{C}$. This fact prevents us from applying certain classical methods of the theory of analytic functions when dealing with class $\mathcal{C}$. For instance, it is impossible to develop a method in class $\mathcal{C}$ which is analogous to representing a regular analytic function as the product of functions each of which possesses one simple zero-point (by the use of the Poisson-Jensen formula). It is evidently desirable that such methods should be applicable, however, and in the next section it is shown how to define a class $\mathcal{N}$ of complex solutions of a certain class of nonlinear partial differential equations which is closed under multiplication, and which bears a close relationship to the class $\mathcal{C}$ of complex solutions of elliptic linear equations.

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[2] That is, to within an imaginary constant multiplied by a certain fixed entire function.

2. The differential equation $\mathrm{N}(u)=0$ and its real solutions. Let $\mathcal{N}$ be the class of functions whose logarithms belong to $\mathcal{C}$—symbolically, $\log \mathcal{N}=\mathcal{C}$. Because $\mathcal{C}$ is closed under addition, $\mathcal{N}$ is closed under multiplication, so that class $\mathcal{N}$ has the property we desire.

Functions of class $\mathcal{N}$ occur in the theory of the two-dimensional heat equation. Let x, y be cartesian coordinates in the plane, and let $U(x, y, t)$ denote the temperature at time t at a point (x, y) in a domain D of the (x, y)-plane. If there are no heat sources or sinks in D, then U satisfies the equation

$$h\rho U_t = (\mu U_x)_x + (\mu U_y)_y, \tag{2.1}$$

where the subscripts denote differentiation with respect to the indicated variable; h, ρ, μ, are respectively the specific heat, the density, and the thermal conductivity.

If we assume that the state is steady and that

$$\mu = \mathrm{X}(U)Q(x, y), \tag{2.2}$$

where X and Q are nonvanishing, sufficiently many times differentiable and dependent only on U and x, y, respectively, then (2.1) becomes

$$[Q(x, y)(\Phi(U))_x]_x + [Q(x, y)(\Phi(U))_y]_y = 0, \tag{2.3}$$

where $\Phi(U)=c_1\int_0^U \mathrm{X}(U)dU+c_2$, c_1, c_2 being constants. If in particular we choose $\Phi(U)=\log U$, that is, $\mathrm{X}(U)=1/U$, then $U=\exp \Phi$ and the equation (1.7) can be written either in the form

$$\Phi_{z\bar z} + a\Phi_z + \bar a\Phi_{\bar z} = 0, \qquad \Phi = \log U, \tag{2.4}$$

or

$$\begin{aligned} \mathrm{N}(U) &\equiv U_{z\bar z} + aU_z + \bar a U_{\bar z} - U_z U_{\bar z}/U = 0, \\ a &= 4^{-1}(Q_x + iQ_y)/Q. \end{aligned} \tag{2.5}$$

We see, therefore, that real parts of functions of class $\mathcal{N}$ represent the temperature distribution under the above described conditions.

REMARK 2.1. If we assume that the law of temperature change is $U(x, y, t)=U^*(x, y)e^{-\nu t}$ and $h\rho\nu=C(x, y)/U^*$ or $C(x, y)\log U^*/U^*$, (2.3) becomes

$$\mathrm{L}(\Phi) \equiv \Phi_{z\bar z} + a\Phi_z + \bar a\Phi_{\bar z} + C/Q = 0, \qquad \Phi = \log U^* \tag{2.6}$$

and

$$\mathrm{L}(\Phi) \equiv \Phi_{z\bar z} + a\Phi_z + \bar a\Phi_{\bar z} + (C/Q)\Phi = 0, \qquad \Phi = \log U^*, \tag{2.7}$$

respectively.

It is evident that theorems about functions of class $\mathcal{C}$ can, in general, be formulated as theorems about functions of class $\mathcal{N}$. Consequently, a not inconsiderable theory of functions of class $\mathcal{N}$ exists, ready made.

There exist however certain exceptions and we shall in the next section discuss the changes which have to be made in attempting to establish an analogue of the Poisson-Jensen formula.

3. **The class $\mathcal{N}$ of complex solutions of $\mathrm{N}(u)=0$.** As already indicated, the introduction of complex solutions of (2.4) enables us to introduce a corresponding class $\mathcal{N}$ of complex solutions of (2.5). Indeed

$$\begin{aligned} w &= \exp \phi = \exp \Phi[\cos \Psi + i \sin \Psi] \\ &= \exp \Phi \cos \Psi + i \exp \Phi \sin \Psi \end{aligned} \tag{3.1}$$

is a solution of (2.5), since ϕ is a solution of (2.4). We note that the $|w|=U$ are again (real) solutions of (2.4), and that, corresponding to every solution U of (2.5), there can be found a complex solution w of (2.5) whose absolute value is U.

Our approach entails certain complications however. If Φ is regular in B, then $U=\exp \Phi$ will be regular in B and will not vanish there. Since in the following we shall deal with solutions U which are zero or become infinite at some points of B, we must consider functions Φ which have singularities in B.

It has already been indicated that in this case the conjugate, and therefore $w=\exp(\Phi+i\Psi)$, are not necessarily single-valued functions. (See [**1b**, p. 151].)

In order to avoid certain complications we shall assume from now on that the functions w under consideration are regular and do not vanish at the origin. Furthermore, we shall avoid ambiguities in the following way: Every point $z^{(k)}$ in B at which w either vanishes or becomes infinite is connected with the boundary of B by means of a portion of a ray through the origin starting at $z^{(k)}$, directed away from the origin, and ending at the first intersection with the boundary. w is single-valued in the domain B_1 which is obtained by cutting B along these lines.

Hereafter let B denote a simply connected domain, whose boundary b is a closed curve with a continuous radius of curvature and no double points. Let us assume further that the differential equation *has no characteristic values in the domain B*, so that the only solution Φ of (2.4) which is regular in B and vanishes on the boundary b of B is $\Phi=0$.

DEFINITION 3.1. A point, $z^{(0)}$, of B is said to be a *zero point* (a *pole*) of the first order of w if w vanishes (becomes infinite) at $z^{(0)}$ and if each of the following expressions

$$h, \log\left|\frac{1}{z-z^{(0)}}\right|\cdot\frac{\partial h}{\partial z}, \log\left|\frac{1}{z-z^{(0)}}\right|\cdot\frac{\partial h}{\partial \bar{z}} \tag{3.2}$$

is bounded in the neighborhood of $z=z^{(0)}$. Here $h=\log|w/(z-z^{(0)})|$ in the case of a zero point, and $h=\log|w(z-z^{(0)})|$ in the case of a pole.

DEFINITION 3.2. A solution of the class $\mathcal{N}$,

$$w = n_B(z, \bar{z}; z^{(0)}), \tag{3.3}$$

of (2.5) is said to be a *normalized zero function* in the domain B, if $n_B(z, \bar{z}; z^{(0)})$ has a zero of the first order at $z=z^{(0)}$, and if

$$\lim_{(z,\bar{z})\to(b,\bar{b})} |n_B(z, \bar{z}; z^{(0)})| = 1, \tag{3.4}$$

uniformly. $(b, \bar{b})$ denotes an (arbitrary) point of the boundary of B.

LEMMA 3.1. *For every point $z^{(0)}\in B$, there exists one and only one normalized zero function $n_B(z, \bar{z}; z^{(0)})$.*

PROOF. Lemma 3.1 is an immediate consequence of the existence of Green's function $\Gamma_B(z, \bar{z}; z^{(0)})$ of the equation (2.5) with respect to B. See [2, p. 8]. If $\gamma_B(z, \bar{z}; z^{(0)})$ is the corresponding complex solution of (2.5) belonging to class $\mathcal{C}$, then

$$n_B(z, \bar{z}; z^{(0)}) = \exp[\gamma_B(z, \bar{z}; z^{(0)})] \tag{3.5}$$

will be the required normalized zero function.

REMARK 3.1.

$$\log|n_B(z, \bar{z}; z^{(0)})| = \Gamma_B(z, \bar{z}; z^{(0)}). \tag{3.6}$$

Every function $f(z)$ which is meromorphic in[3] $E[|z|\leqq 1]$ can be represented there in the form

$$f(z) = \left[\prod_{\kappa=1}^{\kappa_1}\left(\frac{z-\nu_\kappa}{1-\bar{\nu}_\kappa z}\right)\Big/\prod_{\kappa=1}^{\kappa_2}\left(\frac{z-\mu_\kappa}{1-\bar{\mu}_\kappa z}\right)\right]\cdot f_1(z) \tag{3.7}$$

where ν_κ, $\kappa=1, 2, \cdots, \kappa_1$, and μ_κ, $\kappa=1, 2, \cdots, \kappa_2$, are respectively the zeros and the poles of w, and $f_1(z)$ is regular and nonvanishing in $E[|z|<1]$. See [4, p. 137].

We can now generalize this result to the case of solutions of (2.5) which belong to the class $\mathcal{N}$.

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[3] $E[\cdots]$ denotes the set of points whose coordinates satisfy the inequalities indicated in brackets.

THEOREM 3.1. *Let* $w(z, \bar{z})$ *be a (complex) solution of* (2.5) *belonging to class* $\mathcal{N}$ *which is defined in the domain* B *and has continuous boundary values at every point of* b. *Further, assume that* w *has zero points of first order*[4] *at* ν_κ, $\kappa = 1, 2, \cdots, \kappa_1$, *and has poles of first order at* μ_κ, $\kappa = 1, 2, \cdots, \kappa_2$, *and that it vanishes at only a finite number of points on the boundary* b *of* B.

Then w *can be represented in* B_1 *in the form*

$$w(z, \bar{z}) = \left[\prod_{\kappa=1}^{\kappa_1} n_B(z, \bar{z}; \nu_\kappa) \Big/ \prod_{\kappa=1}^{\kappa_2} n_B(z, \bar{z}; \mu_\kappa) \right] \cdot k(z, \bar{z}), \tag{3.8}$$

where $k(z, \bar{z})$ *is a solution of* (2.5) *which belongs to class* $\mathcal{N}$, *is regular, and does not vanish in* B, *and satisfies*

$$\lim_{(z,\bar{z}) \to (b,\bar{b})} |k(z, \bar{z})| = \lim_{(z,\bar{z}) \to (b,\bar{b})} |w(z, \bar{z})|.$$

PROOF. The function $\log|w(z, \bar{z})|$ satisfies equation (2.4) and has logarithmic singularities at the points ν_κ and μ_κ. Let $\Gamma_B(z, \bar{z}; \nu)$ denote Green's function of equation (2.4) with respect to the domain B. Then

$$\log |w(z, \bar{z})| - \sum_{\kappa=1}^{\kappa_1} \Gamma_B(z, \bar{z}; \nu_\kappa) + \sum_{\kappa=1}^{\kappa_2} \Gamma_B(z, \bar{z}; \mu_\kappa) \tag{3.9}$$

is a solution of (2.4) which is regular in B. Since $\Gamma_B(z, \bar{z}; \nu)$ vanishes on the boundary b and $\log|w(z, \bar{z})|$ is continuous on b except at a finite number of points, expression (3.9) has the same boundary values as $\log|w(z, \bar{z})|$. According to §2 there exists a complex solution $\kappa(z, \bar{z})$ of (2.4) in class $\mathcal{C}$ whose real part is (3.9). Writing $k(z, \bar{z}) = \exp[\kappa(z, \bar{z})]$, using Lemma 3.1 and Remark 3.1, we obtain (3.8).

It is clear that this procedure enables us to generalize a whole group of results on analytic functions of a complex variable, for instance the theorem of Blaschke, certain theorems on the relation between the growth of functions $w \in \mathcal{N}$ and the density of their zero points and poles in the neighborhood of a point or curve along which they are singular. Since the development of these results does not present any essential difficulty, we shall not carry out these considerations in detail.

REMARK 3.2. Since the existence of Green's function of equation (1.1) is assured by classical results, the problem of generalizing the Poisson-Jensen formula depends on our ability to associate a suitable

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[4] If the zero is of order n, $n > 1$, the corresponding point ν_κ will appear n times in the sequence.

"conjugate" with a given real solution of $L=0$. In order that the resulting complex functions may have valuable properties of their own or in applications, one has to impose certain conditions on the "conjugate," for example, that it is regular in the same domain in which the given function is regular, and so on. A detailed discussion of various types of "conjugate" functions and their properties is given in [**1b**, §8] and [**1c**, §§2–4].

4. **A remark concerning the solutions of the general equation (1.7).** Using the special form $\Phi = \log U$ we obtained the "canonical representation" (3.8) of functions satisfying this equation of the form (2.5). (2.5) was rendered linear by setting $\Phi = \log U$ (that is, in (2.2), $X(U) = 1/U$). It was then possible to apply the considerations of §3. In many applications, $X(U)$ is not of this form. Clearly the method of §3 can not now be applied. Many of the properties of the functions in class $\mathcal{C}$ of the linearized equation (1.1) can, however, be applied to the investigation of solutions of (2.3), with Φ not necessarily equal to $\log U$. In this section we shall consider such an example.

Let Λ be the function which is inverse to $\Phi(U)$, that is, such that

$$U = \Lambda(\Phi), \tag{4.1}$$

and assume that Λ is an entire function[5] of its complex argument. For instance, in the case considered in §§2 and 3, $\Lambda(\Phi) \equiv \exp \Phi$.

Just as in the case of (2.4), where we introduced complex solutions w of (2.5), we now write

$$u = \Lambda(\phi) \tag{4.2}$$

where ϕ are solutions of class $\mathcal{C}$ of (1.1).

From the fact that in every simply connected domain B which contains the origin and in which $f(z/2)$ is regular, the solution (1.2) is regular, and its converse, the following result was obtained in the paper [**1b**, p. 141]: To every equation L there exists a system of functions

$$\phi_n = 2^{-n}z^n \int_{-1}^{1} e(z, \bar{z}; t)(1 - t^2)^{n-1/2}dt, \qquad n = 0, 1, 2, \cdots, \tag{4.3}$$

such that every solution ϕ of the class $\mathcal{C}$ of $L(\phi) = 0$ which is regular in the circle $x^2 + y^2 < \rho^2$ can be represented there in the form of

$$\phi(z, \bar{z}) = \sum_{n=0}^{\infty} a_n \phi_n(z, \bar{z}) \tag{4.4}$$

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[5] If Λ is not an entire function then it is necessary to make obvious changes in the formulation of the statements of this section.

and can be approximated in every simply connected domain which contains the origin by a "polynomial" $\sum_{n=0}^{m} a_n^{(m)}\phi_n$. (See also [1a, §2].)

We note further that using the classical result of Lindelöf [3, p. 124], we obtain the following conclusion: Let ϕ be a solution of $L(\phi)=0$ and let

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} z^m \bar{z}^n \tag{4.5}$$

be the power series development of ϕ at the origin. Then in every Mittag-Leffler star with the center at the origin in which ϕ is regular it may be represented in the form

$$\lim_{k\to 0} \sum_{n=0}^{\infty} \frac{C_n A_{n0}}{\Gamma(1+kn)} \phi_n, \quad C_n = 2^n\Gamma(n+1)/\pi^{1/2}\Gamma(n+1/2). \tag{4.6}$$

A fortiori: I. *Every solution* $u(z, \bar{z}) = \Lambda(\phi)$, $\phi \in \mathcal{C}$, *of* (2.6) *can be represented in every circle* $x^2+y^2<\rho^2$ *in which it is regular in the form*

$$u(z, \bar{z}) = \Lambda\left[\sum_{n=0}^{\infty} a_n \phi_n(z, \bar{z})\right]. \tag{4.7}$$

II. *In every Mittag-Leffler star with center at the origin in which* u *is regular it can be represented by*

$$u(z, \bar{z}) = \lim_{k\to 0} \Lambda\left[\sum_{n=0}^{\infty} \frac{C_n A_{n0}}{\Gamma(1+kn)} \phi_n(z, \bar{z})\right], \tag{4.8}$$

where A_{n0} *are suitably chosen constants.*[6]

III. Finally *in every simply connected domain which includes the origin, and in which* $u(z, \bar{z})$ *is regular,* u *can be approximated by*

$$\Lambda\left[\sum_{n=0}^{m} a_n^{(m)} \phi_n(z, \bar{z})\right]. \tag{4.9}$$

REMARK 4.1. If $u(z, \bar{z})$ is given in the form of a power series development $\sum B_{mn} z^m \bar{z}^n$, then it is possible to express the A_{n0} in terms of various subsequences of the B_{mn}. Substituting the expressions obtained for A_{n0} into (4.8), we obtain the representation of u in terms of each of these subsequences.

The result III is of interest to applied mathematicians because it leads to a method of determining a solution of (2.3) given by its boundary values. Indeed, for real solutions U of (2.3) we obtain an approximation formula

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[6] The theorem of Lindelöf mentioned above is a special case of the above statement. We obtain it assuming that ϕ is an analytic function of a complex variable and replacing the $C_n\phi_n$ by the powers z^{n-1}, $n=1, 2, \cdots$.

$$\Lambda\left[\sum_{n=0}^{m}\left(A_n^{(m)}\Phi_n(z,\bar{z})+B_n^{(m)}\Psi_n(z,\bar{z})\right)\right] \tag{4.10}$$

analogous to (4.9). Here

$$\Phi_n = \mathrm{Re}\phi_n, \qquad \Psi_n = \mathrm{Im}\phi_n.$$

Suppose now that U is regular in a simply connected domain B and that its values on the boundary b of B are given. We now determine the constants $A_n^{(m)}$ and $B_n^{(m)}$ in (4.10) in such a way that the expression (4.10) approximates, on b, the given boundary values in the "best possible manner." Under appropriate additional conditions it is possible to show that the expressions (4.10) obtained in this manner will converge in B to the required solution when $m \to \infty$.

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BROWN UNIVERSITY

ON SAINT VENANT'S PRINCIPLE

R. v. MISES

The so-called principle of statically equivalent loads, due to Saint Venant, has been referred to for the last fifty years in almost all texts on elasticity. The statements in different books vary only slightly. Let us quote A. E. H. Love's *Treatise on the mathematical theory of elasticity* (4th ed., p. 132): "According to this principle the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part." In this form the statement is not very clear. Forces applied to a body at rest must be in equilibrium in any case. It would not make sense to speak of adding or subtracting a system of forces that is not an equilibrium system. What is meant may be correctly expressed in this way: If the forces acting upon a body are restricted to several small parts of the surface, each included in a sphere of radius ϵ, then the strains and stresses produced in the interior of the body at a finite distance from all those parts are smaller in order of magnitude when the forces for each single part are in equilibrium than when they are not. If this statement is true, it must be capable of a mathematical proof, that is, it must be a consequence of the fundamental differential equations of elasticity theory. But no attempt is made in the usual textbooks to supply a demonstration. Most texts give Boussinesq as a reference for the proof. What Boussinesq really dealt with was the infinite body filling the half space $z>0$ and subjected to normal forces at its boundary $z=0$. If the forces are applied to points ξ, η, 0 where $\xi^2+\eta^2\leqq\epsilon^2$, Boussinesq proved that the stress at a point x, y, z is of order ϵ when the sum of forces is zero and of order ϵ^2 when their moments also vanish. It will be shown in the following that this is not the case, in general, if tangential components of the forces at $z=0$ are admitted. Moreover we shall consider a body of finite dimensions and see that there too Saint Venant's principle in its traditional form does not hold true. The main result, from a practical point of view, is that Saint Venant's principle can be applied if all forces involved are parallel and not tangential to the surface of the body, but not under more general conditions. No objection is raised in the present paper against using the principle in the case of bodies with one or two infinitesimal dimensions, like thin plates, shells or beams, although a proof of its

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validity in these cases or even a precise statement has not yet been given.

1. **The infinite half space.** If the force X, Y, Z is applied in the origin to the body $z>0$, the displacements u, v, w at a point x, y, z can be expressed, as it follows from Boussinesq's results,[1] in the following way. We denote by U and V the harmonic functions

$$U = \frac{xX + yY}{r + z}, \qquad V = Z \log (r + z) \tag{1}$$

where $r^2 = x^2 + y^2 + z^2$, by κ the value $1 - 2\sigma$ where σ is Poisson's ratio, and by μ the shear modulus. Then the displacements are given by

$$\begin{aligned} 4\pi\mu u &= \frac{2}{r} X + \left(\kappa - 1 + z\frac{\partial}{\partial z}\right)\frac{\partial U}{\partial x} - \left(\kappa + z\frac{\partial}{\partial z}\right)\frac{\partial V}{\partial x}, \\ 4\pi\mu v &= \frac{2}{r} Y + \left(\kappa - 1 + z\frac{\partial}{\partial z}\right)\frac{\partial U}{\partial y} - \left(\kappa + z\frac{\partial}{\partial z}\right)\frac{\partial V}{\partial y}, \\ 4\pi\mu w &= \frac{2}{r} Z + \left(-\kappa + z\frac{\partial}{\partial z}\right)\frac{\partial U}{\partial z} - \left(1 - \kappa + z\frac{\partial}{\partial z}\right)\frac{\partial V}{\partial z}. \end{aligned} \tag{2}$$

It can easily be shown that these values satisfy the differential equations of elasticity theory:

$$\begin{gathered} \Delta u + \frac{1}{\kappa}\frac{\partial \theta}{\partial x} = 0, \qquad \Delta v + \frac{1}{\kappa}\frac{\partial \theta}{\partial y} = 0, \qquad \Delta w + \frac{1}{\kappa}\frac{\partial \theta}{\partial z} = 0; \\ \theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \end{gathered} \tag{3}$$

From (2) we derive by differentiation

$$\theta = -\frac{\kappa}{2\pi\mu}\frac{xX + yY + zZ}{r^3}. \tag{4}$$

On the other hand, it follows from (2) if the strain-stress relations are introduced that the stress vector for an element parallel to the x-y-plane is directed toward the origin and has the magnitude

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[1] Boussinesq, J., *Applications des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques*, Paris, 1885. A. E. H. Love, *Treatise on the mathematical theory of elasticity*, 4th ed., Cambridge, 1934, p. 191, gives only the results for a force acting normal to the boundary. The complete expressions can be derived from the formulae developed by E. Trefftz in: Frank-Mises, *Differential- und Integralgleichungen der Mechanik und Physik*, 2nd ed., vol. 2, Braunschweig, 1935 (Reprint, New York, 1944), p. 303.

$$\frac{3z}{2\pi}\,\frac{xX+yY+zZ}{r^4}. \tag{5}$$

Thus the stress is zero in all points $z=0$, $x^2+y^2\neq 0$ and across each plane $z=\text{const.}>0$ the stresses are equivalent to a single force $-X$, $-Y$, $-Z$ passing through the origin.

The mean normal stress is given by

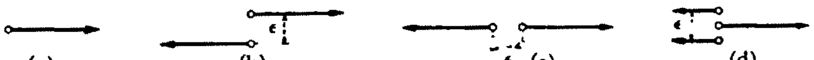
$$\bar{\sigma} = \frac{1}{3}(\sigma_x+\sigma_y+\sigma_z) = \frac{3-\kappa}{3\kappa}\mu\theta = \frac{\kappa-3}{6\pi}\,\frac{xX+yY+zZ}{r^3}. \tag{6}$$

If the force attacks at ξ, η, 0 instead of at the origin, the same formulas can be used, where only x, y have to be replaced by $x-\xi$, $y-\eta$, and r^2 by $(x-\xi)^2+(y-\eta)^2+z^2$. Developing with respect to ξ, η we obtain for the mean normal stress due to a system of forces X_ν, Y_ν, Z_ν attacking at points ξ_ν, η_ν, 0 $(\nu=1, 2, 3, \cdots)$:

$$\begin{aligned}\frac{6\pi}{\kappa-3}r^3\bar{\sigma} = {} & x\sum X_\nu + y\sum Y_\nu + z\sum Z_\nu \\ & + \frac{1}{r^2}\left[(3x^2-r^2)\sum \xi_\nu X_\nu + 3xy\sum \xi_\nu Y_\nu\right. \\ & + 3xz\sum \xi_\nu Z_\nu + 3xy\sum \eta_\nu X_\nu \\ & \left. + (3y^2-r^2)\sum \eta_\nu Y_\nu + 3yz\sum \eta_\nu Z_\nu\right] + \cdots\end{aligned} \tag{7}$$

and similar expressions for other stress values.

If all ξ_ν and η_ν are of the order of magnitude ϵ, we can conclude: The stresses (and strains) in a point x, y, z are of the order ϵ, if the sums of force components $\sum X_\nu$, $\sum Y_\nu$, $\sum Z_\nu$ are zero; they are of the order ϵ^2, if and only if the six linear moments $\sum\xi_\nu X_\nu$, $\sum\xi_\nu Y_\nu$, $\sum\xi_\nu Z_\nu$ and $\sum\eta_\nu X_\nu$, $\sum\eta_\nu Y_\nu$, $\sum\eta_\nu Z_\nu$ also vanish. The case of a system in equilibrium, that is, $\sum\xi_\nu Z_\nu=\sum\eta_\nu Z_\nu=\sum(\xi_\nu Y_\nu-\eta_\nu Z_\nu)=0$, is, in general, in no way distinguished. Only if all forces are parallel to each other, either normal to the boundary surface or inclined under an angle different from zero, the three equilibrium conditions entail the other three conditions. In general, the order of magnitude of the inner strains is reduced to ϵ^2 if and only if the external forces acting upon a small part of the surface are such as to remain in equilibrium when turned through an arbitrary angle (astatic equilibrium).



(a) (b) (c) (d)

finite stress stress of order of magnitude ϵ stress of order ϵ^2

FIG. 1

The results are illustrated in the four simple examples of Fig. 1. All forces are here parallel to the x-direction ($Y=Z=0$). In the case (a) we have a single force which provides a finite stress value according to the first term of (7). In both cases (b) and (c) the sum of forces is zero, but in (b) the sum of $\eta_\nu X_\nu$ and in (c) the sum of $\xi_\nu X_\nu$ is different from zero. It follows from (7) that in either case the stress has the order of magnitude ϵ. If the Saint Venant principle were correct, the stress should be small of higher order in the case (c) where all equilibrium conditions are fulfilled. In fact, in the case (d) only where the three forces form a system in astatic equilibrium with all linear moments zero the stress has the order of magnitude ϵ^2.

2. **Circular disk.** It may be doubted whether the infinite half space gives an adequate picture of what happens in the case of a finite body. Here one cannot speak of applying a single force to the body since the forces as a whole have to form an equilibrium system. It will be sufficient, however, for the present purpose, to analyze one particular shape of finite boundary.

Let us consider the two-dimensional stress distribution in a circular disk of radius R subjected to external forces $F_1, F_2, \cdots, F_n$ which

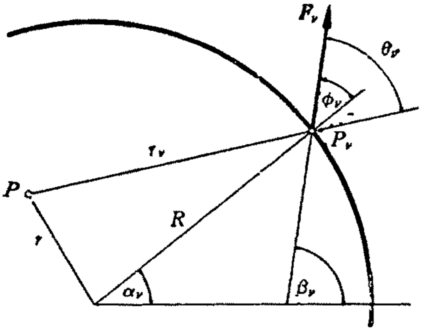


FIG. 2

attack at points $P_1, P_2, \cdots, P_n$ of the circumference. The stresses in a point P are determined by Airy's stress function $S(P)$. If (Fig. 2) r is the distance of P from the center C, r_ν the distance between P and P_ν, ϕ_ν the angle of F_ν with CP_ν, and θ_ν the angle of F_ν with PP_ν, the stress function is given by[2]

$$(8) \qquad S(P) = \frac{1}{\pi} \sum_{\nu=1}^{n} F_\nu \left(r_\nu \theta_\nu \sin \theta_\nu - \frac{r^2}{4R} \cos \phi_\nu \right).$$

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[2] Cf. Love, loc. cit. p. 217. H. Hertz, *Gesammelte Werke*, vol. 1, p. 283.

The stress components with respect to an x-y-coordinate system can be expressed as follows, if β_ν designates the angle of F_ν with the x-axis:

$$
(9)\qquad
\begin{aligned}
\sigma_x &= \frac{1}{\pi}\sum F_\nu\left[\frac{2}{r_\nu}\cos\theta_\nu\cos^2(\theta_\nu-\beta_\nu)-\frac{\cos\phi_\nu}{2R}\right],\\
\sigma_y &= \frac{1}{\pi}\sum F_\nu\left[\frac{2}{r_\nu}\cos\theta_\nu\sin^2(\theta_\nu-\beta_\nu)-\frac{\cos\phi_\nu}{2R}\right],\\
\tau &= -\frac{1}{\pi}\sum F_\nu\frac{2}{r_\nu}\cos\theta_\nu\cos(\theta_\nu-\beta_\nu)\sin(\theta_\nu-\beta_\nu).
\end{aligned}
$$

It can easily be shown that $S(P)$ fulfills the differential equation $\Delta\Delta S=0$ and that the stresses on the boundary $r=R$ are zero, except in the points $P_\nu(r_\nu=0)$. In these points the stresses become infinite and combine so as to balance F_ν.

In the center of the circle we have $r_\nu=R$ and $\theta_\nu=\phi_\nu$ and introducing $\alpha_\nu=\beta_\nu-\phi_\nu$, the polar angle of P_ν, we find for the stress components in C

$$
(10)\qquad
\begin{aligned}
\sigma_x &= \frac{1}{\pi R}\sum F_\nu\cos(\beta_\nu-\alpha_\nu)(1/2+\cos 2\alpha_\nu),\\
\sigma_y &= \frac{1}{\pi R}\sum F_\nu\cos(\beta_\nu-\alpha_\nu)(1/2-\cos 2\alpha_\nu),\\
\tau &= \frac{1}{\pi R}\sum F_\nu\cos(\beta_\nu-\alpha_\nu)\sin 2\alpha_\nu.
\end{aligned}
$$

Let us now assume that there are two groups of forces F_ν and F'_ν, forming angles β_ν and β'_ν with the x-axis. The points of attack P_ν of the first group may lie close to a point P_0 with polar angle α_0 and the points P'_ν of the second group near to P'_0 with polar angle α'_0 so as to have $\alpha_\nu=\alpha_0+\xi_\nu$, $\alpha'_\nu=\alpha'_0+\xi'_\nu$ (ξ, ξ' small). If we develop the expression for σ_x with respect to ξ and ξ', we find

$$
(11)\qquad
\begin{aligned}
\pi R\sigma_x = {} & (1/2+\cos 2\alpha_0)\sum F_\nu\cos(\beta_\nu-\alpha_0)\\
&+(1/2+\cos 2\alpha'_0)\sum F'_\nu\cos(\beta'_\nu-\alpha'_0)\\
&-2\sin 2\alpha_0\sum F_\nu\xi_\nu\cos(\beta_\nu-\alpha_0)\\
&-2\sin 2\alpha'_0\sum F'_\nu\xi'_\nu\cos(\beta'_\nu-\alpha'_0)\\
&+(1/2+\cos 2\alpha_0)\sum F_\nu\xi_\nu\sin(\beta_\nu-\alpha_0)\\
&+(1/2+\cos 2\alpha'_0)\sum F'_\nu\xi'_\nu\sin(\beta'_\nu-\alpha'_0)+\cdots.
\end{aligned}
$$

The first two terms vanish if in each group the sum of forces is zero, since $\beta_\nu-\alpha_0$ and $\beta'_\nu-\alpha'_0$ are the angles the forces form with CP_0 and

CP_0' respectively. In this case, therefore, the highest terms in the expression for σ_x are of the order ξ, ξ'.

If the sums of moments in each group are also zero, we have

$$\begin{aligned} &\sum F_\nu R \sin(\beta_\nu - \alpha_\nu) = R \sum F_\nu \sin(\beta_\nu - \alpha_0 - \xi_\nu) = 0, \\ &\sum F_\nu' R \sin(\beta_\nu' - \alpha_\nu') = R \sum F_\nu' \sin(\beta_\nu' - \alpha_0' - \xi_\nu') = 0. \end{aligned} \tag{12}$$

If these expressions are developed with respect to ξ and ξ', it is seen that with the resultant forces and moments vanishing the sums

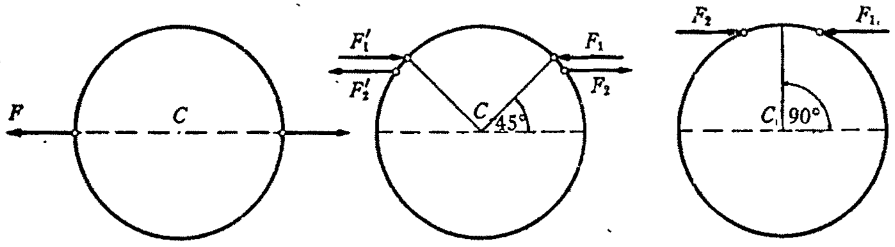
$$\sum F_\nu \xi_\nu \cos(\beta_\nu - \alpha_\nu) \quad \text{and} \quad \sum F_\nu' \xi_\nu' \cos(\beta_\nu' - \alpha_\nu')$$

are zero except for quantities of higher order. In this case the third and fourth terms in (11) vanish, but the fifth and sixth still supply a quantity of the order of magnitude ξ, ξ'. Only if the two additional conditions are fulfilled that

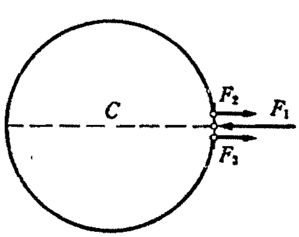
$$\sum F_\nu \xi_\nu \sin(\beta_\nu - \alpha_0) \quad \text{and} \quad \sum F_\nu' \xi_\nu' \sin(\beta_\nu' - \alpha_0')$$

vanish, except for terms of higher order, the development of σ_x will start with terms of the order ξ^2, $\xi\xi'$, ξ'^2.

Again the results may be illustrated on four simple examples shown in Fig. 3. In the case (a) the two single forces of magnitude F produce



(a) finite stress (b) and (c) stress of order of magnitude ϵ



(d) stress of order ϵ^2

FIG. 3

in the center a finite normal stress $\sigma_x=3F/\pi R$ according to (11). In (b) we have two groups of parallel forces with $\alpha_0=45°$, $\alpha_0'=135°$. If the angular distance of the two neighboring points of application is called ϵ, one has $\xi_1=\xi_1'=0$, $\xi_2=-\epsilon$, $\xi_2'=\epsilon$ and with $\beta_1=\beta_2'=180°$, $\beta_2=\beta_1'=0$, equation (11) supplies $\sigma_x=5F\epsilon/2^{1/2}\pi R$. In the case (c) the two forces attacking in the points $\alpha_1=90°-\epsilon$, $\alpha_2=90°+\epsilon$ form an equilibrium system. Equation (11) gives here the value $\sigma_x=F\epsilon/\pi R$ which is of the same order as in (b). Were Saint Venant's principle correct, the stresses in (c) should be small of a higher order. In fact, this is the case with the three forces in example (d) only which form an astatic system (up to terms of higher order). All terms written down on the right-hand side of (11) vanish in the case (d).

3. **Conclusions.** In order to obtain a precise and sufficiently general statement let us consider a finite simply connected body, supported at an adequate number of distinct surface points S_1, S_2, S_3, $\cdots$. Let P_1 be a point of the surface where the load F_1 is applied and P an inner point of the body at finite distances from P_1 and from S_1, S_2, S_3, $\cdots$. Let, finally, σ be some well defined strain or stress quantity in P, for instance, the normal stress in x-direction, or any component of the distortion. Then, with constant F_1, this σ will be a function of the coordinates of P_1. If P_1 is a regular surface point (tangential plane, finite curvature) the function will have finite derivatives. That means, if P_1 moves through a small distance ϵ the change in σ will be of the order of magnitude ϵ. Consequently, two equal and opposite forces attacking at points of distance ϵ will produce a σ-value of the order ϵ. On the other hand, the load F_1 can be replaced by several loads that have the vector sum F_1, all attacking at P_1. Each of them can be shifted to the neighborhood and then reversed. The system of these reversed forces combined with the original F_1 will still produce a σ-value of order ϵ. Thus our first statement reads:

(a) *If a system of loads on an adequately supported body, all applied at surface points within a sphere of diameter ϵ, have the vector sum zero, they produce in an inner point P of the body a strain or stress value σ of the order of magnitude ϵ.*

To this statement we add the results reached in the preceding sections by way of direct computation for two particular cases, the infinite half space and the circular disk. The general proof following the same lines can be given without difficulty.

(b) *If the loads, in addition to having the vector sum zero, fulfill three* *further conditions so as to form an equilibrium system within the sphere of diameter* ϵ; *the* σ*-value produced in* P *will, in general, still be of the order of magnitude* ϵ.

(c) *If the loads, in addition to being an equilibrium system, satisfy three more conditions so as to form a system in astatic equilibrium, then the* σ*-value produced in* P *will be of the order of magnitude* ϵ^2 *or smaller. In particular, if loads applied to a small area are parallel to each other and not tangential to the surface and if they form an equilibrium system, they are also in astatic equilibrium and thus lead to a* σ *of the order* ϵ^2.

In this whole argument the loads as well as the supporting reactions were supposed to be concentrated, finite forces acting at distinct points of the surface. No difficulty arises if, instead, continually distributed surface stresses are assumed with the provision that all integrals of such stresses over finite regions (and the regions that tend to zero) remain finite.

A final remark is in order about the legitimate application of St. Venant's principle (or some equivalent statement) in cases of thin rods, shells, and so on. The only precise and consistent way to deal with thin elastic rods is the theory of the so-called one-dimensional elastica. In this theory the forces acting on the ends of the rod enter the computation only with their resultant vector and resultant moment. This implies, evidently, a principle of "statically equivalent loads." What Saint Venant originally had in mind was doubtlessly the case of a long cylinder with infinite ratio of length to diameter. The purpose of the present paper was to show that an extension of the principle to bodies of finite dimensions is not legitimate.

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METRIC PROPERTIES OF A CLASS OF QUADRATIC DIFFERENTIAL FORMS

P. O. BELL

Introduction. In the present paper a new invariant quadratic differential form Ω is geometrically defined for a general pair of surfaces S, S' whose corresponding points x, x' determine the metric normal to S at x. The ratio of the form Ω to the first fundamental form ds^2 of S, in which Ω and ds^2 are defined for a common arc element of S at x, is found to be independent of the direction of the element if and only if the surface S' is the locus of the center of mean curvature of S; the ratio thus determined is the Gaussian curvature K of S at x. We introduce at a point x of S the concept of conjugate elements of a given arc element of a conjugate net and prove that the form Ω for an arbitrary arc element is identical with the form Kds^2 for either conjugate element if and only if the surface S' is the plane net at infinity. The principal directions at x of the tensor whose components are the coefficients of the form Ω are the classical principal directions of S at x for an arbitrary choice of S'. Finally, we characterize the net of lines of mean-curvature of S and the mean-conjugate net of S as integral nets of equations of the form $\Omega=0$, in which the forms Ω are defined with respect to certain geometrically determined transforms S' of S. The method of the present paper employs dual systems of linear homogeneous equations of the first order in compact forms which facilitate the use of a tensor notation with homogeneous cartesian point and plane coordinates.

1. **The fundamental differential equations.** The rectangular cartesian coordinates of a generic point x of an analytic surface S are defined by single-valued functions of two independent parameters u^1, u^2,

$$x^i = x^i(u^1, u^2), \qquad i = 0, 1, 2.$$

Let $g_{\alpha\beta}$ and $g^{\alpha\beta}$ denote the covariant and contravariant metric tensors of S, respectively, and let $d_{\alpha\beta}$ denote the second fundamental covariant tensor of S. It is known [1, p. 220][1] that the direction cosines z^i of the normal to S at x and the functions x^i are solutions of the differential equations

$$(1.1) \qquad \frac{\partial^2 x}{\partial u^\beta \partial u^\alpha} = \left\{ {\gamma \atop \alpha\beta} \right\} \frac{\partial x}{\partial u^\alpha} + d_{\alpha\beta} z, \quad \frac{\partial z}{\partial u^\alpha} = \binom{\beta}{3\alpha} \frac{\partial x}{\partial u^\beta}, \qquad \alpha, \beta, \gamma = 1, 2,$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

where the functions

$$\left\{ {\gamma \atop \alpha\beta} \right\}$$

are the Christoffel symbols of the second kind formed with respect to the first fundamental form of S and the functions

$$\binom{\beta}{3\alpha}$$

are components of the mixed tensor defined by

$$\binom{\beta}{3\alpha} = - d_{\alpha\gamma} g^{\gamma\beta}.$$

Homogeneous cartesian coordinates of a finite point of space may be obtained by adjoining to the three ordinary rectangular coordinates x^0, x^1, x^2 a fourth coördinate $x^3 = 1$. For a point at infinity the homogeneous cartesian coordinates are of the form x^0, x^1, x^2, 0. The point at infinity on the normal to S at x has homogeneous cartesian coordinates z^0, z^1, z^2, 0. As x moves over S this point at infinity describes the plane net S_z at infinity. We observe that the system (1.1) is satisfied by each pair of homogeneous coordinates x^i, z^i, $i = 0, 1, 2, 3$. *Thus the surface S and the corresponding plane net S_z at infinity generated by the infinite point on the normal to S at x are integral surfaces of the system* (1.1).

Let us define points x_0, x_1, x_2, x_3 by the relations

$$(1.2) \qquad x = x_0, \qquad \frac{\partial x_0}{\partial u^\alpha} = x_\alpha, \quad \alpha = 1, 2, \qquad z = x_3.$$

These relations enable us to put the system (1.1) in the form of the following system of linear homogeneous differential equations of the first order,

$$(1.3) \qquad \frac{\partial x_i}{\partial u^\alpha} - \binom{h}{i\alpha} x_h = 0, \qquad i, h = 0, 1, 2, 3; \; \alpha = 1, 2,$$

where

$$\binom{h}{0\alpha} = \delta^h_\alpha, \quad \binom{0}{i\alpha} = 0, \quad \binom{3}{3\alpha} = 0;$$

$$\binom{\gamma}{\alpha\beta} = \left\{ {\gamma \atop \alpha\beta} \right\}, \quad \binom{3}{\alpha\beta} = d_{\alpha\beta}, \quad \binom{\beta}{3\alpha} = - d_{\alpha\gamma} g^{\gamma\beta}, \qquad \alpha, \beta, \gamma = 1, 2.$$

Let $|x|$ denote the determinant whose elements are the functions $x_i^{(p)}$ and let ξ_i^j denote the normalized cofactor of x_j^i in $|x|$, defined by the relations

$$\xi_i^r x_h^i = \delta_h^r, \qquad h, r = 0, 1, 2, 3, \tag{1.4}$$

in which the right members are the Kronecker deltas. The functions ξ_i^r, $i=0, 1, 2, 3$, form a set of homogeneous cartesian plane coordinates of the plane determined by the three points x_h, $h \neq r$. Differentiating (1.4) with respect to u^α and making use of (1.3), we obtain

$$x_h^i \frac{\partial \xi_i^r}{\partial u^\alpha} + \binom{r}{h\alpha} | x | = 0. \tag{1.5}$$

On forming the inner product of the left member of this equation with ξ_k^h and dividing by $|x|$, we find that the plane coordinates ξ_k^r are solutions of the system of equations

$$\frac{\partial \xi^r}{\partial u^\alpha} + \xi^h \binom{r}{h\alpha} = 0, \qquad \alpha = 1, 2. \tag{1.6}$$

A relation of the form

$$x' = z + kx, \tag{1.7}$$

where k is an arbitrary function of u^1, u^2, defines the general coordinates of a point x' which is collinear with x and z and generates a surface S' as u^1, u^2 vary. For the sake of convenience we denote the points x, x_1, x_2, x' by y_0, y_1, y_2, y_3, respectively, so that the fundamental differential equations (1.3) may be written in the form

$$\frac{\partial y_i}{\partial u^\alpha} - \binom{h}{i\alpha}' y_h = 0; \qquad i, h = 0, 1, 2, 3; \alpha = 1, 2, \tag{1.8}$$

in which

$$\binom{h}{0\alpha}' = \delta_\alpha^h, \quad \binom{0}{\alpha\beta}' = -k\binom{3}{\alpha\beta}, \quad \binom{\gamma}{\alpha\beta}' = \left\{ {\gamma \atop \alpha\beta} \right\}, \quad \binom{0}{3\alpha}' = \frac{\partial k}{\partial u^\alpha},$$

$$\binom{\alpha}{3\alpha}' = k + \binom{\alpha}{3\alpha}, \quad \binom{3}{\alpha\beta}' = \binom{3}{\alpha\beta}, \quad \binom{3}{3\alpha}' = 0, \qquad \alpha, \beta, \gamma = 1, 2;$$

$$\binom{\beta}{3\alpha}' = \binom{\beta}{3\alpha}, \qquad \alpha, \beta = 1, 2, \alpha \neq \beta.$$

Let $|y|$ denote the determinant whose elements are the functions y_i^p, i, $p=0, 1, 2, 3$, and let η_i^j denote the normalized cofactor of y_j^i

in $|y|$. The functions η^r_i, $i=0, 1, 2, 3$, are homogeneous plane coordinates of the plane determined by the three points y_h, $h \neq r$. The differential equations satisfied by the functions η^r_i are easily found to be

$$\frac{\partial \eta^r}{\partial u^\alpha} + \eta^h \left(\begin{matrix} r \\ h\alpha \end{matrix}\right)' = 0. \tag{1.9}$$

2. The invariant quadratic differential form. Let x', X' denote the points of S' whose curvilinear coordinates are u^1, u^2 and u^1+du^1, u^2+du^2, respectively, and let π, p denote the corresponding tangent planes of S. Let l denote the line joining the points x' and X' and let y and Y denote the intersections of l with the planes π and p, respectively. We prove the following theorem.

THEOREM 1. *The principal part of the cross ratio* (x', y, X', Y) *is the quadratic differential invariant*

$$\Omega = a_{\alpha\beta} du^\alpha du^\beta,$$

where $a_{\alpha\beta}$ *is the tensor defined by*

$$a_{\alpha\beta} = k d_{\alpha\beta} - h_{\alpha\beta},$$

in which $d_{\alpha\beta}$ *is the second fundamental tensor of* S *and* $h_{\alpha\beta}$ *is the first fundamental tensor of the spherical representation of* S.

Except for terms of order at least two, the point coordinates of X' and the plane coordinates of π and p are given by

$$X' = y_3 + \left(\begin{matrix} h \\ 3\alpha \end{matrix}\right)' y_h du^\alpha, \qquad \left(\begin{matrix} 3 \\ 3\alpha \end{matrix}\right)' = 0,$$

and

$$\pi = \eta^3, \qquad p = \eta^3 - \eta^h \left(\begin{matrix} 3 \\ h\alpha \end{matrix}\right)' du^\alpha,$$

respectively. The homogeneous cartesian coordinates of y, except for terms of order at least two, are obviously given by the form

$$y = \left(\begin{matrix} h \\ 3\alpha \end{matrix}\right)' y_h du^\alpha.$$

The coordinates of Y are defined by a relation of the form

$$Y = y + \Omega y_3$$

such that the condition $Y^i p_i = 0$ is fulfilled. The function Ω is, therefore, the root of the equation

$$\left(\Omega y_3^i + \binom{h}{3\beta}' y_h^i du^\beta\right)\left(\eta_i^3 - \eta_i^h \binom{3}{h\alpha}' du^\alpha\right) = 0.$$

Since $y_j^i \eta_i^p = \delta_j^p$, multiplication of the factors of the left member yields

$$\Omega = \binom{h}{3\beta}' \binom{3}{h\alpha}' du^\beta du^\alpha. \tag{2.1}$$

On making use of the relations between corresponding coefficients of (1.7) and (1.3) we obtain the form

$$\Omega = a_{\alpha\beta} du^\alpha du^\beta, \tag{2.2}$$

where $a_{\alpha\beta} = kd_{\alpha\beta} - g^{\delta\gamma} d_{\delta\beta} d_{\gamma\alpha}$. It is known [1, p. 253] that the first fundamental tensor of Gauss's spherical representation of S is defined by

$$h_{\alpha\beta} = g^{\delta\gamma} d_{\delta\beta} d_{\gamma\alpha}. \tag{2.3}$$

The proof of the theorem is, therefore, complete.

3. **New geometric characterizations of the form Kds^2.** The plane at infinity is the surface S' for which $k=0$. The associated tensor $a_{\alpha\beta}$ is, therefore, defined by

$$a_{\alpha\beta} = - h_{\alpha\beta}, \tag{3.1}$$

and *the invariant* $-\Omega$ *is identical with the first fundamental form of the spherical representation of* S.

It is known that for a unique conjugate parametric net, namely, the mean-conjugate net, the first fundamental form of the spherical representation is expressible in the form

$$d\bar{s}^2 = \sigma^2(g_{11}(du^1)^2 - 2g_{12}du^1du^2 + g_{22}(du^2)^2).$$

Let us determine all of the parametric nets on an unspecialized surface S for which Ω is expressible in the form

$$\Omega = \phi(g_{11}(du^1)^2 - 2g_{12}du^1du^2 + g_{22}(du^2)^2). \tag{3.2}$$

The conditions to be fulfilled are represented by the relations

$$h_{11}/g_{11} = - h_{12}/g_{12} = h_{22}/g_{22}. \tag{3.3}$$

It is known [1, p. 253] that the tensor $h_{\alpha\beta}$ may be expressed in terms of the first and second fundamental tensors by means of the relation

$$h_{\alpha\beta} = d_{\alpha\beta}K_m - g_{\alpha\beta}K \tag{3.4}$$

in which K and K_m are the Gaussian curvature and mean curvature of

S defined by

$$K = (d_{11}d_{22} - d_{12}^2)/g, \quad K_m = 2(g_{11}d_{22} - g_{12}d_{12})/g. \tag{3.5}$$

Equations (3.3), if $h_{\alpha\beta}$ is replaced by the right members of (3.4), assume the forms

$$\begin{aligned} K_m(g_{22}d_{11} - g_{11}d_{22}) &= 0, \\ K_m(d_{11}g_{12} + g_{11}d_{12}) &= 2Kg_{11}g_{12}, \\ K_m(d_{22}g_{12} + g_{22}d_{12}) &= 2Kg_{12}g_{22}. \end{aligned} \tag{3.6}$$

On substituting the right members of (3.5) in equations (3.6) and simplifying, we obtain the relations

$$\begin{aligned} g_{22}d_{11} - g_{11}d_{22} &= 0, \\ d_{11}d_{12} = d_{12}d_{22} &= 0. \end{aligned} \tag{3.7}$$

These relations are satisfied if $d_{11}=d_{22}=0$, $d_{12}\neq 0$, that is, if the asymptotic curves of S are parametric. If, however, the asymptotic curves of S are not the parametric curves of S, the first of equations (3.7) is the condition that the parametric net be a duametric net [**2**, p. 308]; the second and third conditions insure that the parametric net be a conjugate net. Hence, in this case, the parametric net is the unique conjugate duametric net of S, that is, *the mean-conjugate net of S*. We have, therefore, that Ω *assumes the form* (3.2) *when S′ is the plane net at infinity if, and only if, the parametric net of S is either the asymptotic net or the mean conjugate net.* For the case of the asymptotic parametric net $\phi=K$ and for the mean conjugate parametric net $\phi=-K$.

The result described above in terms of the asymptotic parametric net leads to the following geometric determination of the invariant Kds^2. Let C_λ, $C_{-\lambda}$ denote the curves which pass through x of the conjugate net defined by

$$(du^2)^2 - \lambda^2(du^1)^2 = 0 \tag{3.8}$$

whose directions at x are λ, $-\lambda$, respectively. Let x and P denote the points of S whose curvilinear coordinates are u^1, u^2 and u^1+du^1, u^2+du^2, respectively. The u^1 and u^2 asymptotic curves which pass through P intersect the curve $C_{-\lambda}$ in the points Q, Q' whose curvilinear coordinates are u^1-du^1, u^2+du^2; and u^1+du^1, u^2-du^2, respectively. We shall call the elements xQ and xQ' the conjugate elements of xP. We are able now to describe our characterization of Kds^2 by means of the following theorem.

THEOREM 2. *The form Ω for an arbitrary arc element xP of a conjugate net is identical with the form Kds^2 for either conjugate element xQ or xQ' if and only if the surface S' is the plane net at infinity.*

An analogous characterization of $-Kds^2$ can obviously be formulated with reference to the mean-conjugate parametric net of S, but we shall not describe this result here.

Another geometric characterization of Kds^2 arises from the determination of the surface S' such that the invariant Ω of S, S' is expressible in the form

$$\Omega = \sigma ds^2. \tag{3.9}$$

Just one such surface S' exists since the equations

$$kd_{\alpha\beta} - h_{\alpha\beta} = \sigma g_{\alpha\beta} \tag{3.10}$$

possess a unique solution (k, σ). For, on substituting the right member of (3.4) for $h_{\alpha\beta}$ in (3.10) we find the equations

$$(k - K_m)d_{\alpha\beta} = (\sigma - K)g_{\alpha\beta}, \tag{3.11}$$

which hold for an unspecialized surface S if, and only if,

$$k = K_m, \qquad \sigma = K.$$

Hence, we have the following theorem.

THEOREM 3. *The differential form Ω of S, S' and the first fundamental form of S, defined with respect to a common direction, are related by the equation*

$$\Omega = \sigma ds^2$$

if and only if S' is the locus of the center of mean curvature of S. The associated function σ is the Gaussian curvature of S at x.

4. The principal directions, the lines of mean curvature, and the mean conjugate net of S. Since we have

$$a_{\alpha\beta} = kd_{\alpha\beta} - h_{\alpha\beta}, \qquad h_{\alpha\beta} = d_{\alpha\beta}K_m - g_{\alpha\beta}K,$$

we may write

$$a_{\alpha\beta} = (k - K_m)d_{\alpha\beta} + Kg_{\alpha\beta}. \tag{4.1}$$

It is obvious, in view of the form of (4.1), that if $k \neq K_m$, the principal directions for the tensor $a_{\alpha\beta}$ are identical with the classical principal directions of S.

Let us assume, for the sake of convenience, that the lines of curvature are the parametric curves of S. The results which we present are, however, independent of the choice of the parametric net. The author [2, p. 309] has called a curve a line of mean curvature of S if at each of its points the normal curvature $1/\rho$ of S in the direction of the curve is equal to the arithmetic mean of the principal normal curvatures of S at the point, that is, if

$$\frac{1}{\rho} = \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right).$$

From equations (6.5) and (6.6) of [2, p. 310] we find that if $g_{12} = d_{12} = 0$, the lines of mean curvature of S are the integral curves of

$$\Delta(g_{11}(du^1)^2 - g_{22}(du^2)^2) = 0 \tag{4.2}$$

in which $\Delta = g_{11}d_{22} - g_{22}d_{11}$. The significance of the presence of Δ in equation (4.2) is that the lines of mean curvature form a determinate net, except when $\Delta = g_{12} = d_{12} = 0$; in this exceptional case S is a sphere.

We inquire if there exists a surface S' for which Ω is expressible in the form

$$\Omega = \sigma\Delta(g_{11}(du^1)^2 - g_{22}(du^2)^2). \tag{4.3}$$

The affirmative answer is readily reached, for we find that the equations

$$\begin{aligned} kd_{11} - h_{11} &= \sigma\Delta g_{11}, \\ kd_{12} - h_{12} &= 0, \\ kd_{22} - h_{22} &= -\sigma\Delta g_{22} \end{aligned} \tag{4.4}$$

can be satisfied simultaneously. The second equation of (4.4) is satisfied identically in k, σ since $g_{12} = d_{12} = h_{12} = 0$. The other two equations may be written in the forms

$$\begin{aligned} kd_{11} - g_{22}d_{11}^2/g &= \sigma\Delta g_{11}, \\ kd_{22} - g_{11}d_{22}^2/g &= -\sigma\Delta g_{22}. \end{aligned} \tag{4.5}$$

Solving, we find

$$\begin{aligned} k &= (d_{11}^2 g_{22}^2 + g_{11}^2 d_{22}^2)/g(g_{22}d_{11} + g_{11}d_{22}), \\ \sigma &= d_{11}d_{22}/g(g_{22}d_{11} + g_{11}d_{22}). \end{aligned} \tag{4.6}$$

Making use of the expressions

$$\begin{aligned} K &= d_{11}d_{22}/g, \\ K_m &= (g_{11}d_{22} + g_{22}d_{11})/g \end{aligned}$$

for the Gaussian and mean curvatures of S, we express k, σ in terms of K and K_m as follows:

$$k = K_m - 2K/K_m, \qquad \sigma = K/gK_m. \tag{4.7}$$

Hence we have the following theorem.

THEOREM 4. *If and only if S' is characterized in relation to S by* (1.7) *with k expressed in terms of the Gaussian and mean curvatures of S by the relation*

$$k = K_m - 2K/K_m,$$

the lines of mean curvature of S are the integral curves of the associated differential equation $\Omega = 0$.

Let us recall that a curve of the mean conjugate net is characterized by the property that at each of its points the radius of normal curvature of S in the direction of the curve is the arithmetic mean of the principal radii of normal curvature of S at the point, that is,

$$\rho = (\rho_1 + \rho_2)/2.$$

Retaining the lines of curvature as parametric curves, the mean conjugate net is the integral net of the equation

$$\Delta(d_{11}(du^1)^2 - d_{22}(du^2)^2) = 0. \tag{4.8}$$

It is not difficult to prove, by the method we have employed above, that a surface S' exists such that the associated form Ω is expressible by the relation

$$\Omega = \sigma\Delta(d_{11}(du^1)^2 - d_{22}(du^2)^2). \tag{4.9}$$

In fact, we find that

$$k = K_m/2, \qquad \sigma = 1/2g. \tag{4.10}$$

We are now in a position to state our concluding result.

THEOREM 5. *If and only if k is the arithmetic mean of the principal normal curvatures of S at x, the integral net of the associated differential equation $\Omega = 0$ is the mean conjugate net of S.*

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SOME INVARIANTS OF CERTAIN PAIRS OF HYPERSURFACES

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Introduction. It is known [8, 9][1] that if two surfaces in ordinary space have a common tangent plane at an ordinary point, then the ratio of their total curvatures at this point is a projective invariant, and the theorem holds true similarly for hyperspaces.[2] In connection with this theorem and the investigation of Bouton [2], Buzano [3] and Bompiani [1] have shown the existence of a projective invariant, together with metric and projective characterizations, determined by the neighborhood of the second order of two surfaces S, S^* at two ordinary points O, O^* in ordinary space under the conditions that the tangent planes of the surfaces S, S^* at the points O, O^* be distinct and have OO^* for the common line. Furthermore, the other case in which the tangent planes of the surfaces S, S^* at the points O, O^* are coincident[3] has been considered in recent papers of the author [6, 7].

It is the purpose of the present paper to generalize the results of the two cases mentioned above.

Let V_{n-1}, V_{n-1}^* be two hypersurfaces in a space S_n of n dimensions, and t_{n-1}, t_{n-1}^* the tangent hyperplanes of the hypersurfaces V_{n-1}, V_{n-1}^* at two ordinary points O, O^*. For the subsequent discussion it is convenient to assume in Chapter I that the tangent hyperplanes t_{n-1}, t_{n-1}^* are coincident. We can (§1), as in ordinary space, determine a projective invariant by the neighborhood of the second order of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*; and the projective and metric characterizations of this invariant are given in the next two sections.

Chapter II treats of the case in which the tangent hyperplanes t_{n-1}, t_{n-1}^* are distinct, and the common tangent flat space t_{n-2} of t_{n-1}, t_{n-1}^* contains the line OO^*. We first (§4) show by analysis the existence of two projective invariants determined by the neighbor-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] The simple projective characterizations of this invariant were given by C. Segre [10] for two plane curves and by P. Buzano [4] for two surfaces in space S_n $(n>2)$. On the other hand, A. Terracini [11] also interpreted projectively this invariant by virtue of the conception of density of dualistic correspondences.

[3] It should be noted that for two plane curves having a common tangent at two ordinary points no projective invariant can be determined by the neighborhood of the second order of the two curves at these points. See my paper [5].

hood of the second order of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*; and then (§§5, 6) give them simple projective and metric characterizations. From the fact that one of the two invariants is reduced to 1 when the immersed space S_n is of three dimensions, it follows that our result in this chapter stands actually for a generalization of that of Buzano and Bompiani.

CHAPTER I. TWO HYPERSURFACES WITH COMMON TANGENT HYPERPLANE AT TWO ORDINARY POINTS

1. **Derivation of an invariant.** Let V_{n-1}, V_{n-1}^* be two hypersurfaces in a space S_n of n dimensions with common tangent hyperplane t_{n-1} at two ordinary points O, O^*. Let $x_1, \cdots, x_{n+1}$ denote the homogeneous projective coordinates of a point in the space S_n. If we choose the points O, O^* to be the vertices $(1, 0, \cdots, 0)$, $(0, \cdots, 0, 1, 0)$ of the system of reference, and the common tangent hyperplane t_{n-1} to be the coordinate hyperplane $x_{n+1}=0$ of the system, then the power series expansions of the hypersurfaces V_{n-1}, V_{n-1}^* in the neighborhood of the points O, O^* may be written in the form

$$(1) \qquad V_{n-1}: \quad \frac{x_{n+1}}{x_1} = \sum_{i,k=2}^{n} l_{ik} \frac{x_i}{x_1} \frac{x_k}{x_1} + \cdots,$$

$$(2) \qquad V_{n-1}^*: \quad \frac{x_{n+1}}{x_n} = \sum_{i,k=1}^{n-1} m_{ik} \frac{x_i}{x_n} \frac{x_k}{x_n} + \cdots.$$

In order to find a projective invariant of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*, we have to consider the most general projective transformation of coordinates which shall leave the points O, O^* and the hyperplane t_{n-1} unchanged:

$$(3) \qquad \begin{aligned} x_i &= \sum_{r=1}^{n+1} a_{ir} x_r' \qquad (i = 1, \cdots, n), \\ x_{n+1} &= a_{n+1,n+1} x_{n+1}', \end{aligned}$$

where

$$(4) \qquad a_{21} = \cdots = a_{n1} = 0, \qquad a_{1n} = \cdots = a_{n-1,n} = 0,$$

$$(5) \qquad D = \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n-1} \\ a_{32} & a_{33} & \cdots & a_{3,n-1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{vmatrix} \neq 0.$$

The effect of this transformation on equations (1), (2) is to produce two other equations of the same form whose coefficients, indicated by accents, are given by the formulas

$$
(6)\qquad
\begin{aligned}
a_{11}a_{n+1,n+1}l'_{ik} &= \sum_{r,s=2}^{n} a_{ri}a_{sk}l_{rs} && (i,\ k = 2, \cdots, n),\\
a_{nn}a_{n+1,n+1}m'_{ik} &= \sum_{r,s=1}^{n-1} a_{ri}a_{sk}m_{rs} && (i,\ k = 1, \cdots; n-1).
\end{aligned}
$$

From equations (4), (5), (6) it is easily seen that the determinants

$$
L = \begin{vmatrix} l_{22} & l_{23} \cdots l_{2n} \\ l_{32} & l_{33} \cdots l_{3n} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ l_{n2} & l_{n3} \cdots l_{nn} \end{vmatrix}, \qquad
M = \begin{vmatrix} m_{11} & m_{12} & \cdots m_{1,n-1} \\ m_{21} & m_{22} & \cdots m_{2,n-1} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ m_{n-1,1} & m_{n-1,2} & \cdots m_{n-1,n-1} \end{vmatrix},
$$

and their transformed ones L', M' are connected by the relations

$$
(7)\qquad
\begin{aligned}
a_{11}^{n-1}\,a_{n+1,n+1}^{n-1}\,L' &= a_{nn}^{2}D^{2}L,\\
a_{nn}^{n-1}\,a_{n+1,n+1}^{n-1}M' &= a_{11}^{2}D^{2}M.
\end{aligned}
$$

Further elimination of a_{ik} from equations (6), (7) shows immediately that *the quantity*

$$
(8)\qquad I = \frac{L}{M}\left(\frac{m_{11}}{l_{nn}}\right)^{(n+1)/3}
$$

is a projective invariant determined by the neighborhood of the second order of the hypersurfaces V_{n-1}, V^*_{n-1} *at the points* O, O^*.

2. A projective characterization of the invariant I. Let the polar spaces of the line OO^* with respect to the asymptotic hypercones of the hypersurfaces V_{n-1}, V^*_{n-1} at the points O, O^* be respectively denoted by t_{n-2}, t^*_{n-2}, which determine a space t_{n-3} of $n-3$ dimensions in the common tangent hyperplane $x_{n+1}=0$. If the $n-2$ vertices, other than O and O^*, of the system of reference in the hyperplane $x_{n+1}=0$ be chosen in the space t_{n-3}, then the invariant I may be reduced to

$$
(9)\qquad I = \frac{L_{nn}}{M_{11}}\left(\frac{m_{11}}{l_{nn}}\right)^{(n-2)/3},
$$

where L_{nn}, M_{11} are the minors of l_{nn}, m_{11} in the determinants L, M respectively.

For the purpose of finding a projective characterization of the invariant I we first observe the space S_3 determined by the vertices $(1, 0, \cdots, 0)$, $(0, \cdots, 0, 1, 0)$, $(0, \cdots, 0, 1)$ and any one, say for instance $O_2(0, 1, 0, \cdots, 0)$, of the system of reference in the space t_{n-3}. The space S_3 intersects the hypersurfaces V_{n-1}, V_{n-1}^* in two surfaces S, S^*. Since the tangent planes of the surfaces S, S^* at the points O, O^* are coincident we have a projective invariant, denoted by J,

$$J = \frac{l_{22}}{m_{22}}\left(\frac{m_{11}}{l_{nn}}\right)^{1/3}, \tag{10}$$

whose projective characterization has been obtained [6].

Let Q (Q^*) be any quadric in the space S_3 which has OO_2 (O^*O_2), OO^* (OO^*) for generators and whose curve of intersection with the element of the second order of the surface S (S^*) at the point O (O^*) has a cusp at O (O^*). If the cone projecting from the point O_2 the curve of intersection of the two quadrics Q, Q^* be tangent to the common tangent plane OO^*O_2 along a line through the point O_2, then this line must be one of the lines (cf. [6])

$$\begin{gathered} x_n \pm (\pm 1)^{1/2}\left(\frac{m_{11}m_{22}}{l_{22}l_{nn}}\right)^{1/4} x_1 = 0, \\ x_3 = \cdots = x_{n-1} = x_{n+1} = 0. \end{gathered} \tag{11}$$

We may now uniquely determine a point P on the line OO^* such that the cross ratio of the three points O, O^*, P, and the intersection of the line (11) with OO^* is equal to $J^{1/4}$. On the other hand, the asymptotic hypercones of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^* determine a pencil of hyperquadrics in the hyperplane $x_{n+1}=0$, among which there exist n hypercones, two of them being the asymptotic hypercones. The line OO^* intersects each of the other $n-2$ hypercones in a pair of points. Let Q_i ($i=1, \cdots, n-2$) be any one of each pair of these points and D_i the cross ratio of the four points O, O^*, Q_i, P on the line OO^*, then we may easily show that *the invariant I can be expressed in terms of the $n-2$ cross ratios $D_1, D_2, \cdots, D_{n-2}$ as follows*:

$$I = (\pm 1)^{n-2}(D_1D_2 \cdots D_{n-2})^2. \tag{12}$$

3. A metric characterization of the invariant I. It is deemed worth while to give in this section a simple metric characterization of the invariant I. For this purpose we choose an orthogonal Cartesian coordinate system in such a way that the point O be the origin, the line OO^* be the X_{n-1}-axis, and the common tangent hyperplane t_{n-1} be the coordinate hyperplane $X_n=0$. Then the power series expansions of the hypersurfaces V_{n-1}, V_{n-1}^* in the neighborhood of the points O, O^* may be put into the form

$$V_{n-1}:\ X_n = \sum_{i,k=1}^{n-1} \lambda_{ik} X_i X_k + \cdots, \tag{13}$$

$$\begin{aligned} V_{n-1}^*:\ X_n = {} & \sum_{i,k=1}^{n-2} \mu_{ik} X_i X_k + 2 \sum_{i=1}^{n-2} \mu_{i,n-1} X_i (X_{n-1} - h) \\ & + \mu_{n-1,n-1}(X_{n-1} - h)^2 + \cdots, \end{aligned} \tag{14}$$

where h is the distance between the points O, O^*.

Let $y_0, y_1, \cdots, y_n$ be the homogeneous coordinates of a point defined by the formulas

$$X_i = y_i / y_0 \qquad (i = 1, \cdots, n), \tag{15}$$

and let us consider the most general projective transformation of coordinates which shall leave the point O and the common tangent hyperplane t_{n-1} invariant, and change the point O^* into the vertex $(0, \cdots, 0, 1, 0)$ of the new coordinate system:

$$\begin{aligned} y_0 &= \sum_{i=0}^{n} a_{0i} y_i', \\ y_i &= \sum_{r=1}^{n} a_{ir} y_r' \qquad (i = 1, \cdots, n-1), \\ y_n &= a_{nn} y_n', \end{aligned} \tag{16}$$

where

$$a_{1,n-1} = \cdots = a_{n-2,n-1} = 0, \qquad a_{n-1,n-1} = h a_{0,n-1}, \tag{17}$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-2} \\ a_{21} & a_{22} & \cdots & a_{2,n-2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n-2,1} & a_{n-2,2} & \cdots & a_{n-2,n-2} \end{vmatrix} \neq 0. \tag{18}$$

By transformations (15) and (16), equations (13), (14) shall be carried into two others of the form

$$V_{n-1}:\ \frac{y_n'}{y_0'} = \sum_{i,k=1}^{n-1} p_{ik} \frac{y_i'}{y_0'} \frac{y_k'}{y_0'} + \cdots, \tag{19}$$

$$V_{n-1}^*:\ \frac{y_n'}{y_{n-1}'} = \sum_{i,k=0}^{n-2} q_{ik} \frac{y_i'}{y_{n-1}'} \frac{y_k'}{y_{n-1}'} + \cdots, \tag{20}$$

where the coefficients p_{ik}, q_{ik} are given by the equations:

$$a_{00}a_{nn}p_{ik} = \sum_{r,s=1}^{n-1} a_{ri}a_{sk}\lambda_{rs} \qquad (i,\, k = 1, \cdots, n-1); \tag{21}$$

$$a_{nn}a_{0,n-1}q_{ik} = \sum_{r,s=0}^{n-2} \alpha_{ri}\alpha_{sk}\mu_{rs} \qquad (i,\, k = 0, 1, \cdots, n-2), \tag{22}$$

$$\begin{gathered} \alpha_{00} = -\,ha_{00}, \qquad \alpha_{i0} = 0, \qquad \alpha_{0i} = a_{n-1,i} - ha_{0i}, \qquad \alpha_{ri} = a_{ri}, \\ \mu_{00} = \mu_{n-1,n-1},\ \ \mu_{0r} = \mu_{r0} = \mu_{n-1,r} = \mu_{r,n-1} \ \ (i,\, r = 1, \cdots, n-2). \end{gathered} \tag{23}$$

Let

$$\Phi = \begin{vmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1,n-1} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2,n-1} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{n-1,1} & \lambda_{n-1,2} & \cdots & \lambda_{n-1,n-1} \end{vmatrix}, \quad \Psi = \begin{vmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1,n-1} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2,n-1} \\ \cdot & \cdot & \cdot & \cdot \\ \mu_{n-1,1} & \mu_{n-1,2} & \cdots & \mu_{n-1,n-1} \end{vmatrix},$$

$$P = \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1,n-1} \\ p_{21} & p_{22} & \cdots & p_{2,n-1} \\ \cdot & \cdot & \cdot & \cdot \\ p_{n-1,1} & p_{n-1,2} & \cdots & p_{n-1,n-1} \end{vmatrix}, \quad Q = \begin{vmatrix} q_{00} & q_{01} & \cdots & q_{0,n-2} \\ q_{10} & q_{11} & \cdots & q_{1,n-2} \\ \cdot & \cdot & \cdot & \cdot \\ q_{n-2,0} & q_{n-2,1} & \cdots & q_{n-2,n-2} \end{vmatrix},$$

then from equations (17), (18), (21), (22), (23) we obtain

$$a_{00}^{n-1}a_{nn}^{n-1}P = a_{n-1,n-1}^{2}\Delta^{2}\Phi, \qquad a_{nn}^{n-1}a_{0,n-1}^{n-1}Q = h^{2}a_{00}^{2}\Delta^{2}\Psi. \tag{24}$$

Making use of the result obtained in §1 and observing equations (19), (20) we see that the projective invariant I associated with the hypersurfaces V_{n-1}, V_{n-1}^{*} at the points O, O^{*} is

$$I = \frac{P}{Q}\left(\frac{q_{00}}{p_{n-1,n-1}}\right)^{(n+1)/3}. \tag{25}$$

Furthermore, substituting (21), (22), (24) in (25) and reducing by equations (17) it follows that *the invariant I now takes the form*

$$I = \frac{\Phi}{\Psi}\left(\frac{\mu_{n-1,n-1}}{\lambda_{n-1,n-1}}\right)^{(n+1)/3}. \tag{26}$$

Let K, K^{*} be the curvatures of the hypersurfaces V_{n-1}, V_{n-1}^{*} at the points O, O^{*}; and R, R^{*} the curvatures at the points O, O^{*} of the plane sections of the hypersurfaces V_{n-1}, V_{n-1}^{*} made by the plane of the line OO^{*} and the normal to the common tangent hyperplane t_{n-1} at any point on the line OO^{*}. By a known formula it is easy to demonstrate that

$$K/K^* = \Phi/\Psi, \qquad R/R^* = \lambda_{n-1,n-1}/\mu_{n-1,n-1}, \tag{27}$$

and therefore that

$$I = \frac{K}{K^*}\left(\frac{R^*}{R}\right)^{(n+1)/3}. \tag{28}$$

Hence we have the following theorem.

THEOREM. *Let V_{n-1}, V_{n-1}^* be two hypersurfaces in a space S_n of n dimensions having a common tangent hyperplane t_{n-1} at two ordinary point O, O^*; K, K^* the curvatures of the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*; and R, R^* the curvatures at the points O, O^* of the plane sections of the hypersurfaces V_{n-1}, V_{n-1}^* made by the plane of the line OO^* and the normal to the common tangent hyperplane t_{n-1} at any point on the line OO^*. Then $(K/K^*)(R^*/R)^{(n+1)/3}$ is a projective invariant associated with the hypersurfaces V_{n-1}, V_{n-1}^* at the points O, O^*.*

CHAPTER II. TWO HYPERSURFACES WITH DISTINCT TANGENT HYPERPLANES AT TWO ORDINARY POINTS

4. Derivation of invariants. Let V_{n-1}, V_{n-1}^* be two hypersurfaces in a space S_n of n dimensions such that the tangent hyperplanes t_{n-1}, t_{n-1}^* at two ordinary points O, O^* are distinct, and the common tangent flat space t_{n-2} of t_{n-1}, t_{n-1}^* contains the line OO^*. If we choose the points O, O^* to be the vertices $(0, 1, 0, \cdots, 0)$, $(0, \cdots, 0, 1, 0)$ of a homogeneous projective coordinate system of reference, and the tangent hyperplanes t_{n-1}, t_{n-1}^* to be the coordinate hyperplanes $x_1 = 0$, $x_{n+1} = 0$ respectively, then the power series expansions of the hypersurfaces V_{n-1}, V_{n-1}^* in the neighborhood of the points O, O^* may be written in the form

$$V_{n-1}\colon\quad \frac{x_1}{x_2} = \sum_{i,k=3}^{n+1} l_{ik}\,\frac{x_i}{x_2}\,\frac{x_k}{x_2} + \cdots, \tag{29}$$

$$V_{n-1}^*\colon\quad \frac{x_{n+1}}{x_n} = \sum_{i,k=1}^{n-1} m_{ik}\,\frac{x_i}{x_n}\,\frac{x_k}{x_n} + \cdots. \tag{30}$$

Considering the most general projective transformation of coordinates which shall leave the points O, O^* and the hyperplanes t_{n-1}, t_{n-1}^* unchanged, we may easily show as in §1 that *the quantities*

$$I = \frac{LMl_{nn}m_{22}}{L_{n+1,n+1}M_{11}}, \qquad J = \left(\frac{M}{L}\right)^{n-3}\left(\frac{L_{n+1,n+1}m_{22}}{M_{11}l_{nn}}\right)^{n+1} \tag{31}$$

are projective invariants determined by the neighborhood of the second order of the hypersurfaces V_{n-1}, V^*_{n-1} *at the points* O, O^*, *where* $L_{n+1,n+1}$, M_{11} *are respectively the minors of* $l_{n+1,n+1}$, m_{11} *in the determinants*

$$L = \begin{vmatrix} l_{33} & l_{34} & \cdots & l_{3,n+1} \\ l_{43} & l_{44} & \cdots & l_{4,n+1} \\ \cdot & \cdot & \cdot & \cdot \\ l_{n+1,3} & l_{n+1,4} & \cdots & l_{n+1,n+1} \end{vmatrix}, \quad M = \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1,n-1} \\ m_{21} & m_{22} & \cdots & m_{2,n-1} \\ \cdot & \cdot & \cdot & \cdot \\ m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,n-1} \end{vmatrix},$$

and L', M', $L'_{n+1,n+1}$, M'_{11} *are denoted by similar expressions.*

5. **Projective characterizations of the invariants** I, J. By suitable choice of the system of reference the invariants I, J of equations (31) can be simplified. In fact, if we choose $n-1$ vertices of the system in the common tangent flat space t_{n-2}, and the other two $O_{n+1}(0, \cdots, 0, 1)$, $O_1(1, 0, \cdots, 0)$ respectively on the polars t, t^* of the flat space t_{n-2} with respect to the asymptotic hypercones of the hypersurfaces V_{n-1}, V^*_{n-1} at the points O, O^*, the invariants I, J then take the simple form

$$(32) \qquad \begin{aligned} I &= l_{nn} l_{n+1,n+1} m_{11} m_{22}, \\ J &= \left(\frac{L_{n+1,n+1}}{M_{11}}\right)^4 \left(\frac{m_{11}}{l_{n+1,n+1}}\right)^{n-3} \left(\frac{m_{22}}{l_{nn}}\right)^{n+1}. \end{aligned}$$

It should be noticed that the invariant J is reduced to 1 as $n=3$.

The polars t, t^* determine a space S_3, which intersects the hypersurfaces V_{n-1}, V^*_{n-1} in two surfaces S, S^*. These two surfaces S, S^* are evidently in the class considered by Buzano and Bompiani, and the corresponding invariant may be easily found from Bompiani's note [1] to coincide just with the invariant I. Thus we reach the conclusion:

The invariant I *associated with the hypersurfaces* V_{n-1}, V^*_{n-1} *at the points* O, O^* *is the invariant of Buzano at the points* O, O^* *of the surfaces* S, S^* *in which the hypersurfaces* V_{n-1}, V^*_{n-1} *are intersected by the space* S_3 *determined by the polars* t, t^*.

To characterize projectively the other invariant J we consider any hyperplane π_α through the common tangent flat space t_{n-2}:

$$(33) \qquad x_{n+1} = \alpha x_1 \qquad (\alpha \neq 0),$$

which intersects the hypersurfaces V_{n-1}, V^*_{n-1} in two hypersurfaces V_{n-2}, V^*_{n-2} of $n-2$ dimensions. Since these two hypersurfaces V_{n-2}, V^*_{n-2} have a common tangent hyperplane at the points O, O^* we may determine an invariant, denoted by I_α, as in §1:

$$I = \alpha^{2(n-3)/3}\frac{L_{n+1,n+1}}{M_{11}}\left(\frac{m_{22}}{l_{nn}}\right)^{n/3}. \tag{34}$$

On the other hand, it is useful to consider the hypercones C, C^* projecting respectively from the vertices $O_1(1, 0, \cdots, 0)$, $O_{n+1}(0, \cdots, 0, 1)$ the asymptotic hypercones at the points O, O^* of the hypersurfaces V_{n-1}, V_{n-1}^*. These two hypercones C, C^* determine a pencil of hyperquadrics in the space S_n: among which there exist $n-1$ hypercones, two of them being C, C^*. The line O_1O_{n+1} intersects each of the other $n-3$ hypercones in a pair of points. Let Q_i $(i=1, \cdots, n-3)$ be any one of each pair of these points, P the point of intersection of the line O_1O_{n+1} with the hyperplane π_α, and D_i the cross ratio of the four points O_1, O_{n+1}, Q_i, P on the line O_1O_{n+1}; then it follows that *the invariant J can be expressed in terms of the invariant I_α and the $n-3$ cross ratios $D_1, D_2, \cdots, D_{n-3}$ as follows*:

$$J = I_\alpha^{3}(D_1D_2\cdots D_{n-3})^{2}. \tag{35}$$

6. **Metric characterizations of the invariants I, J.** For the purpose of finding simple metric characterizations of the invariants I, J, we choose an orthogonal Cartesian coordinate system in such a way that the point O is the origin, the line OO^* is the X_{n-1}-axis, and the tangent hyperplane t_{n-1} is the coordinate hyperplane $X_1=0$. Then the power series expansions of the hypersurfaces V_{n-1}, V_{n-1}^* in the neighborhood of the points O, O^* may be put into the form

$$V_{n-1}:\ X_1 = \sum_{i,k=2}^{n} \lambda_{ik}X_iX_k + \cdots, \tag{36}$$

$$\begin{aligned} V_{n-1}^*:\ X_n &= \mu X_1 + \sum_{i,k=1}^{n-2} \mu_{ik}X_iX_k + 2\sum_{i=1}^{n-2}\mu_{i,n-1}X_i(X_{n-1}-h) \\ &\quad + \mu_{n-1,n-1}(X_{n-1}-h)^2 + \cdots, \end{aligned} \tag{37}$$

where h is the distance between the points O, O^*, and $\mu=\cot\omega$, ω being the angle of the tangent hyperplanes t_{n-1}, t_{n-1}^*.

In order to express the two invariants I, J in terms of the coefficients of expansions (36), (37) we have first as in §3 to consider the homogeneous coordinates $y_0, y_1, \cdots, y_n$ of a point defined by formulas (15) and the most general projective transformation of coordinates, which shall leave the point O and the tangent hyperplane t_{n-1} invariant and carry the point O^* and the tangent hyperplane t_{n-1}^* into the vertex $(0, \cdots, 0, 1, 0)$ and the coordinate hyperplane

$y'_n=0$ of the new coordinate system respectively. An easy calculation, which shall be omitted here, suffices to demonstrate the result as follows:

$$I = h^4 \frac{\Phi\Psi\lambda_{n-1,n-1}\mu_{n-1,n-1}}{\Phi_{nn}\Psi_{11}}, \quad J = \left(\frac{\Psi}{\Phi}\right)^{n-3}\left(\frac{\Phi_{nn}\mu_{n-1,n-1}}{\Psi_{11}\lambda_{n-1,n-1}}\right)^{n+1}, \tag{38}$$

where Φ_{nn}, Ψ_{11} denote respectively the minors of λ_{nn}, μ_{11} in the determinants

$$\Phi = \begin{vmatrix} \lambda_{22} & \lambda_{23} & \cdots & \lambda_{2n} \\ \lambda_{32} & \lambda_{33} & \cdots & \lambda_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{n2} & \lambda_{n3} & \cdots & \lambda_{nn} \end{vmatrix}, \qquad \Psi = \begin{vmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1,n-1} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2,n-1} \\ \cdot & \cdot & \cdot & \cdot \\ \mu_{n-1,1} & \mu_{n-1,2} & \cdots & \mu_{n-1,n-1} \end{vmatrix}.$$

Finally, we shall make use of the normals ON, ON^* at the point O of the common tangent flat space t_{n-2} in the tangent hyperplanes t_{n-1}, t^*_{n-1}. Let K_2, K_2^* be respectively the curvatures at the points O, O^* of the plane sections of the hypersurfaces V_{n-1}, V^*_{n-1} made by the planes OO^*N^*, OO^*N. Further, let K_n, K_n^* be the curvatures of the hypersurfaces V_{n-1}, V^*_{n-1} at the points O, O^*; and K_{n-1}, K^*_{n-1} the curvatures at the points O, O^* of the hypersurfaces V_{n-2}, V^*_{n-2} of $n-2$ dimensions in which the tangent hyperplanes t^*_{n-1}, t_{n-1} intersect the hypersurfaces V_{n-1}, V^*_{n-1} respectively. Then

$$\begin{aligned} K_n &= 2^{n-1}\Phi, & K_n^* &= 2^{n-1}(1+\mu^2)^{-(n+1)/2}\Psi, \\ K_{n-1} &= 2^{n-2}(1+\mu^2)^{(n-2)/2}\Phi_{nn}, & K^*_{n-1} &= 2^{n-2}\Psi_{11}, \\ K_2 &= 2(1+\mu^2)^{1/2}\lambda_{n-1,n-1}, & K_2^* &= 2\mu_{n-1,n-1}, \end{aligned} \tag{39}$$

and hence we arrive at *the following metric characterizations of the invariants I, J*:

$$I = \frac{h^4}{16}\,\frac{K_nK_n^*K_2K_2^*}{K_{n-1}K^*_{n-1}\sin^{2(n-1)}\omega}, \quad J = \left(\frac{K_n^*}{K_n}\right)^{n-3}\left(\frac{K_{n-1}K_2^*}{K^*_{n-1}K_2}\right)^{n+1}. \tag{40}$$

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REGULARITY PROPERTIES OF A CERTAIN CLASS OF SURFACES

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Introduction. Convex curves in the plane and convex surfaces in space have many at first sight unexpected regularity properties.[1] The stringency of the convexity condition makes it desirable to find a geometric property of convex curves or surfaces which has similar analytical implications but applies to a wider class of geometrical objects. The first and most studied generalization of convexity is finiteness of the order: a curve or surface is said to be of order n if its intersection with any straight line consists of at most n connected sets[2] (segments, points, or rays). However, this condition proves to be much too weak.[3] On the other hand, the dual condition, finiteness of class, will turn out to be more restrictive than necessary.

The present paper will show that for curves *finiteness of the total curvature* is an entirely satisfactory condition. It includes in particular all curves of finite class. More generally, hypersurfaces in E^n with finite total curvature seem to share the most important differentiability properties with convex hypersurfaces.

The present results on curves with finite total curvature and the modern theory of functions of a real variable lead to the following surprising fact concerning surfaces in E^3: *If the paratingens*[4] *of Φ at a given point p leaves out at least one line L, and if, locally, the plane sections of Φ parallel to L have uniformly bounded total curvature, then the following facts hold simultaneously at almost all points q of Φ: the surface Φ has a tangent plane Π at q. All plane sections ($\neq \Pi$) of Φ through p have a curvature*[5] *at p, and these curvatures satisfy the theorems of Meusnier and Euler.*

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[1] For curves compare Jessen [1], for surfaces Busemann-Feller [1], and for hypersurfaces Alexandrov [1]. Numbers in brackets refer to the references cited at the end of the paper.

[2] This formulation is due to Hjelmslev [2] and has the obvious advantage not to exclude polygons, polyhedrons, ruled surfaces, $\cdots$.

[3] Compare Marchaud [1], Haupt [1], and §2 of the present paper.

[4] This concept is due to Bouligand, see Bouligand [1]. The paratingens of Φ represented parametrically by $p(u, v)$ at (u_0, v_0) consists of all non-oriented lines G which are limits of sequences of non-oriented lines $G(p(u'_\nu, v'_\nu), p(u_\nu, v_\nu))$ with $(u'_\nu, v'_\nu) \neq (u_\nu, v_\nu)$ and $u'_\nu, u_\nu \to u_0$, $v'_\nu, v_\nu \to v_0$. For the definition of total curvature see the following §1.

[5] More precisely the ordinary curvature as defined at the end of §1.

It will be seen that the hypotheses can be somewhat reduced. Theorems (4.5) and (4.8) are believed to be of interest for the theory of real variables.

1. Curves of finite total curvature.

For $a \leqq t \leqq b$ let the point $p(t) = (x(t), y(t))$ of the (x, y)-plane describe a continuous curve C which is locally a Jordan arc: that is to say, for every t_0 let there be an $\eta(t_0) > 0$ such that the mapping $p(t) \to t$ is topological for $|t - t_0| \leqq \eta(t_0)$, or $a \leqq t \leqq a + \eta(a)$, or $b - \eta(b) \leqq t \leqq b$,[6] respectively.

The oriented straight line G is called a *paratangent* of C at t_0 if there exist sequences $t_k' \to t_0$, $t_k'' \to t_0$ with $t_k' < t_k''$ and such that the oriented line $G(p(t_k'), p(t_k'')) \equiv G(t_k', t_k'')$ tends to G. If, in particular, the two sequences can be chosen so that $t_k' \leqq t_0 \leqq t_k''$, then the paratangent G is called a *tangent* of C at t_0. If $t_k'' \leqq t_0$ $(t_k'' = t_0)$, then G is called a *left-hand* paratangent (tangent) of C at t_0; right-hand tangents and paratangents are defined similarly.[7]

(1.1) *The paratangent (tangent) of C at t_0 is unique if and only if the one-sided paratangents (tangents) of C at t_0 are unique and coincide.*

The statement is obvious, but it should be noticed that it holds only for oriented tangents and paratangents.

The *mean value theorem* holds for these tangents in the following form:

(1.2) *If $t_1 < t_2$ and $p(t_1) \neq p(t_2)$, then a value t_0 between t_1 and t_2 exists such that a suitable tangent of C at t_0 is parallel either to $G^+ = G(t_1, t_2)$ or to its opposite orientation G^-.*[8]

A loop joining two points (for example, a strophoid) will show that a tangent parallel to G^+ need not exist. The statement (1.2) is easily proved by considering the points of C for which the distance from G^+ is maximal.

For every ϕ, $0 \leqq \phi < 2\pi$, define $N(\phi)$ (possibly $N(\phi) = \infty$) as the number of values t for which a tangent of C at t forms the angle ϕ with the positive x-axis. *If $N(\phi)$ is summable, $\kappa = \int_0^{2\pi} N(\phi)\,d\phi$ is called the total curvature of C, otherwise put $\kappa = \infty$.*

If the tangent $T(t)$ of C is unique for every t and varies continuously with t, then the end point of the unit vector with origin $(0, 0)$ and parallel to $T(t)$ traverses a continuous curve on the unit circle whose (finite or infinite) length equals κ.[9] This justifies the above nomenclature. It will now be shown that curves of finite total curvature have an important property of convex curves.

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[6] The end points will frequently require similar obvious changes. but these will in general not be mentioned.

[7] One-sided tangents or paratangents will be used as common word for right-hand and left-hand tangents or paratangents.

[8] For curves given in the form $y = f(x)$ this result (compare (3.1)) is due to Hjelmslev [1].

(1.3) THEOREM. *A curve of finite total curvature has everywhere unique one-sided paratangents.*

Suppose that there exist two left-hand paratangents T' and T'' at t_0. Then at least one of the two angular intervals between T' and T'' has the property that to every line L contained in it there exist infinitely many chords of $p(t)$ parallel to L. By the mean value theorem there exist therefore infinitely many points $t_\nu \to t$ such that a tangent at t_ν is parallel or opposite to L. Therefore, for every ϕ of one of the two angular intervals between T' and T'' either $N(\phi)=\infty$ or $N(-\phi)=\infty$ so that $N(\phi)$ can not be finite almost everywhere.

For any curve C let t_0 be a value for which the one-sided paratangents are unique. Then it is easily seen that the paratingens of C at t_0, that is, the set of all paratangents at t_0, is determined as follows: Let T_1 and T_2 be the two one-sided paratangents, T_1' and T_2' their positive subrays with origin $p(t_0)$. Then either the angle between T_1' and T_2' has measure π, in which case the paratingens is the whole plane and we say that C has at t_0 a *cusp*; or T_1' and T_2' bound exactly one angle of measure $\omega(t_0)<\pi$. This closed angle together with its vertical angle form the paratingens of C at t_0 whose measure is defined as $\omega(t_0)$.

The importance of the last theorem is shown by the following simple facts:

(1.4) *If the curve* C: $p(t)$, $a \leqq t \leqq b$, *has everywhere unique one-sided paratangents then*:

(a) *Every paratangent of* C *is a tangent.*

(b) *If* $t_\nu \to t_0 - 0$ *and* T_ν *is a tangent of* C *at* t_ν, *then* T_ν *tends to the left-hand tangent of* C *at* t_0.

(b′) *If* $t_\nu \to t_0$, *and the tangent* T *of* C *at* t_0 *is unique, then* $T_\nu \to T$.

(c) *For every* $\eta>0$ *the set of those values* t *for which* $\omega(t_0)>\eta$ *is finite.*

(c′) *The number of* t*'s for which the paratangent of* C *is not unique is at most countable.*

To see (a) let $t_\nu' < t_\nu''$ and $t_\nu' \to t_0$, $t_\nu'' \to t_0$, $G(t_\nu', t_\nu'') \to G$. If $t_\nu' \leqq t_0 \leqq t_\nu''$ for infinitely many ν nothing is to be proved. If $t_\nu' \leqq t_\nu'' \leqq t_0$, then

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[9] This is a geometric formulation of a known theorem in the theory of real variables; compare, for example, Saks [1].

$G(t_\nu', t_\nu'')$ tend by hypothesis to the same limit as $G(t_\nu', t_0)$, so that C is a tangent.

To prove (b) choose t_ν' and t_ν'' with $t_\nu' \leqq t_\nu \leqq t_\nu'' < t_0$ such that $0 < t_\nu'' - t_\nu' < 1/\nu$ and the smaller angle between $G(t_\nu', t_\nu'')$ and T_ν has measure less than $1/\nu$. Then $t_\nu' \to t_0$ and $G(t_\nu', t_\nu'')$ tends by (a) to the left-hand tangent of C at t_0.

If (c) were not true, then a sequence t_ν with $\omega(t_\nu) > \eta$ would exist. It may be assumed that $t_\nu \to t_0 - 0$. The two one-sided tangents of C at t_ν tend by (b) to the left-hand tangent of C at t_0, which contradicts the hypothesis $\omega(t_\nu) > \eta$. If C has the finite total curvature κ, the statement (c′) may be improved to $\sum \omega(t_\nu) \leqq \kappa$, where the t_ν are the values of t where the paratangent is not unique. In particular, there are fewer than κ/π cusps.

The uniqueness of the one-sided paratangents allows us also to associate with every tangent T of C at t an angle $\psi(t)$ between the tangent and the positive x-axis ($\psi(t)$ not necessarily between 0 and 2π) which, although multivalued, varies with t in very much the same manner as when C has a continuous tangent; let $\tau_1 < \tau_2 < \cdots < \tau_r$ denote the cusps of C; and put $a = \tau_0$, $b = \tau_{r+1}$. Then $\tau_0 < \tau_1$, $\tau_r < \tau_{r+1}$. Consider a subarc C_i: $\tau_i \leqq t \leqq \tau_{i+1}$ of C. Define as tangents of C_i at τ_i and τ_{i+1} the right-hand and left-hand tangents of C at τ_i and τ_{i+1}, respectively. By (1.4b) there is for every τ, $\tau_i \leqq \tau \leqq \tau_{i+1}$, a $\delta(\tau) > 0$ such that the smaller angle between any two paratangents of C for $|\tau - t| < \delta(\tau)$ $(0 \leqq \tau - \tau_i < \delta(\tau_i),\ 0 \leqq \tau_{i+1} - \tau < \delta(\tau_{i+1}))$ is less than π. A finite number of these intervals belonging to values $t_1 = \tau_i < t_2 < \cdots < t_m = \tau_{i+1}$ will cover the interval (τ_i, τ_{i+1}) so that only consecutive intervals $|t_i - t| < \delta(\tau_i)$ have common points. If we fix the value $\psi(\tau_i) = \psi(t_1)$ arbitrarily, then the angle $\psi(\tau)$ belonging to all t in $|t_1 - t| < \delta(\tau_i)$ is uniquely determined by the requirement that $|\psi(t) - \psi(\tau_i)| < \pi$. The same requirement and the already defined value of $\psi(t)$ for common points of $|t_i - t| < \delta(t_i)$, $i = 1, 2$, determine $\psi(t)$ uniquely for $|t_2 - t| < \delta(t_2)$. Continuing in the same way one defines $\psi(t)$ for all t on C_i. Notice that $\psi(\tau_i + 0) = \psi(\tau_i)$ and $\psi(\tau_{i+1} - 0) = \psi(\tau_{i+1})$.

Now choose $\psi(a)$ so that $0 \leqq \psi(a) < 2\pi$. Then the angles $\psi(t)$ are determined for $a \leqq t \leqq \tau_1$. As values $\psi(\tau_1)$ take those with $\psi(\tau_1 - 0) \leqq \psi(\tau_1) \leqq \psi(\tau_1 - 0) + \pi$. As initial value on C_1 choose $\psi(\tau_1 - 0) + \pi$, then $\psi(t)$ is determined on C_2, and so forth. A consequence of this determination of $\psi(t)$ is that for any t the limits $\psi(t-0)$ and $\psi(t+0)$ exist and that for any $\psi(t)$ belonging to t either

$$(1.5)\quad \psi(t-0) \leqq \psi(t) \leqq \psi(t+0) \text{ or } \psi(t-0) \geqq \psi(t) \geqq \psi(t+0).$$

(1.6) *If C has total curvature κ, then for any $a=t_1<t_2<\cdots<t_n=b$ and any choice of the values $\psi(t_i)$*

$$\sum_{\nu=1}^{n-1} \left| \psi(t_{i+1}) - \psi(t_i) \right| \leqq \kappa.$$

For by (1.5)

$$\sum \left| \psi(t_i) - \psi(t_{i+1}) \right| \leqq \sum \left| \psi(t_i + 0) - \psi(t_{i+1} - 0) \right| + \sum \left| \psi(t_i + 0) - \psi(t_i - 0) \right|.$$

Now $\psi(t)$ takes on every value between $\psi(t_i+0)$ and $\psi(t_{i+1}-0)$ at least once in (t_i, t_{i+1}) and every value between $\psi(t_i-0)$ and $\psi(t_i+0)$ exactly once at t_i, which proves (1.6).

Let C_i, $\tau_i \leqq t \leqq \tau_{i+1}$, denote the same subarc of C as above. For every τ_0 the paratingens of C_i at t_0 leaves out at least one line L'. Choose the perpendicular L to L' as an x'-axis and L' as a y'-axis. Then for a suitable $\delta>0$, the subarc $|t_0-t| \leqq \delta$ has a representation $y'=f(x')$, $-\delta_1 \leqq x' \leqq \delta_2$, $\delta_i>0$, and the difference quotients $(f(x_1')-f(x_2'))/(x_1'-x_2')$ are bounded[10] in $(-\delta_1, \delta_2)$, since the paratingens of C at t_0 is a closed set. Therefore the arc $|t_0-t| \leqq \delta$ of C has the finite length $\int_{-\delta_1}^{\delta_2}\{1+f'^2(x')\}^{1/2}dx'$. A trivial application of the Heine-Borel theorem yields that C_i has finite length, so that C is also rectifiable. Introduce the arc length s as parameter on C, and let $\psi(t)$ become the function $\phi(s)$. Let $\phi^*(s)$ denote any of the values $\phi(s)$. Then $\phi^*(s)$ has by (1.6) bounded variation so that $\phi^{*\prime}(s)$ exists for almost all s. If $\bar{\phi}(s)$ denotes any other value of $\phi(s)$, then because of (1.5), $\bar{\phi}'(s)$ exists and equals $\phi^{*\prime}(s)$ whenever $\phi^{*\prime}(s)$ exists. Therefore we shall simply speak of $\phi'(s)$. Whenever $\phi'(s)$ exists, the tangent $T(s)$ of C is unique.

The geometric equivalent to the existence of $\phi'(s)$ is this: Let $\nu(s)$ be the normal to $T(s)$ at $p(s)$, and denote by $\nu(s+h)$ the normal to any tangent of C at $(s+h)$, and by $d_s(h)$ the distance (possibly ∞) of the intersection of $\nu(s)$ and $\nu(s+h)$ from $p(s)$. Then $1/d_s(h) \rightarrow |\phi'(s)|$, and $|\phi'(s)|$ is the so-called *strong curvature* of C at s.

(1.7) *A curve of finite total curvature has almost everywhere* (*with respect to the arclength s*) *a finite strong curvature.*

The (*ordinary*) *curvature* of C at a point s where C has a unique tangent $T(s)$ is defined as follows: Let $r_s(h)$ denote the radius (possibly ∞) of the circle which has tangent $T(s)$ at $p(s)$ and passes through $p(s+h)$. If $1/r_s(h)$ converges, then the limit is the curvature of C at s. For any curve the existence of the strong curvature $\lim (1/d_s(h))$ implies the existence of the curvature $\lim (1/r_s(h))$; for convex curves the converse holds.[11] This is not necessarily true for curves of finite total curvature.

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[10] Compare Bouligand [1, p. 123].

2. **Examples.** Before passing on to surfaces, we exhibit three examples of curves. (a) We show that curves of finite total curvature are really essentially more general than convex curves. This will be accomplished by constructing a curve with $N(\phi)\leqq 3$ which is not the union of a countable set of convex arcs and a set of measure 0. (b) We prove the statement at the end of §1. (c) We show that finiteness of the order is not an adequate generalization of convexity.

Let $\kappa_1, \kappa_2, \cdots$ be any finite or countable number of closed disjoint subarcs of a semicircle κ. Denote by κ_ν' the image of κ_ν under the reflection in the line $G(a_\nu, b_\nu)$ connecting the end points a_ν, b_ν of κ_ν, and replace κ_ν by κ_ν' for every ν. The curve C' thus obtained from κ has $N(\phi)\leqq 3$. If the κ_ν are chosen so that the set $\kappa-\sum\kappa_\nu$ is a nowhere dense subset of κ with positive measure, C' will not be the union of a countable number of convex arcs and a set of measure 0.

It requires a little more to find an example for (b). With the same notation as before but arbitrary κ_ν take a subarc κ_ν^* of κ_ν' whose end points a_ν' and b_ν' are close to but different from a_ν and b_ν, respectively, and replace κ_ν^* by its image κ_ν'' under reflection in $G(a_\nu', b_\nu')$. For the curve C'' thus obtained from C' the function $N(\phi)$ is at most 6.

Now let κ be the semicircle $x\geqq 0$ of the circle $x^2+y^2=1$, and E any curve which passes through (0, 1) and lies except for (0, 1) entirely in the unit circle and has there $y=1$ as tangent, and curvature 1. Then any curve between κ and E through (0, 1) will have curvature 1 at (0, 1). Now choose $a_\nu=(\cos(\pi/2^{2\nu-1}), \sin(\pi/2^{2\nu-1}))$, $b_\nu=(\cos(\pi/2^{2\nu}), \sin(\pi/2^{2\nu}))$ and a_ν' and b_ν' so close to a_ν and b_ν that the arc κ_ν'' lies between κ and E. Then C'' has curvature 1 at (0, 1). But it does not have a strong curvature at (0, 1); for a parallel P_ν through a_ν to the radius of κ from (0, 0) to b_ν is a normal to C'' at a_ν. The equation of P_ν is $y-\sin(\pi/2^{2\nu-1})=\tan(\pi/2^{2\nu})(x-\cos(\pi/2^{2\nu-1}))$, hence P_ν intersects the x-axis, which is the normal to C'' at (0, 1), at

$$x_\nu = \cos(\pi/2^{2\nu-1}) - \frac{\sin(\pi/2^{2\nu-1})}{\tan(\pi/2^{2\nu})} \to -1.$$

Curves of order 3 are the union of at most 4 convex arcs[12] and have therefore all differentiability properties of convex curves.

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[11] See Jessen [1].

[12] See Marchaud [1, p. 88, Théorème IV].

Haupt[13] constructed an example of a curve of order 4, where at a set μ of positive measure the paratingens is the whole plane. The strong curvature can, of course, not exist at any point of μ. An example of a curve of order at most 8 with the same properties can easily be constructed as follows: With the above notations choose the κ_ν again so that $\kappa-\sum\kappa_\nu$ is nowhere dense and has positive measure. Construct a semicircle $\bar{\kappa}_\nu$ with two suitably chosen points c_ν, d_ν of κ_ν as end points such that $\bar{\kappa}_\nu$ is contained in the circular segment S_ν bounded by κ_ν and the chord from a_ν to b_ν. Let $\bar{C}$ be the curve obtained from κ by replacing the arc from c_ν to d_ν by $\bar{\kappa}_\nu$. Since any straight line intersects at most two segments S_ν, and has at most 4 common points with the subarc of $\bar{C}$ in S_ν, it follows that $\bar{C}$ has at most order 8 (actually it has order 6). The paratingens is the whole plane at every point of $\kappa-\sum\kappa_\nu$. If the corners of $\bar{C}$ at c_ν and d_ν are smoothed off by changes taking place in a sufficiently small neighborhood of c_ν and d_ν, a curve $\bar{C}'$ is obtained, where the tangent exists and is continuous, hence the paratingens is everywhere a straight line, but the strong curvature exists at no point of $\kappa-\sum\kappa_\nu$. The question whether a curve of finite order has almost everywhere an ordinary curvature is open.

3. **Auxiliary facts on real variables.** The results of §1 can be applied to surfaces Φ: $p(u, v)=(x(u, v), y(u, v), z(u, v))$, $-\infty<u, v<\infty$, in a Cartesian (x, y, z)-space with the following properties:

I. *For every pair* (u_0, v_0) *there is a* $\delta(u_0, v_0)>0$ *so that the mapping* $p(u, v)\rightarrow(u, v)$ *is topological for* $|u-u_0|<\delta$ *and* $|v-v_0|<\delta$.

II. *For every* (u_0, v_0) *there is at least one line* L *not contained in the paratingens*[14] *of* Φ *at* (u_0, v_0).

III. *There is a positive* $\delta'\leqq\delta$ *such that planes parallel to* L *intersect the part* $|u-u_0|\leqq\delta'$, $|v-v_0|\leqq\delta'$ *of* Φ *in curves of uniformly bounded total curvature.*

Some observations to explain the content of conditions II and III are in order:

There is a positive $\delta^*\leqq\delta$ such that with L as z'-axis and the plane Π perpendicular to L as (x', y')-plane the part $|u-u_0|<\delta^*$, $|v-v_0|<\delta^*$ of Φ is represented in the form $z'=f(x', y')$, where (x', y') traverses some open set Γ in Π containing $x'=0$, $y'=0$. Clearly the number δ^* may in addition be chosen so that the difference quotients are bounded, that is,

$$|f(x_1', y_1')-f(x_2', y_2')|/\{(x_1'-x_2')^2+(y_1'-y_2')^2\}^{1/2}<M$$

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[13] See Haupt [1].

[14] See footnote 4.

for $(x_i', y_i') \in \Gamma$. The planes parallel to L intersect therefore for $\delta' < \delta^*$ the part $|u-u_0| \leqq \delta'$, $|v-v_0| \leqq \delta'$ of Φ in Jordan arcs of the form $p(u(t), v(t))$ so that the concept of total curvature becomes applicable. The set $|u-u_0| < \delta'$, $|v-v_0| < \delta'$ corresponds to an open set Γ' in Π which contains $x'=0$, $y'=0$. Therefore an $\eta > 0$ exists such that the domain $|x'| < \eta$, $|y'| < \eta$ is contained in Γ'; it corresponds to some open set Δ containing (u_0, v_0) in the (u, v)-plane.

This leads to a function $f(x', y')$ with the following properties:

(a) $f(x', y')$ is defined and continuous for $|x'| \leqq \eta$, $|y'| \leqq \eta$.

(b) $|f(x_1', y_1') - f(x_2', y_2')| / \{(x_1' - x_2')^2 + (y_1' - y_2')^2\}^{1/2} < M < \infty$.

(c) All sections $ax' + by' + c = 0$ with $z' = f(x', y')$ have total curvature less than $\kappa < \infty$.

Before proceeding to prove the result mentioned in the introduction, some facts on functions of a real variable will be recalled. Let $f(x)$ be defined and continuous for $a \leqq x \leqq b$. The mean value theorem (1.2) becomes then:[9]

(3.1) *If* $a \leqq x_1 < x_2 \leqq b$ *then an* x_0 *between* x_1 *and* x_2 *and a tangent of* $y = f(x)$ *(tangent in the above sense) at* x_0 *exist whose slope* $f^*(x_0)$ *satisfies the relation* $f^*(x_0) = \{f(x_2) - f(x_1)\} / \{x_2 - x_1\}$.

The first and second *Schwarz derivatives* $D'f(x_0)$ and $D''f(x_0)$ of f at x_0 are defined by

$$D'f(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h},$$

$$D''f(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2}.$$

If $f'(x_0)$ exists, then $D'f(x_0)$ exists, but not conversely. However,[15] if $D'f(x_0)$ exists in a set of positive measure E then $f'(x)$ exists almost everywhere (a.e.) in E.

Let the difference quotients of $f(x)$ be bounded by M and let the total curvature of $y = f(x)$ be at most κ. If $f^*(x)$ denotes the slope of any tangent of $y = f(x)$ at x, it follows from §1 that $f^*(x)$ has variation $V \leqq \kappa(1 + M^2)^{1/2}$. The derivative $f'(x)$ exists a.e. in (a, b). At all points where $f'(x)$ exists, the existence of $f^{*\prime}(x)$ for one function $f^*(x)$ implies the existence of the derivative for every other slope function, and the derivatives are equal; they will be denoted by $f''(x)$.

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[15] This is a special case of the general and deeper theorem of Marcinkiewicz and Zygmund [1] that the existence of the kth Riemann derivative of $f(x)$ in a set E of positive measure implies the existence a.e. in E of the kth de la Valée Poussin derivative, because the kth Riemann derivative coincides for $k = 1, 2$, with the Schwarz derivative.

If $f''(x)$ exists then $D''f(x_0)$ and the so-called *second de la Vallée Poussin derivative*

$$d''f(x_0) = \lim \frac{2}{h}\left\{\frac{f(x_0+h)-f(x_0)}{h} - f'(x_0)\right\}$$

exist and equal $f''(x_0)$. The existence of $f''(x_0)$ and $d''f(x_0)$ are equivalent to the existence of the strong and ordinary curvatures of $y=f(x)$ at x_0[16] respectively.

$d''f(x)$ may exist in a set E of positive measure, without $f''(x)$ existing anywhere in E.[17] If $d''f(x_0)$ exists, then $D''f(x_0)$ exists but not conversely, however:[18]

(3.2) *If $D''f(x)$ exists in a set E of positive measure, then $d''f(x)$ exists a.e. in E.*

If $f(x, y)$ is defined and continuous for $|x| \leqq \eta$, $|y| \leqq \eta$ all these concepts can be applied to derivatives of $f(x, y)$ in a given direction α and lead to concepts like $f_\alpha^*(x_0, y_0)$, $f_\alpha'(x_0, y_0)$, $D_\alpha' f(x_0, y_0)$, $f_\alpha''(x_0, y_0)$, $D_\alpha'' f(x_0, y_0)$ and $d_\alpha'' f(x_0, y_0)$, where, for example,

$$D_\alpha' f(x_0, y_0) = \lim \frac{1}{2h}\{f(x_0 + h\cdot\cos\alpha,\ y_0 + h\cdot\sin\alpha) - f(x_0 - h\cdot\cos\alpha,\ y_0 - h\cdot\sin\alpha)\},$$

$$d_\alpha' f(x_0, y_0) = \lim \frac{2}{h}\left\{\frac{f(x_0 + h\cdot\cos\alpha,\ y_0 + h\cdot\sin\alpha) - f(x_0, y_0)}{h} - f_\alpha'(x_0, y_0)\right\}.$$

For fixed α the set where any one of these derivatives exists is measurable.

For any interval I: $x' < x < x''$, $y' < y < y''$ put

$$\Delta(I, f) = f(x', y') + f(x'', y'') - f(x', y'') - f(x'', y').$$

The *variation of f* in the rectangle S: $|x| < \delta$, $|y| < \delta$ is defined as $V(f, S) = \sup_{[I_\nu]} \sum |\Delta(I_\nu, f)|$ where $[I_\nu]$ traverses all finite sets of nonoverlapping intervals in S. To establish $V(f, S) \leqq M$ it suffices to show $V(f, S') < M$ for every rectangle S': $|x| < \delta' < \delta$, $|y| < \delta'' < \delta$. If $V(f, S)$ is finite, $f(x, y)$ is called of bounded variation in S in the sense of Lebesgue. According to a fundamental theorem of Lebesgue[18] $V(f, S) < \infty$ implies that almost everywhere in S

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[16] Compare Busemann-Feller [1, p. 7].

[17] See Denjoy [1].

[18] See Lebesgue [1] or Saks [1].

$$\lim_{\nu \to \infty} \Delta(I_\nu, f)/|I_\nu| = F(x_0, y_0), \qquad |I_\nu| = \text{area of } I_\nu, \tag{3.3}$$

exists and is finite, provided $(x_0, y_0) \in \bar{I}_\nu$ and the sequence $\{I_\nu\}$ is regular, that is, if h_ν and k_ν are the sides of I_ν, then

$$0 < \liminf \frac{h_\nu}{k_\nu} \leqq \limsup \frac{h_\nu}{k_\nu} < \infty. \tag{3.4}$$

4. The existence of the second differential.

After these preparations we show first:

(4.1) *If f is defined and continuous for $|x| \leqq \delta$, $|y| \leqq \delta$ and if for some α, $0 < \alpha < \pi/2$, the slopes of the sections $x \sin \alpha \pm y \cos \alpha = \text{const.}$ of $z = f(x, y)$ have variation at most B, then $f(x, y)$ has in S: $|x| < \delta$, $|y| < \delta$ at most variation $2\delta B/\cos \alpha$ and $z = f(x, y)$ has a tangent plane a.e., that is, the derivatives f'_β exist simultaneously for all β and*

$$f'_\beta = f'_0 \cos \beta + f'_{\pi/2} \sin \beta. \tag{4.2}$$

Proof. For given δ', $\delta'' < \delta$ choose a rational number $r = p/q$ so that $\delta'' \leqq r\delta' \cdot \tan \alpha < \delta$. For any N that is divisible by q let $I^N_{\mu\nu}$ denote the interval $[(\mu-1)/N-1]\delta' < x < (\mu/N-1)\delta$, $\{(\nu-1)/N-r\}\delta' \tan \alpha < y < (\nu/N-r)\delta' \tan \alpha$, $\mu = 1, \cdots, 2N$; $\nu = 1, \cdots, 2Nr = N'$. Then the diagonals of $I^N_{\mu\nu}$ are along lines $x \cdot \sin \alpha \pm y \cdot \cos \alpha = \text{const.}$ Since $f(x, y)$ is continuous it suffices to show that

$$\sum_{\mu,\nu=1}^{2N,N'} |\Delta(I^N_{\mu\nu}, f)| \leqq 2\delta' B/\cos \alpha \text{ for every } N \text{ divisible by } q. \tag{4.3}$$

Let $p_{\mu\nu}(s, t)$ denote the point $x = \delta'[(2\mu - 1 - 2N + s)/(2N)]$, $y = \delta' \tan \alpha[(2\nu - 1 - 2rN + t)/(2N)]$ so that $p_{\mu\nu}(0, 0)$ is the center of $I^N_{\mu\nu}$. Then

$$\begin{aligned}\Delta(I^N_{\mu\nu}, f) = {} & \{[f(p_{\mu\nu}(1, 1)) - f(p_{\mu\nu}(0, 0))] - [f(p_{\mu\nu}(0, 0)) \\ & - f(p_{\mu\nu}(-1, -1))]\} - \{[f(p_{\mu\nu}(1, -1)) - f(p_{\mu\nu}(0, 0))] \\ & - [f(p_{\mu\nu}(0, 0)) - f(p_{\mu\nu}(-1, 1))]\}.\end{aligned}$$

Hence by (3.1)

$$\begin{aligned}\Delta(I^N_{\mu\nu}, f) = {} & (\delta'/(2N \cos \alpha))\{f^*_\alpha(p_{\mu\nu}(\theta^1, \theta^1)) \\ & - f^*_\alpha(p_{\mu\nu}(-\theta^2, -\theta^2))\} - (\delta'/(2N \cos \alpha)) \\ & \cdot \{f^*_{-\alpha}(p_{\mu\nu}(\theta^3, -\theta^3)) - f^*_{-\alpha}(p_{\mu\nu}(-\theta^4, \theta^4))\},\end{aligned}$$

where $0<\theta^i<1$ and the θ^i depend on μ and ν. Therefore

$$\begin{aligned}\sum_{\mu,\nu=1}^{2N,N'} \left| \Delta(I_{\mu\nu}^{N}, f)\right| \leqq (\delta'/(2N\cos\alpha)) \sum_{\lambda=1}^{2N} \sum_{\mu-\nu=\lambda} &\left| f_\alpha^*(p_{\mu\nu}(\theta^1, \theta^1)) \right. \\ &\left. - f_\alpha^*(p_{\mu\nu}(-\theta^2, -\theta^2)) \right| \\ &+ (\delta'/(2N\cos\alpha)) \sum_{\lambda=1}^{N'} \sum_{\mu+\nu=\lambda} \left| f_\alpha^*(p_{\mu\nu}(\theta^3, -\theta^3)) \right. \\ &\left. - f_\alpha^*(p_{\mu\nu}(-\theta^4, \theta^4)) \right|.\end{aligned}$$

Since the variation of $f_{\pm\alpha}^*$ along the lines $x\cdot\sin\alpha \pm y\cdot\cos\alpha=\text{const.}$ is at most B and $\mu-\nu=\lambda$ implies that $p_{\mu\nu}(\theta^1, \theta^1)$ and $p_{\mu\nu}(-\theta^2, -\theta^2)$ lie on the line $x\cdot\sin\alpha - y\cdot\cos\alpha = \delta\lambda - \sin\alpha - r\cdot\cos\alpha$ the relation (4.3) follows.

The derivative $f_\alpha'(x, y)$ exists a.e. along a fixed line $x\cdot\sin\alpha - y\cdot\cos\alpha = \text{const.}$ Since the two-dimensional set where f_α' exists is measurable, it follows that $f_\alpha'(x, y)$ exists a.e. in $|x| \leqq \delta$, $|y| \leqq \delta$. Denote by N_1 the set of measure 0 in S where either f_α' or $f_{-\alpha}'$ or the set derivative $F(x, y)$ of $f(x, y)$ does not exist. If $(x_0, y_0) \in S - N_1$ and $k = h\cdot\cos\alpha$, $l = h\cdot\sin\alpha$, then

$$\begin{aligned} F(x_0, y_0) &= \lim \frac{1}{kl} \{ f(x_0+k, y_0+l) + f(x_0, y_0) \\ &\qquad\qquad - f(x_0+k, y) + f(x_0, y_0+l) \} \\ &= \lim \frac{1}{kl} \{ - f(x_0+k, y_0-l) - f(x_0, y_0) \\ &\qquad\qquad + f(x_0+k, y) + f(x_0, y_0-l) \}. \end{aligned} \tag{4.4}$$

Hence

$$\begin{aligned} 2F(x_0, y_0) = \lim (k\sin\alpha)^{-1} \Bigg\{ &\frac{f(x_0+k, y_0+l) - f(x_0, y_0)}{h} \\ &- \frac{f(x_0+k, y_0-l) - f(x_0, y_0)}{h} \\ &- 2\sin\alpha \frac{f(x_0, y_0+l) - f(x_0, y_0-l)}{2l} \Bigg\}. \end{aligned}$$

The first two terms in the braces tend to $f_\alpha'(x_0, y_0)$ and $f_{-\alpha}'(x_0, y_0)$ respectively, therefore the last term converges, that is, $D_{\pi/2}'f(x_0, y_0)$ exists. Similarly $D_0'f(x_0, y_0)$ exists. It follows from the last section that f_0' and $f_{\pi/2}'$ exist a.e. in $S-N_1$, so that f_0', $f_{\pi/2}'$, $f_{\pm\alpha}'$ and F exist in S except for a set $N_2 \supset N_1$ of measure 0.

Let $(x_0, y_0) \in S-N_2$, $0<\beta<\pi$, $\beta \neq \pi/2$, and put $h\cdot\cos\beta = k$, $h\cdot\sin\beta = l$. By transforming (4.4) it follows that

$$F(x_0, y_0) = \lim\, (h\cdot\sin\beta\cos\beta)^{-1}\left\{\frac{1}{k}\,(f(x_0+k,\, y_0+l) - f(x_0, y_0)) - \frac{\cos\beta}{k}\,(f(x_0+k,\, y_0) - f(x_0, y_0)) - \frac{\sin\beta}{l}\,(f(x_0,\, y_0+l) - f(x_0, y_0))\right\}.$$

The last part in the braces tends to $-\cos\beta f'_0 - \sin\beta f'_{\pi/2}$, hence the first part converges and (4.2) follows.

This theorem leads to the following main result on functions $f(x, y)$ for the present paper.

(4.5) Theorem. *If f is defined and continuous for $|x| \leqq \delta$, $|y| \leqq \delta$, and if the slopes of the sections $y=$ const. and $x\cdot\sin\alpha \pm y\cdot\cos\alpha =$ const., for some fixed α, $0<\alpha<\pi/2$, have variation at most B, then the following facts hold simultaneously almost everywhere in S ($|x|<\delta$, $|y|<\delta$).*

(1) *The set derivative $F(x, y)$ of $f(x, y)$ exists.*

(2) *$z=f(x, y)$ has a tangent plane.*

(3) *$d''_\beta f(x, y)$ exists for all β in the following strong sense:*

$$d''_\beta f(x_0, y_0) = \lim_{\bar\beta\to\beta,\, h\to 0} \frac{2}{h}\left\{\frac{f(x_0+h\cdot\cos\bar\beta,\, y_0+h\cdot\sin\bar\beta) - f(x_0, y_0)}{h} - f_{\bar\beta}(x_0, y_0)\right\} \tag{4.6}$$

and

$$d''_\beta f = d''_0 f\cos^2\beta + 2F\cdot\cos\beta\sin\beta + d''_{\pi/2}\sin^2\beta. \tag{4.7}$$

Proof. By (4.1) and previous remarks the following statements are true except for a set N_3 of measure 0: f'_β exists for all β and $f'_\beta = f'_0\cos\beta + f'_{\pi/2}\sin\beta$; the derivatives $d''_0 f$ and $d_{\pm\alpha} f$ exist. It is important that in the first part of this proof only the existence of $d''_0 f$ is used and not the fact that the slopes of $y=$ const. have variation at most B. Let $(x_0, y_0) \in S-N_3$. Put $f_0(s, t) = f(x_0+s,\, y_0+t)$, $f_0 = f(0, 0) = f(x_0, y_0)$ and $k = h\cdot\cos\alpha$, $l = h\cdot\sin\alpha$. Then

$$\begin{aligned}\sin\alpha\cos\alpha\, F(x_0, y_0) &= \lim\, h^{-2}\{f_0(k, l) + f_0 - f_0(k, 0) - f(0, l)\}\\ &= \lim\, h^{-2}\{f_0 + f_0(-k, -l)\\ &\qquad - f_0(-k, 0) - f_0(0, -l)\},\end{aligned}$$

hence

$$\begin{aligned}2\cdot\sin\alpha\cos\alpha F(x_0, y_0) = \lim\bigl\{&h^{-2}(f_0(k, l) + f_0(-k, -l) - 2f_0)\\ &- k^{-2}\cos^2\alpha(f_0(k, 0) + f_0(-k, 0) - 2f_0)\\ &- l^{-2}\sin^2\alpha(f_0(0, l) + f(0, l) - 2f_0)\bigr\}.\end{aligned}$$

The first two terms in the braces tend to $D_\alpha' f = d_\alpha'' f$ and $-\cos^2\alpha D_0'' f = -\cos^2\alpha d_0'' f$, hence the last term converges, so that $D_{\pi/2}'' f(x_0, y_0)$ exists. By (3.2) the derivative $d_{\pi/2}'' f$ will exist except at the set $N_4 \supset N_3$ of measure 0.

Let $(x_0, y_0) \in S - N_4$ and $\bar\beta \to \beta^*$, then by (4.2)

$$f_{\bar\beta}'(x_0, y_0) = f_0' \cos\bar\beta + f_{\pi/2}' \sin\bar\beta \to f_{\beta^*}'(x_0, y_0).$$

If $\bar k = h\cdot\cos\bar\beta$, $\bar l = h\cdot\sin\bar\beta$, and $0 < \beta^* < \pi$, $\beta^* \neq \pi/2$, then the regularity condition (3.4) is satisfied for $h \to 0$ and $\bar\beta \to \beta^*$. Therefore

$$\begin{aligned}2\sin\bar\beta\cos\bar\beta F(x_0, y_0) = \lim_{\bar\beta\to\beta^*, h\to 0}\Biggl\{&\frac{2}{h}\left(\frac{f_0(\bar k, \bar l) - f_0}{h} - f_{\bar\beta}'\right)\\ &- \frac{2\cos^2\bar\beta}{\bar k}\left(\frac{f_0(\bar k, 0) - f_0}{\bar k} - f_0'(x_0, y_0)\right)\\ &- \frac{2\sin^2\beta}{\bar l}\left(\frac{f_0(0, \bar l) - f_0}{\bar l} - f_{\pi/2}'(x_0, y_0)\right)\Biggr\}.\end{aligned}$$

The last part of the terms in braces tends to $d_0'' f(x_0, y_0)\cos^2\beta^* + d_{\pi/2}\sin^2\beta^*$, respectively. Therefore the first part has a limit, which proves (4.7) for all β^* and (4.6) for $\beta^* \neq 0, \pi/2$. The proof fails for these two values because the regularity condition (3.4) does not hold.

However, we now apply our result to the coordinate system $\xi = x\cdot\sin\alpha/2 + y\cdot\cos\alpha/2$, $\eta = -x\cdot\cos\alpha/2 + y\cdot\sin\alpha/2$. The lines $y = \text{const.}$ and $x\cdot\sin\alpha + y\cdot\cos\alpha = \text{const.}$ have then the equations $\xi\cdot\sin\alpha/2 \pm \eta\cdot\cos\alpha/2 = \text{const.}$ By hypothesis the slopes of these lines have variation at most B; moreover, it was proved that $d'' f$ exists along the lines $\eta = \text{const.}$ As was pointed out in the beginning, the first part of the proof used only the existence a.e. of $d'' f$ along $y = \text{const.}$, hence it follows that (4.6) holds for all directions except those parallel to the ξ- or η-axis; (4.6) holds therefore in particular in the direction of the old x- and y-axes. This completes the proof of (4.5). A simple application of (4.6) and (4.7) yields

(4.8) COROLLARY. *Under the assumptions of* (4.5) *the function* $f(x, y)$ *has almost everywhere a second differential in the sense of Stolz, that is, the relation*

$$
(4.9)\qquad \begin{aligned} f(x_0 + h, y_0 + k) = f(x_0, y_0) + hf_0'(x, y) + kf_{\pi/2}' \\ + (h^2d_0''f + 2h\cdot k\cdot F + k^2d_{\pi/2}''f)/2 + o(h^2 + k^2)\end{aligned}
$$

holds for almost all x_0, y_0.

Relation (4.9) implies that

$$
\begin{aligned} F(x, y) = \lim_{h\to 0, k\to 0} \{f(x_0 + h, y_0 + k) + f(x_0, y_0) \\ - f(x_0 + h, y_0) - f(x_0, y_0 + k)\}\end{aligned}
$$

without any regularity condition. If f_0' and $f_{\pi/2}'$ exist in a neighborhood of (x_0, y_0) it follows that the mixed partial derivatives $\partial^2 f/\partial x\cdot\partial y$ and $\partial^2 f/\partial y\cdot\partial x$ exist at (x_0, y_0) and equal $F(x_0, y_0)$. A similar interpretation can be given to $F(x_0, y_0)$ in the general case, however this question is alien to the geometric purpose of the present paper.

The theorem of Euler on the curvatures of the normal sections of $z=f(x, y)$ at (x_0, y_0) follows from (4.2) and (4.7) alone.[19] In the present case both the theorems of Euler and of Meusnier follow immediately from (4.9). For it follows from II of §3 that the paratingens at (x_0, y_0) does not contain the normal to the tangent plane Π of $z=f(x, y)$ at (x_0, y_0). If $(\bar{x}, \bar{y}, \bar{z})$ are new coordinates with Π as $(\bar{x}, \bar{y})$-plane and (x_0, y_0) as origin, then $z=f(x, y)$ will in a neighborhood of $(0, 0)=(x_0, y_0)$ be represented in the form $\bar{z}=\bar{f}(\bar{x}, \bar{y})$. Then $o((x-x_0)^2+(y-y_0)^2)=o(\bar{x}^2+\bar{y}^2)$. The relation (4.9) is equivalent to the statement that $f(x, y)$ coincides, in a neighborhood of (x_0, y_0), except for $o((x-x_0)^2+(y-y_0)^2)$, with a paraboloid $a_1+a_2x+a_3y+a_4x^2+a_5xy+a_6y^2$. This same paraboloid written in the new coordinates will approximate $\bar{z}=f(\bar{x}, \bar{y})$ up to $o((x-x_0)^2+(y-y_0)^2)=o(\bar{x}^2+\bar{y}^2)$. In the new coordinates (4.9) reads as

$$
(4.9')\qquad \bar{f}(\bar{x}, \bar{y}) = (\bar{x}^2d_0''\bar{f} + 2\cdot\bar{x}\cdot\bar{y}\cdot\bar{F} + \bar{y}^2d_{\pi/2}''\bar{f})/2 + o(\bar{x}^2 + \bar{y}^2)
$$

from which it follows that

$$
(4.10)\qquad d_\beta''\bar{f}(0, 0) = \cos^2\beta d_0''\bar{f} + 2\cdot\cos\beta\sin\beta\bar{F} + \sin^2\beta d_{\pi/2}''\bar{f}.
$$

But now $d_\beta''f(0, 0)$ is the curvature of the section $\bar{x}\cdot\sin\beta+\bar{y}\cdot\cos\beta=0$ with $\bar{z}=f(\bar{x}, \bar{y})$, so that (4.10) is Euler's theorem.

Meusnier's theorem can easily be derived from (4.9′) in the following form (cf. Bouligand [**1**, p. 120]): Suppose that the surface $z=f(x, y)$ has a second differential in the sense of Stolz (cf. (4.9)) at $P=(x_0, y_0, f(x_0, y_0))$. *Then any two curves on the surface with the*

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[19] Compare Busemann-Feller [**1**, pp. 28–29].

same tangent and same osculating plane at P have there the same curvature; and the circles of curvature of all plane sections with a given tangent at P lie on a sphere.

We apply these results to the neighborhood $|x'|\leqq\eta$, $|y'|\leqq\eta$ of (a), (b), (c) of §3. Because of (b) the Lebesgue area of the surface $z'=f(x', y')$ over $|x'|\leqq\eta$, $|y'|\leqq\eta$ is finite; it equals the area of the corresponding piece $\bar{\Delta}$ of the original surface Φ, where $\bar{\Delta}$ is the closure of the open set Δ in the (μ, ν)-plane corresponding to $|x'|<\eta$, $|y'|<\eta$. In order to avoid the known difficulties regarding the additivity of area, observe that Φ is, according to Lindelöf's theorem, covered by a countable number of sets $\bar{\Delta}$. This yields the result of the introduction, which may be stated more precisely as follows:

(4.11) THEOREM. *If the surface Φ satisfies the hypotheses* I, II, III *of* §3, *then Φ is the union of a countable number of pieces $\bar{\Delta}$ of finite area, such that for almost all points q of $\bar{\Delta}$ the following statements hold simultaneously: Φ has a tangent plane Π at q. All curves on $\bar{\Delta}$ through q which have an osculating plane, but different from π, have a curvature at q; and these curvatures satisfy the theorems of Euler and Meusnier.*

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INTEGRAL DISTANCES

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In the present note we are going to prove the following result:

For any n we can find n points in the plane not all on a line such that their distances are all integral, but it is impossible to find infinitely many points with integral distances (not all on a line).[1]

PROOF. Consider the circle of diameter 1, $x^2+y^2=1/4$. Let $p_1, p_2, \cdots$ be the sequence of primes of the form $4k+1$. It is well known that

$$p_i^2 = a_i^2 + b_i^2, \qquad a_i \neq 0, \qquad b_i \neq 0,$$

is solvable. Consider the point (on the circle $x^2+y^2=1/4$) whose distance from $(-1/2, 0)$ is b_i/p_i. Denote this point by (x_i, y_i). Consider the sequence of points $(-1/2, 0)$, $(1/2, 0)$, (x_i, y_i), $i=1, 2, \cdots$. We shall show that any two distances are rational. Suppose this has been shown for all $i<j$. We then prove that the distance from (x_j, y_j) to (x_i, y_i) is rational. Consider the 4 concyclic points $(-1/2, 0)$, $(1/2, 0)$, (x_i, y_i), (x_j, y_j); 5 distances are clearly rational, and then by Ptolemy's theorem the distance from (x_i, y_i) to (x_j, y_j) is also rational. This completes the proof. Thus of course by enlarging the radius of the circle we can obtain n points with integral distances.

It is very likely that these points are dense in the circle $x^2+y^2=1/4$, but this we can not prove. It is easy to obtain a set which is dense on $x^2+y^2=1/4$ such that all the distances are rational. Consider the

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[1] Anning gave 24 points on a circle with integral distances. Amer. Math. Monthly vol. 22 (1915) p. 321. Recently several authors considered this question in the Mathematical Gazette.

point x_1 whose distance from $(-1/2, 0)$ is $3/5$; the distance from $(0, 1/2)$ is of course $4/5$. Denote $(-1/2, 0)$ by P_1, $(1/2, 0)$ by P_2, and let α be the angle $P_2P_1X_1$. α is known to be an irrational multiple of π. Let x_i be the point for which the angle $P_1P_2X_i$ equals $i\alpha$; the points X_i are known to be dense on the circle $X^2+y^2=1/2$, and all distances between x_i and x_j are rational because if $\sin \alpha$ and $\cos \alpha$ are rational, clearly $\sin i\alpha$ and $\cos i\alpha$ are also rational.

To give another configuration of n points with integral distances, let m^2 be an odd number with d divisors, and put

$$m^2 = x_i^2 - y_i^2.$$

This equation has clearly d solutions. Consider now the points

$$(m, 0), \qquad (0, y_i) \qquad\qquad i = 1, 2, \cdots .$$

It is immediate that all the distances are integral.

These configurations are all of very special nature. Several years ago Ulam asked whether it is possible to find a dense set in the plane such that all the distances are rational. We do not know the answer.

Now we prove that we cannot have infinitely many points $P_1, P_2, \cdots$ in the plane not all on a line with all the distances P_iP_j being integral.

First we show that no line L can contain infinitely many points $Q_1, Q_2, \cdots$. Let P be a point not on L, Q_i and Q_j two points very far away from P and very far from each other. Put $d(PQ_i)=a$, $d(Q_iQ_j)=b$, $d(PQ_j)=c$. ($d(A, B)$ denotes the distance from A to B.)

$$c \leqq a + b - 1. \tag{1}$$

Let Q_iR be perpendicular to PQ_j. We have

$$a < d(PR) + (d(Q_iR))^2/d(PR), \qquad b < d(Q_jR) + (d(Q_iR))^2/d(Q_jR).$$

Thus from (1)

$$(d(Q_iR))^2\left(\frac{1}{d(PR)} + \frac{1}{d(Q_jR)}\right) > 1$$

which is clearly false for a and b sufficiently large. ($d(Q_iR)$ is clearly less than the distance of P from L.) This completes the proof.

There clearly exists a direction P_1X such that in every angular neighborhood of P_1X there are infinitely many P_i.

Let P_2 be a point not on the line P_1X.

Denote the angle XP_1P_2 by α, $0<\alpha<\pi$. Evidently the P_i cannot form a bounded set. Let Q be one of the P_i sufficiently far away from

P_1, where the angle QP_1X equals ϵ (ϵ sufficiently small). Denote $d(P_1, P_2)=a$, $d(P_1, Q)=b$, $d(P_2, Q)=c$. We evidently have

$$c^2 = a^2 + b^2 - 2ab \cos (\alpha - \epsilon).$$

a, b, c all are integers. From this we shall show that if b and c are sufficiently large, ϵ sufficiently small, then

$$c = b - a \cos \alpha. \tag{2}$$

Put

$$c = b - a \cos \alpha + \delta, \qquad \delta > 0.$$

Then

$$(b - a \cos \alpha + \delta)^2 = b^2 - 2ab \cos \alpha + a^2 \cos^2 \alpha + 2\delta(b - a \cos \alpha) + \delta^2 > a^2 + b^2 - 2ab \cos (\alpha - \epsilon)$$

if b is sufficiently large and ϵ sufficiently small. Similarly we dispose of the case $\delta < 0$. Thus (2) is proved.

From (2) we have

$$a^2 + b^2 - 2ab \cos (\alpha - \epsilon) = b^2 - 2ab \cos \alpha + a^2 \cos^2 \alpha$$

or

$$\cos (\alpha - \epsilon) - \cos \alpha = \frac{a^2 \sin^2 \alpha}{2b}.$$

Thus we clearly obtain

$$\epsilon < c_1/b.$$

Thus clearly all the points Q_i have distance less than c_2 from the line P_1X. Let Q_1, Q_2, Q_3 be three such points not on a line, where $d(Q_iQ_j)$ are large. Let Q_1Q_3 be the largest side of the triangle $Q_1Q_2Q_3$. Let Q_2R be perpendicular to Q_1Q_3. We have as before

$$d(Q_1Q_3) \leqq d(Q_1, Q_2) + d(Q_2Q_3) - 1;$$

also

$$d(Q_1Q_2) - d(Q_1R) < \epsilon, \qquad d(Q_2Q_3) - d(Q_3R) < \epsilon$$

an evident contradiction; this completes the proof.

By a similar argument we can show that we cannot have infinitely many points in n-dimensional space not all on a line, with all the distances being integral.

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THE CONTROL OF AN ELASTIC FLUID

H. BATEMAN

1. INTRODUCTION

1.1. Introduction. Mathematicians should pause periodically in their own work and peruse the progress in astronomy, biology, chemistry, economics, engineering, and physics to see if recent advances in these fields suggest problems of mathematical interest. One reason why the Gibbs Lectureship was founded was, indeed, to facilitate a fruitful friendliness between mathematicians and other scientists.

The subject of control is now very important and promises to be so in the future. Much has been written about the control of the air, the control of ships, airplanes, balloons, bombs, gliders, robots and torpedoes. The regulation of rotation became important in the early days of the telescope and steam engine. The related problem of stability is important now for electric motors, marine engines, hydraulic turbines and the generating plants for the distribution of gas and electricity for there is generally an economical speed of operation. In radio telegraphy a certain speed may be needed in order to get a desired frequency.

Controls are necessary in the chemical industries and in mining. They are useful in entertainment and were much needed when arc lights were used for illumination. Fountains which begin to play automatically at sunset are used at exhibitions. Appold's home in London had many automatic devices to interest visitors.

The control of conditions under which observations are made is of great importance to the astronomer, the physicist and the aeronautical engineer. The designer of an engine plans to regulate the flow, pressure, temperature and composition of his working fluid so that the engine will run smoothly and economically.

The control of combustion may be important not only for economical reasons but also to avoid the production of smoke. On the other hand this production may be desirable sometimes when a smoke screen is needed. In such a case there should be flexibility of control. The subject of control is important also in refrigeration, air conditioning and the preparation of food. Great attention is being paid to human comfort. We are in an era of air conditioning on a large scale

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and this requires the solution of many problems of control. It is now understood throughout the land that the provision of the proper atmospheric conditions for the comfort of workmen and the performance of good work is even more important than the regulation of the supply of air and fuel to an engine. Precise weather is needed for precision work and for the manufacture of instruments of precision such as gauges. Proper air conditioning is needed for the production of quality fabrics. The proper temperature must be maintained when stained glass windows are being made. In small arms munition works where dry explosives are handled there is inevitably a certain amount of dust and for safety the amount must be regulated. A gas company must regulate the pressure of gas which it distributes and must also regulate the composition so that an escape of gas may be readily detected by the odour of the escaping gas. Controls are needed for the safety of miners and of workmen in many industries. In the purification of drinking water the rate of supply of chlorine must be regulated.

The subject of control is clearly an enormous one and it is well to bear in mind that advances made in one branch of the subject are sometimes useful in another. A recent aerodynamical torque transmitter for the regulation of a marine engine[1] is based on a principle used in the Remarex carbon dioxide recorder in which there are two pairs of vaned discs, one pair running in air and the other in the flue gas to be tested for CO_2 content. The torque transmitter is intended to prevent the marine engine from racing when, owing to the pitching of the ship, the propeller leaves the water.

Thus advances in marine and aeronautical engineering may depend on advances in chemical engineering. They may depend also on advances in electrical engineering. In wind tunnel research it is important to be able to regulate the velocity of the air moving through the tunnel and one way of doing this has been provided by the extensive work on amplidynes made by the General Electric Company. A large adjustable-speed wind tunnel drive based on the use of amplidynes is described in a paper by Clymer.[2]

In acoustical research it is often necessary to control the vibrations of air in a room. In reverberation work, for instance, the generator of sound may be required to produce a pure tone for a certain length of time. If a loud speaker is used and the drive is furnished by an electric current the frequency must be controlled. Sometimes a pure tone is obtained electrically with the aid of an electrical filter. The theory of acoustic and electric filters belongs to the larger subject of the control of vibrations which is important also as it is desirable to eliminate as far as possible the noise and unpleasant vibrations associated with the use of machinery. Much attention has been paid in recent years to the problem of the muffler, the damping of the torsional vibrations of crankshafts and the avoidance of dangerous oscillations in hydraulic transmission lines.

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[1] *Aerodynamic marine-engine governor*, Engineering vol. 157 (1944) pp. 447–448.

[2] C. C. Clymer, *Large adjustable-speed wind-tunnel drive*, Transactions of the American Institute of Electrical Engineers vol. 61 (1942) pp. 156–158. Many applications of the amplidyne in closed cycle controllers or regulating systems are described by F. E. Crever, *Fundamental principles of amplidyne applications*, ibid. vol. 62 (1943) pp. 603–606.

1.2. Quantities which it may be desirable to control.[3]

1. The temperature T.
2. The total density, ρ, or its reciprocal the specific volume.
3. The speed q or the component velocities u, v, w.
4. The mass flow per unit area q.
5. The pressure p.
6. The heat content or enthalpy H.
7. The rate of chemical action, evaporation or condensation.
8. The coefficient of heat transfer.

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[3] For the control of temperature reference may be made to E. Griffiths, *Thermostats and temperature-regulating instruments*, Griffin, London, 1943; Th. J. Rhodes, *Industrial instruments for measurement and control*, McGraw-Hill, New York and London, 1941; R. L. Weber, *Temperature measurement and control*, Blakiston, Philadelphia, 1941. For the control of various physical quantities see M. Jakob, P. Gmelin and J. Kronert, *Physikalische Kontrolle und Regulierung des Betriebes*, Part I, Leipzig, 1932, Lithoprint, Edwards Brothers, Ann Arbor, Mich., 1943. For matters relating to heat transfer and evaporation see W. H. McAdams, *Heat transmission*, McGraw-Hill, New York, 1942; also A. Fono and C. H. Fielding's papers in Engineering vol. 149 (1940) pp. 79–82. In the theory of hydrogen cooling a mathematical theory of the shaft sealing system is given by D. S. Snell, *The hydrogen-cooled turbine*, Transactions of the American Institute of Electrical Engineers vol. 59 (1940) pp. 35–50. For the recent work on detonation and the physics of flames reference may be made to the book of Bernard Lewis and G. v. Elbe, *Combustion, flames and explosion of gases*, Cambridge, University Press, 1938, and to their paper, *Stability and structure of burner flames*, Journal of Chemistry and Physics vol. 11 (1943) pp. 75–97. For problems of control in the chemical industry reference may be made to the article by H. Seiferheld, *Die Regeltechnik in der chemischen Grossindustrie*, Zeitschrift für Technische Physik, vol. 18 (1937) p. 409 and to a paper by M. Ruhemann, *Equilibrium of liquid and vapour in a rectifying pan*, Physica vol. 4 (1937) pp. 1157–1168, in which use is made of some equations given by F. Bošnjaković. For the electrical precipitation of particles and fumes in gases reference may be made to the work of F. G. Cottrell, *Electrical dust and fume precipitation*, Bulletin of the American Institute of Mining Engineers vol. 67 (1912) pp. 667–675, discussion, pp. 675–680.

9. The velocity of propagation of ignition, detonation or wave motion.

10. The increase or rate of increase of the entropy.

1.3. Types of motion which it may be desirable to produce or avoid.

1. Motions characterized by a certain degree of turbulence such as:

a. Laminar motion in which there is no turbulence.

b. Flow that seems steady but is really turbulent.

c. Tumultuous flow.

d. Swirling flow which is desirable for efficient mixing and good combustion.

e. Flow in which the stream breaks away from the boundary at a selected place.

f. A standard type of turbulence for comparable results in different wind tunnels.

g. Pulsating flow.

h. Flow in which there is a decided rotation about a fixed axis or a mean direction of motion.

2. Motions in which there is a specified relation between p and ρ (barotropic flow) or between ρq and q.

3. Motions accompanied by regular vibrations which may or may not be audible.

a. Regular vibrations may be desirable in experimental work or in some types of engine. Thus in the Kadency engine[4] the intake is timed to occur at the moment when the pressure in the cylinder has fallen below the atmospheric pressure. In the Constantinesco patents regular vibrations in a fluid are used for various types of control in which accurate timing is essential.

b. Both regular and irregular vibrations may be undesirable on account of the noise they produce or because they make a flame unsteady and lead to its extinction or to flash back.

1.4. Methods of control. As the subject of control belongs largely to chemical and electrical engineering only a brief outline of methods can be given here and these will be restricted largely to cases of aerodynamical interest. A few references are given to books from which the reader can obtain information on chemical and electrical methods.

Some of the most important methods of control are:

1. Clever design of fixed boundaries so as to produce desired results with little attention. Thus diffusors, guide vanes and honeycombs may produce the desired type of flow in a wind channel. A

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[4] S. J. Davies, *An analysis of certain characteristics of a Kadency engine*, Engineering vol. 149 (1940) pp. 515–517, 557–559, 617–620.

spoiler may prevent the flow of air over a roof from injuring the roof. Stationary parts of an engine may be designed to produce a type of flow which is compatible with high efficiency. The wings, body and control surfaces of an airplane may be designed so as to provide low drag and good maneuverability.

2. Devices for altering the form of the boundary of a fluid. Of these the valve is the most important and the design of a suitable valve is often one of the chief steps in the development of a new invention. A list has been formed of nearly eighty different kinds of valves. A valve is generally a device for regulating the rate of flow of a fluid but it may also be used to regulate the pressure or composition of a gas in an enclosure. Mathematically it is usually considered in connection with the regulating device but there are some cases in which equations can be set up for the valve alone. In an attempt to elucidate the action of the throttle valve Joule and Thomson (Lord Kelvin) made their famous porous plug experiment which tests the accuracy of the thermodynamical assumption that the internal energy of a gas depends only on its temperature. Thomson's discussion of the experiment brought into prominence the idea of heat content or enthalpy. Valves may be regarded as including adjustable slots in wings[5] and devices for sucking air from the boundary layer or for blowing air into the boundary layer.[6] The ports of a bunsen burner are also valves. Another device for altering the form of the boundary of the fluid is the fan or blower which produces a forced draught. A modification of this is the windmill or air turbine. A steam turbine or turbine working with gas, mercury vapor or some combination of fluids is another modification.

3. Devices for altering the thermal or electrical condition of a fluid at a boundary. The supply of heat at a boundary or the absorption of heat at a boundary is a most effective way of controlling the motion of a fluid. There are also many electrical devices by means of which an aerodynamic or hydraulic system may be coupled with an electrical system that is furnished with some means of control which may or may not be automatic.

4. Devices for introducing solid particles or liquid in the form of a spray into the body of the working fluid. Gases in the form of jets may also be introduced as in the blowpipe torch and in furnaces. Overfire air jets have been found to be effective for smoke elimination

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[5] F. Handley Page, *The Handley-Page wing*, Aeronautical Journal vol. 25 (1921) p. 363.

[6] O. Schrenk, *Boundary layer removal by suction*, National Advisory Committee on Aeronautics, Technical Memorandum No. 974, 1941.

as in a recent paper by Engdahl and Holton.[7] Davis[8] has applied a theory of turbulent air jets developed by Tollmien to the problem of the furnace. The ignition of gaseous mixtures by hot moving particles has been studied by Silver[9] and Paterson.[10] The cooling of gases by sprayed water is one of the methods employed in air conditioning; it has the advantage that the humidity of the air may thereby be controlled at the same time. A history of air conditioning is given by W. H. Carrier.[11] There are cases in which a supply of heat may lead to large fluctuation in temperature. The stability of a simple thermal device which has been called an "academic oven" has been considered by Turner.[12] The analysis depends on a transcendental equation involving both exponential and trigonometrical functions. Oscillations in thermal regulators have been considered also by Himmler.[13]

The problem of stability and of the avoidance of large oscillations becomes important whenever the working fluid is coupled with a mechanical or electrical regulating device. The centrifugal governor invented by Huygens[14] as a possible means of regulating a clock was adapted for windmills and water wheels before it was used by James Watt for the steam engine. A theory of the governor of Huygens has been given by Poor,[15] the theory of Watt's governor and related devices has an extensive literature beginning, perhaps, with the work of

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[7] R. B. Engdahl, *Overfire air jets effective for smoke elimination*, Heating, Piping and Air Conditioning, September 1943, p. 481.

[8] R. F. Davis, *The mechanics of flame and air jets*, Engineering vol. 144 (1937) pp. 608–610, 667–668; Proceedings of the Institute of Mechanical Engineers vol. 137 (1938) pp. 11–72.

[9] R. S. Silver, *The ignition of gaseous mixtures by hot particles*, Philosophical Magazine (7) vol. 23 (1936) pp. 633–657.

[10] S. Paterson, *The ignition of inflammable gases by hot moving particles*, Philosophical Magazine (7) vol. 28 (1939) pp. 1–23.

[11] W. H. Carrier, *Air conditioning*, Encyclopedia Brittanica; see also W. H. Carrier, R. E. Cherne and W. A. Grant, *Modern air conditioning, heating and ventilation*, Chicago, Pitman, 1940; C. O. Mackey, *Air conditioning principles*, Scranton, Pa., 1941. Lord Kelvin is credited with a proposal for the use of mechanical cooling as a means of improving human comfort. William Appold devised apparatus for control of temperature and humidity. See J. P. Gassiot, *On Appold's apparatus for regulating temperature and keeping the air in a building at any desired degree of moisture*, Proc. Roy. Soc. London vol. 15 (1867) pp. 144–146.

[12] L. B. Turner, *Self-oscillation in a retroacting thermal conductor*, Proc. Cambridge Philos. Soc. vol. 32 (1936) pp. 663–675.

[13] C. Himmler, *Die Pendelungen bei warmetechnischen Regelvorgängen*, Zeitschrift für Technische Physik vol. 11 (1929) pp. 579–584.

[14] C. Huygens, *Horologii oscillatorii*, Part 5, Paris, 1673; *Horologium*, 1658.

[15] V. C. Poor, *The Huygens governor*, Amer. Math. Monthly vol. 32 (1925) pp. 115–121.

Navier and Poncelet.[16] For a survey of this literature reference may be made to the article of von Mises and to the book of Tolle.[17]

2. THE ALGEBRAIC PROBLEM

2.1. Conditions of stability. In direct regulation the stability of the dynamical, electrical or hydraulic regulating device often can be discussed by the method of small oscillations. The system is generally a compound one which is partly of one type and partly of another. For instance, in the case of the steam engine, the compound system consists of the steam, the valve and the centrifugal governor and so the processes taking place are described by a system of differential equations. In indirect regulation the system is again compound. When a deviation from the norm (or rated value of the quantity to be controlled) passes out of the region of insensitivity, the indicator actuates a motor through the amplifier and a disturbance is produced which tends to annul the disturbance shown by the indicator. When the transient force is no longer operative, the most desirable type of motion of the system is a damped oscillation or a simple decay without oscillation such as is sometimes produced when a jet is used to control the speed of rotation as in Michelson's[18] measurements of the velocity of light by means of a revolving mirror. In the so-called exact regulation the indicator returns to the normal setting after a single swing past it. It is generally, but not always, advantageous to eliminate all the variables but one which may be denoted by x, then, if $D \equiv d/dt$

$$(p_0 D^n + p_1 D^{n-1} + \cdots + p_n)x = f(t).$$

The transient function $f(t)$ may be different from zero only for $0 < t < T$. Then, for $t > T$, x is a sum of terms satisfying the equation with $f(t)$ replaced by zero but it is not certain that all possible solutions of the homogeneous equation enter into the expression for the particular quantity x.

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[16] J. V. Poncelet, *Cours de méchanique, appliquée aux machines*, Cours de l'école d'application de Metz, 1826. R. v. Mises, *Dynamische Probleme der Maschinenlehre*, Encyklopädie Mathematischen Wissenshaften vol. 4, part 10, pp. 153–355.

[17] M. Tolle, *Die Regelung von Kraftmaschinen*, 3d ed., Berlin, 1921. Tolle gives in particular a theory for the combination of a centrifugal governor and a relay. The theory is presented and amplified for the case of two relays and the effect of the steam by G. W. Higgs-Walker, *Some problems connected with steam turbine governing*, Proceedings of the Institute of Mechanical Engineers vol. 146 (1941) pp. 117–125.

[18] A. A. Michelson, *Measurements of the velocity of light between Mount Wilson and Mount San Antonio*, Astrophysical Journal vol. 65 (1927) pp. 1–14; A. A. Michelson, F. G. Pease and F. Pearson, ibid. vol. 82 (1935) pp. 26–61.

A regulator is said to be stable when, after a transient disturbance, the indicator returns to the region of insensitivity and does not go beyond this. In the case when the free motion involves an undamped or growing motion, the regulator is said to hunt or to be unstable. A sufficient condition for stability is that the roots of the algebraic equation

$$F(z) \equiv p_0 z^n + p_1 z^{n-1} + \cdots + p_n = 0 \tag{2.1}$$

should have only negative real parts. This condition may not be quite necessary because it may happen that a root with positive or zero real part does not happen to give a term in the expression for x. This possibility must be considered because something of an analogous nature seems to occur in some cases when the existence of a double root might make the sufficiency of the foregoing criterion seem doubtful.

The mathematical theory of stability based on the theory of small oscillations may be hard to use on account of lack of knowledge of the constants of the dynamical or electrical system. These can be estimated in many cases as in the theory of airplane stability but it is wise to have means of checking the results or of obtaining results when the computations are too difficult.

An oscillograph for the analysis of governor performance was built by J. E. Allen,[19] and the East Pittsburgh Research Laboratories have built an instrument for analyzing governor performance which satisfies the specifications that have been laid down. Instruments of this nature have been made elsewhere.[20] Dougill has devised an instrument for testing regulators in operation and has used it to test the governors in the gas works.[21]

2.2. Pseudo-negative roots. Liénard[22] calls a quantity pseudo-negative when its real part is negative. The criterion for pseudo-negative roots of an algebraic equation is a special case of the criterion that the roots of an algebraic equation should lie within a specified circle in the complex plane, a line being regarded as a degenerate circle. A line may also be transformed into a circle by means of a transformation

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[19] J. E. Allen, *Oscillograph analyses governor performance*, Power vol. 78 (1934) pp. 610–612.

[20] W. O. Oebon, *A turbine governor performance analyzer*, American Institute of Electrical Engineers vol. 69 (1941) pp. 963–967.

[21] G. Dougill, *Retort house and exhauster governing of gas works*, Engineering vol. 144 (1937) p. 697.

[22] A. Liénard, *Signe de la partie reélle des racines d'une équation algébrique*, J. Math. Pures Appl. (9) vol. 15 (1936) pp. 235–250.

$$Z = (Az + B)/(Cz + D)$$

which does not change the degree of the equation

Cauchy[23] devised a method for finding the criterion and Hermite[24] carried the analysis much further considering particularly the conditions that the roots should lie in the upper half of the complex plane. He used a symmetrical polynomial

$$H(z', z) = i[F(z')F_0(z) - F(z)F_0(z')]/(z - z')$$

associated with the function $F(z)$ and an associated function $F_0(z)$ derived from $F(z)$ by changing i into $-i$ in all the coefficients. The decomposition into squares of an associated quadratic form then indicated the number of pseudo-negative roots, this number being dependent in fact on the signature of the quadratic form.

At a meeting of the London Mathematical Society[25] in 1868, James Clerk Maxwell asked if any member present could point out a way of determining in what cases all the possible parts of the imaginary roots of an algebraic equation are negative. He said that in studying the motion of certain governors for regulating machinery he had found that the stability of the motion depended on this condition, which is easily obtained for a cubic, but becomes difficult in the higher degrees. W. K. Clifford said in reply that by forming an equation whose roots are the sums of the roots of the original equation taken in pairs and by determining the condition of the real roots of this equation being negative, we should obtain the condition required.

Routh[26] used Clifford's idea when formulating conditions for a quartic equation with real coefficients,

$$F(z) = az^4 + bz^3 + cz^2 + dz + e = 0. \tag{2.2}$$

He says: "Let us form that symmetrical function of the roots which is the product of the sums of the roots taken two and two. If this be called X/a^3, we find $X = bcd - ad^2 - eb^2$. Suppose we know the roots to be imaginary, say $\alpha \pm ip$, $\beta \pm iq$. Then

$$X/a^3 = 4\alpha\beta[(\alpha + \beta)^2 + (p + q)^2][(\alpha + \beta)^2 + (p - q)^2].$$

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[23] A. Cauchy, *Calcul des indices des fonctions*, J. École Polytech. vol. 15 (1837) pp. 176–229, *Oeuvres* (2), vol. 1, pp. 416–466.

[24] C. Hermite, *Extrait d'une lettre, Sur le nombre des racines d'une équation algébrique compris des limites données*, J. Reine Angew. Math. vol. 52 (1856) pp. 39–51.

[25] See the discussion of the paper by J. J. Walker, *On the anharmonic sextic*, Proc. London Math. Soc. (1) vol. 2 (1868) pp. 60–61.

[26] E. J. Routh, *Rigid dynamics*, vol. 2, 1897, pp. 192–193; *A treatise on the stability of motion*, London, 1877; *Advanced rigid dynamics*, 6th ed., 1907, pp. 256–307.

Thus, $\alpha\beta$ always takes the sign of X/a and $\alpha+\beta$ always takes the sign of $-b/a$. The signs of both α and β can therefore be determined; and if a, b, X have the same sign, the real parts of the roots are all negative." Routh also formed the equation $G(z)=0$ whose roots are the sums in pairs of the roots of $F(z)=0$ and in the case when the coefficients p_r in the general equation (2.1) are all real he came to the following conclusion:

In order that (2.1) may have all its roots pseudo-negative, it is necessary and sufficient that the equations $F(z)=0$, $G(z)=0$ should be complete with coefficients all of one sign. This means that no p should be zero and that if $p_0>0$ then $p_r>0$. If, moreover, the coefficients of $G(z)$ are q_r, $r=0, 1, \cdots, n^2/2-n/2$ then if $q_0>0$, we should also have $q_r>0$. These conditions give $n^2/2+n/2$ inequalities while the expected number of conditions is only n so the foregoing conditions are not all independent.

In his work on governors in which he considered particularly the governors designed by Foucault and Lord Kelvin, Maxwell found that the stability depended upon the conditions for pseudo-negative roots of an equation of the fifth degree $z^5+pz^4+qz^3+rz^2+sz+t=0$. He found the necessary conditions $pq>r$, $ps<t$ but could not prove that these conditions were sufficient.

The simplicity of these conditions suggested that there might be necessary and sufficient conditions in the general case which could be formulated in a simple form. The subject of the stability of motion was soon afterwards proposed as a subject for the Adams Prize at the University of Cambridge and the prize was won by E. J. Routh of Peterhouse. He made use of the methods of Cauchy and Charles Sturm and a set of test functions was formed by a cascade process. Writing

$$F(z) = E(z) + O(z) = A(z^2) + zB(z^2)$$

so as to resolve $F(z)$ into its even and odd parts, he made use of the functions $s_0(y)$, $s_1(y)$, $s_2(y)$, $\cdots$, $s_n(y)$, where

$$s_0(y) = p_0y^n - p_2y^{n-1} + \cdots, s_1(y) = p_1y^{n-1} - p_3y^{n-3} + \cdots,$$

$s_2(y)$ is the remainder with sign changed when use is made of the process for finding the G.C.M. of $s_0(y)$ and $s_1(y)$, $s_3(y)$ is derived from $s_1(y)$ and $s_2(y)$ in a similar way, and so on. It is then clear as in Sturm's theorem that when $s_r(y)=0$, $s_{r+1}(y)$ and $s_{r-1}(y)$ have opposite signs. If E denotes the excess of the number of changes of sign from $+$ to $-$ in $s_0(y)/s_1(y)$ over that from $-$ to $+$, then by Cauchy's theorem the whole number of radical points on the positive side of the axis of y is

$(n+E)/2$. If $E=-n$ the roots are all pseudo-negative. The roots of the equations $A(x)=0$, $B(x)=0$ are in this case all negative and occur alternately. In the case of the biquadratic

$$F(z) = z^4 + pz^3 + qz^2 + rz + s = (z^2 + x_1)(z^2 + x_2) + pz(z^2 + x_0)$$

$q=x_1+x_2$, $r=px_0$, $s=x_1x_2$ and Routh's test functions are

$$s,\ p,\ pq - r = p(x_1 + x_2 - x_0),\ r(pq - r) - p^2s = p^2(x_2 - x_0)(x_0 - x_1).$$

The third test function is positive when and only when x_0 lies between x_1 and x_2. The test functions p and $pq-r$ are both positive when $p>0$ and $x_1+x_2-x_0$ is positive. Now if x_1-x_0 is negative, x_2 must be positive. When s is positive, x_1 and x_2 must be either both positive or both negative, hence if x_2 is positive so also is x_1. Routh's criteria for pseudo-negative roots imply then that x_0, x_1, x_2 are all positive and that x_0 lies between x_1 and x_2. When these conditions are all satisfied s is positive, $r(pq-r)-p^2s$ is positive, $r(pq-r)$ and $q-r/p$ are positive. The ratio r/p of the two last quantities is positive and so q must be positive. If p is also positive r is positive. Routh's conditions are all satisfied and the equation $F(z)=0$ has pseudo-negative roots. Liénard attributes this converse theorem to E. Jouguet and says that Chipart has extended it to an equation of any degree. A general proof is given in Liénard's paper.

Related polynomials such as $A(-x)$, $B(-x)$ occur naturally in Rayleigh's theory of the driving point reaction in dynamics and in the theory of electric filters.

About 1893 the Swiss engineer Aurel Stodola[27] investigated the stability of regulating devices for turbines, particularly those used in hydroelectric plants. He referred to Thomson and Tait's *Natural philosophy*[28] for the relation between stability and pseudo-negative roots and on this account Corral[29] has called the question of pseudo-negative roots the problem of Lord Kelvin. Previously[30] he had followed Orlando[31] in calling it the problem of Hurwitz because

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[27] A. Stodola, *Über die Regulierung von Turbinen*, Schweizerische Bauzeitung vol. 22 (1893) pp. 113–117, 121–122, 126–128, 134–135; vol. 23 (1894) pp. 108–112, 115–117.

[28] W. Thomson (Lord Kelvin) and P. G. Tait, *Natural philosophy*, vol. 1, 1879, p. 39.

[29] J. J. Corral, *Nueva solucion del problema de Lord Kelvin sobre ecuaciones de coefficientes reales*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales de Madrid vol. 22 (1928) pp. 25–31.

[30] J. J. Corral, *Nuevos teoremas que resuelven el problema de Hurwitz*, Madrid, Imprenta Clasica Española, 1921.

[31] L. Orlando, *Sul problema di Hurwitz*, Rendiconti Accademia Lincei (5) vol. 19 (1910) pp. 801–805; Math. Ann. vol. 71 (1911) pp. 233–245.

Stodola's compatriot Adolf Hurwitz[32] had investigated the subject and had succeeded in obtaining criteria in the determinantal form

$$p_0 > 0, \qquad p_1 > 0, \qquad \begin{vmatrix} p_1 & p_0 \\ p_3 & p_2 \end{vmatrix} > 0, \qquad \begin{vmatrix} p_1 & p_0 & 0 \\ p_3 & p_2 & p_1 \\ p_5 & p_4 & p_3 \end{vmatrix} > 0,$$

$$\begin{vmatrix} p_1 & p_0 & 0 & 0 \\ p_3 & p_2 & p_1 & p_0 \\ p_5 & p_4 & p_3 & p_2 \\ p_7 & p_6 & p_5 & p_4 \end{vmatrix} > 0, \quad \text{and so on.}$$

The equivalence of the conditions of Routh and Hurwitz was shown by Bompiani,[33] and Orlando obtained a proof by induction of the necessity and sufficiency of Hurwitz's conditions.

The study of equations with complex coefficients is also useful as there are some cases in which the conditions of Routh and Hurwitz are not the simplest possible conditions for pseudo-negative roots. In Appell's *Mécanique rationnelle* a discussion is given of dynamical equations such as

$$x'' + p_1x' + q_1x - p_2y' - q_2y = 0,$$
$$y'' + p_1y' + q_1y + p_2x' - q_2x = 0,$$

in which there are gyrostatic terms. These equations are essentially those considered by Sir Horace Lamb in his work on kinetic stability[34] and by E. Jouguet in his work on secular stability.[35] The algebraic equation obtained in the usual way is

$$(z^2 + p_1z + q_1)^2 + (p_2z + q_2)^2 = 0$$

but if we put $x+iy=Z$, as Lamb does, there is a single dynamical equation

$$Z'' + (p_1 + ip_2)Z' + q_1 + iq_2 = 0$$

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[32] A. Hurwitz, *Über die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Theilen besitzt,* Math. Ann. vol. 46 (1895) pp. 273–284; *Werke,* vol. 2, pp. 533–545.

[33] E. Bompiani, *Sulle condizioni sotto le quali un equazione a coefficienti reale ammette solo radici con parte reale negative,* Giornale di Matematica vol. 49 (1911) pp. 33–39.

[34] H. Lamb, *On kinetic stability,* Proc. Roy. Soc. London. Ser. A. vol. 80 (1908) pp. 168–177.

[35] E. Jouquet, *Sur la stabilité seculaire quand les forces positionnelles n'admettent pas de potentiel,* C. R. Acad. Sci. Paris vol. 207 (1938) pp. 267–270.

which gives rise to an algebraic equation

$$z^2 + (p_1 + ip_2)z + q_1 + iq_2 = (z + x_1 + iy_1)(z + x_2 + iy_2) = 0.$$

The conditions for pseudo-negative roots are now

$$x_1 + x_2 > 0 \quad \text{and} \quad x_1x_2[(x_1 + x_2)^2 + (y_1 - y_2)^2] > 0$$

while the corresponding conditions derived by considering the biquadratic equation are $x_1+x_2>0$ and

$$x_1x_2[(x_1 + x_2)^2 + (y_1 - y_2)^2][(x_1 + x_2)^2 + (y_1 + y_2)^2] > 0$$

and there is an extra factor in the expression used for the second criterion.

The conditions for the quadratic may be expressed in terms of the quantities I_1, I_2, I_3 which are invariant when the equation is changed into a new equation by a substitution of the form $z=Z+ia$, where a is real. If

$$(Z + ia)^2 + (p_1 + ip_2)(Z + ia) + q_1 + iq_2 \equiv Z^2 + (P_1 + iP_2)Z + Q_1 + iQ_2$$

then $P_1=p_1$, $P_2=p_2-2a$, $Q_1=q_1+ap_2-a^2$, $Q_2=q_2-ap_1$ and so there are 3 invariants

$$I_1 = P_1 = p_1, \qquad I_2 = Q_1 + P_2^2/4 = q_1 + p_2^2/4,$$
$$I_3 = Q_2 - P_1P_2/2 = q_2 - p_1p_2/2.$$

If, in particular, we choose a so that $P_2=0$, the equation takes the simple form

$$Z^2 + I_1Z + I_2 + iI_3 = 0.$$

If Z_1, Z_2 are the roots of this equation and if T_1, T_2 are the roots of the conjugate equation

$$T^2 + I_1T + I_2 - iI_3 = 0,$$

the equation whose roots are Z_1+T_1, Z_2+T_2, Z_1+T_2, Z_2+T_1 is

$$S^4 + 4I_1S^3 + (5I_1^2 + 4I_2)S^2 + (2I_1^3 + 8I_1I_2)S + 4I_1^2I_2 - 4I_3^2 = 0.$$

This is also the equation whose roots are z_1+t_1, z_2+t_2, z_1+t_2, z_2+t_1 where z_1, z_2 are the roots of the original equation and t_1, t_2 are the roots of its conjugate equation. It should be noticed that the terms in the equation for S involve the invariants and S only, moreover, by using two of these terms expressions

$$x_1 + x_2 = I_1, \qquad x_1x_2[(x_1 + x_2)^2 + (y_1 - y_2)^2] = 4(I_1^2I_2 - I_3^2)$$

are obtained for the quantities that furnish criteria for the roots to be pseudo-negative.

The quadratic equation may be reduced to a canonical form

$$\frac{1}{z + iw_1} + \frac{1}{z + iw_2} + \frac{1}{z + u + iv} = 0$$

where w_1, w_2, u and v are real quantities. The equation is then of stable type (with pseudo-negative roots) when $u > 0$ for the roots are those of the derived function of the cubic

$$(z + iw)(z + iw)(z + u + iv) = 0$$

and so by the theorem of Gauss[36] and Lucas[37] lie within the triangle formed by the points $z = -iw_1$, $z = -iw_2$, $z = -u - iv$ in the complex z-plane. The roots are in fact the foci of the ellipse which touches the sides of this triangle at its middle points.

The equations for determining u, v, w_1, w_2 are

$$3p_1 = 2u, \qquad 3p_2 = 2(v + w_1 + w_2), \qquad 3q_1 = -w_1w_2 - v(w_1 + w_2),$$
$$3q_2 = u(w_1 + w_2)$$

and so $w_1+w_2=2q_2/p_1$, $w_1w_2=-3q_1-(3p_1p_2q_2-4q_2^2)/p_1^2$. The quantities w_1, w_2 are thus the roots of the quadratic equation

$$p_1^2w^2 - 2q_2p_1w + 4q_2^2 - 3p_1p_2q_2 - 3p_1^2q_1 = 0$$

which has real roots when $p_1^2q_1+p_1p_2q_2-q_2^2>0$ or $I_1^2I_2-I_3^2>0$. When this condition is satisfied the sign of u is positive when $I_1>0$.

The extension of Clifford's method which was used for the quadratic may be applied also to the cubic

$$z^3 + (p_1 + ip_2)z^2 + (q_1 + iq_2)z + r_1 + ir_2 = 0.$$

The 5 invariants are $I_1=p_1$, $3I_2=p_2^2+3q_1$, $3I_3=3q_2-2p_1p_2$, $3I_4=9r_1-3p_1q_1+3p_2q_2-2p_1^2p_2^2$, $27I_5=27r_2-9p_2q_1-2p_2^3$. When a substitution $z=Z+ia$ is chosen so that in the new equation the coefficient of Z^2 is real, the new equation is

$$Z^3 + I_1Z^2 + (I_2 + iI_3)Z + (I_4 + I_1I_2)/3 + iI_5 = 0.$$

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[36] C. F. Gauss, *Oeurves*, vol. 3, 1886, p. 112; vol. 8, 1900, p. 32.

[37] F. Lucas, *Géometrie des polynômes*, J. École Polytech. vol. 29 (1879) pp. 1–33. See also M. Marden, *The location of the zeros of the derivative of a polynomial*, Amer. Math. Monthly vol. 42 (1935) pp. 277–286.

If the roots of this equation are Z_1, Z_2, Z_3 and those of the conjugate equation T_1, T_2, T_3 the equation for $S=Z+T$ may be readily found by elimination and its roots are the 9 quantities Z_1+T_1, Z_1+T_2, Z_1+T_3, Z_2+T_1, Z_2+T_2, Z_2+T_3, Z_3+T_1, Z_3+T_2, Z_3+T_3. The product of the roots of this equation is particularly interesting as it furnishes a quantity

$$K = J(I_1I_2 - J)^2 - 3JI_1I_3I_5 + I_1^2I_2I_3I_5 - JI_2I_3^2 - I_1^3I_5^2,$$
$$J = I_4 + I_1I_2,$$

which is positive when all the roots are pseudo-negative. The necessary and sufficient conditions for pseudo-negative roots are $I_1>0$, $K>0$, and $I>0$ where these quantities are such that when positive they imply that $X_1+X_2+X_3$, $X_1X_2X_3$ and $X_2X_3+X_3X_1+X_1X_2$ are all positive. To find the invariant I it is helpful to use the notation

$$U_1 = X_1 + iY_1, \quad U_2 = X_2 + iY_2, \quad U_3 = X_3 + iY_3, \quad V_1 = X_1 - iY_1,$$
$$V_2 = X_2 - iY_2, \quad V_3 = X_3 - iY_3, \quad H_{rs} = U_r + V_s.$$
$$\begin{aligned} I &= (H_{22}H_{33} + H_{33}H_{11} + H_{11}H_{22})(H_{23}H_{31} + H_{31}H_{12} \\ &\quad + H_{12}H_{23})(H_{32}H_{13} + H_{13}H_{21} + H_{21}H_{32}) \\ &= 4(X_2X_3 + X_3X_1 + X_1X_2)\left| H_{23}H_{31} + H_{31}H_{12} + H_{12}H_{23} \right|^2. \end{aligned}$$

The quantities U_1, U_2, U_3 are identical with $-Z_1$, $-Z_2$ and $-Z_3$; also

$$I = (W - W_1)(W - W_2)(W - W_3)$$

where

$$\begin{aligned} W_1 &= U_2U_3 + U_3U_1 + U_1U_2 + V_2V_3 + V_3V_1 + V_1V_2 \\ &\qquad + (U_1 + U_2 + U_3)(V_1 + V_2 + V_3), \\ W_1 &= U_1V_1 + U_2V_2 + U_3V_3, \qquad W_2 = U_2V_3 + U_3V_1 + U_1V_2, \\ W_3 &= U_3V_2 + U_1V_3 + U_2V_1. \end{aligned}$$

These quantities W_1, W_2, W_3 and a second set of quantities W_1', W_2', W_3' obtained by changing the cyclic order of V_1, V_2, V_3 to V_3, V_2, V_1 are the roots of a sextic equation

$$(W^3 - AW^2 + BW - C)^2 = d^2DD'$$

where D is the discriminant of the equation for U and D' is the discriminant of the equation for V. This form is indicated by the fact that when the equation for U_1, U_2, U_3 has equal roots or when the equation for V_1, V_2, V_3 has equal roots the two sets of three sums become the same. Also we have identically $W_1'+W_2'+W_3'=W_1+W_2$

$+W_3$, $W_2'W_3'+W_3'W_1'+W_1'W_2' = W_2W_3+W_3W_1+W_1W_2$.

To find d we put $W=0$ and note that $W_1W_2W_3-W_1'W_2'W_3' = 2d(DD')^{1/2}$. But

$$W_1W_2W_3-W_1'W_2'W_3'$$
$$=(U_2-U_3)(U_3-U_1)(U_1-U_2)(V_2-V_3)(V_3-V_1)(V_1-V_2)=(DD')^{1/2},$$

consequently $2d=1$. We also have the relation

$$\begin{aligned}2C=&\,W_1W_2W_3+W_1'W_2'W_3'\\ =&\,U_1U_2U_3(V_1^3+V_2^3+V_3^3)+V_1V_2V_3(U_1^3+U_2^3+U_3^3)+6U_1U_2U_3V_1V_2V_3\\ &+(U_2^2U_3+U_3^2U_2+U_3^2U_1+U_1^2U_3+U_1^2U_2+U_2^2U_1)(V_2^2V_3+V_3^2V_2\\ &+V_3^2V_1+V_1^2V_3+V_1^2V_2+V_2^2V_1)\\ =&\,R(P'^3-3P'Q'+3R')+R'(P^3-3PQ+3R)+6RR'\\ &+(PQ-3R)(P'Q'-3R')\end{aligned}$$

where P, Q, R are the coefficients in the equation for Z_1, Z_2, Z_3, P', Q', R' the coefficients in the equation for T_1, T_2, T_3.

Hence $2C=RP'^3+R'P^3-6P'Q'R-6PQR'+PQP'Q'+18RR'$. Also

$$\begin{aligned}B=&\,W_2W_3+W_3W_1+W_1W_2=(U_1^2+U_2^2+U_3^2)(V_2V_3+V_3V_1+V_1V_2)\\ &+(U_2U_3+U_3U_1+U_1U_2)(V_2V_3+V_3V_1+V_1V_2+V_1^2+V_2^2+V_3^2)\\ =&\,Q'(P^2-2Q)+Q(P'^2-Q')=P^2Q'+P'^2Q-3QQ'.\end{aligned}$$

Hence

$$\begin{aligned}(W-W_1)(W-W_2)(W-W_3)=&\,W^3-PP'W^2+(P^2Q'+P'^2Q-3QQ')W\\ &-(RP'^3+R'P^3-6P'Q'R-6PQR'+PQP'Q'+18RR')/2-(DD')^{1/2}/2.\end{aligned}$$

With the value $W=Q+Q'+PP'$ the expression for I is

$$\begin{aligned}I=&\,(Q+Q'+PP')^3-PP'(Q+Q'+PP')^2\\ &+(P^2Q'+P'^2Q-3QQ')(Q+Q'+PP')\\ &-(RP'^3+R'P^3-6P'Q'R-6PQR'+PQP'Q'+18RR')/2-(DD')^{1/2}/2.\end{aligned}$$

This is the expression obtained in a former paper.[38] It is well known that

$$D = P^2Q^2 - 4P^3R + 18PQR - 27R^2$$

while D' can be expressed in a similar way in terms of P', Q', R'. It should be mentioned that the conditions for pseudo-negative roots for the case of the cubic equation with complex coefficients should be derivable from the conditions given by P. Bohl[39] that the roots of a trinomial equation may have moduli less than p.

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[38] H. Bateman, *Stability of the parachute and helicopter*, National Advisory Committee for Aeronautics, Report No. 80, 1920.

2.3. The case of equal roots. In the solution of linear differential equations with constant coefficients a double root of the associated algebraic equation indicates the existence of secular terms such as $a \sin (mt)+bt \cos (mt)$ or $(A+Bt)e^{-kt}$ in the general solution. This fact was a kind of bogie in the theory of the small oscillations of a dynamical system because it was thought at one time that there was a kind of instability associated with the presence of repeated roots. In 1858 Weierstrass[40] completed the theory of normal coordinates and showed that in some cases at least secular terms do not occur in the final solution of the equations of motion. Further remarks of interest were made by Somoff,[41] Routh[42] and Stokes.[43] A review of the subject has been given recently by Melikov.[44]

Experience shows that it is better to work with the original system of differential equations than with the single equation obtained by eliminating all the variables but one. For instance, in the case of the well known system

$$a(x'' + k^2x) = b(y'' + k^2y) = c(z'' + k^2z) = bcx + cay + abz = s,$$

say, the algebraic equation for m in an exponential factor e^{mt} occurring in the solutions is $(m^2+k^2)^2(m^2+h^2)=0$ where $h^2=k^2-bc/a-ca/b-ab/c$. The equation for s is, however, $s''+h^2s=0$ and so secular terms do not appear when s is calculated first and x, y, z derived from s.

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[39] P. Bohl, *Zur Theorie der trinomischen Gleichungen*, Math. Ann. vol. 56 (1908) pp. 556–569.

[40] K. Weierstrass, *Über ein die homogenen Functionen zweiten Grades betreffendes Theorem, nebst Anwendung desselben auf die Theorie der kleinen Schwingungen*, Monatsberichte der Akademie der Wissenschaft zu Berlin, 1858, pp. 207–220; *Mathematische Werke*, vol. 1, Berlin, 1894, pp. 233–246. See also F. Purser, *Occurrences of equal roots in Lagrange's determinantal equation*, Report of the British Association for the Advancement of Science, 1878, pp. 463–464.

[41] K. Somoff, *Oscillations of systems of particles, algebraic problem*, Mémoires de l'Academie des Sciences de Saint Petersburg (Akademiia Nauk), no. 14, 1879, 30 pp. The related paper of C. Jordan, *Sur les oscillations infiniment petites des systèmes matériels*, C. R. Acad. Sci. Paris vol. 74 (1872) pp. 1395–1399, is reviewed unfavorably in Fortschritte der Mathematik vol. 4 (1872) pp. 471–472.

[42] E. J. Routh, *Rigid dynamics*, part 2, pp. 84, 190.

[43] G. G. Stokes, *Explanation of a dynamical paradox*, Messenger of Mathematics vol. 1 (1872) pp. 1–3; *Mathematical and physical papers*, vol. 4, pp. 334–335.

[44] K. V. Melikov, *Über das Theorem von Weierstrass und Routh*, Annals of the Institute of Mines, Leningrad, vol. 10 (1936) pp. 71–76.

The case of equal roots is connected of course with the phenomenon of resonance and there are many cases in practice in which a number of identical dynamical systems are coupled together particularly in the construction of acoustical, electrical and mechanical filters. Much depends on the nature of the coupling and even when equal roots do not occur in the final analysis there are interesting phenomena. The use of symmetrical arrangements is sometimes advantageous on account of the simplicity of the analysis.

In gas producing plants in which there is one exhaust regulator for two coke ovens there seems to be an idea that symmetry must be avoided on account of a possible interaction or resonance between two coke ovens which would make the regulator unstable. Thus Dougill[45] remarks: "The interaction which so often occurred when two retort houses of equal size were connected to a common main which led to one exhaust governor could be remedied by provision of a time lag, preferably in the exhaust governor."

The question may be raised whether the troubles encountered can really be attributed to the equality in size and an answer to this question cannot be given without a careful analysis of the precise setup. In the meantime, however, it may be of interest to examine some of the complications which arise when use is made of a time lag in dynamical or electrical systems.

3. THE TRANSCENDENTAL PROBLEM

3.1. Time lag in control systems. Long ago the delayed action of a regulating system was recognized as one of the primary causes of the hunting of governed engines.[46] The effect of time lag has consequently been studied by many investigators, particularly by D. R. Hartree, A. Porter, A. Callender, A. B. Stevenson,[47] H. König,[48] J. G. Ziegler and N. B. Nichols.[49] Time lag was also considered by H. L. Hazen[50] in his work on servo mechanisms and by N. Minorsky[51] in his study of control problems.

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[45] G. Dougill, *Retort house and exhauster governing of gas works*, Engineering vol. 144 (1937) p. 144.

[46] See for instance, J. Swinburne, *The hunting of governed engines*, Engineering vol. 58 (1894) p. 247; Practice, *The "hunting" of steam engine governors*, Engineering vol. 71 (1901) p. 216.

[47] A. Callender, D. R. Hartree and A. Porter, *Time-lag in a control system*, Trans. Roy. Soc. London Ser. A. vol. 235 (1936) pp. 415–444. D. R. Hartree, A. Porter, A. Callender and A. B. Stevenson, *Time-lag in a control system*. II, Proc. Roy. Soc. London Ser. A. vol. 161 (1937) pp. 460–476. A. Callender and A. B. Stevenson, Proceedings of the Society of the Chemical Industry (Chemical Engineering Group) vol. 18 (1936) p. 108. See also L. Nisolle, *Sur la stabilité des régulateurs à impulsions retardies ou amorties*, C. R. Acad. Sci. Paris vol. 211 (1940) pp. 762–765.

[48] H. König, *Periodische und aperiodische Schwingungen an empfindlichen Regelan ordnungen*, Zeitschrift für Technische Physik vol. 18 (1937) pp. 426–431. See also D. Stein, *Untersuchung der Stabilitatsbedingungen bei verzögerter Regelung*, Elektrische Nachrichten Technik vol. 20 (1943) pp. 205–213.

The effects of time lag have been considered usually by three distinct methods:

(1) By the use of Taylor's theorem and a neglect of small terms so that linear differential equations are obtained.

(2) By the use of differential difference equations or equations of mixed differences.

(3) By the use of integral equations of the Poisson-Volterra type.

The first method is explained in a general discussion of control problems by the editorial staff of The Engineer and by N. Minorsky who regards the differential equation as an asymptotic form and gives four different types. In a simple case a body is supposed to oscillate under the influence of a restoring force $R(t-k)$ proportional to the body's displacement at a previous instant and also under the influence of a damping depending partly on the body's instantaneous velocity and partly on its velocity at a previous time $t-h$. The equation of motion is supposed, indeed, to be of the type[52]

$$x'' + Qx' + Nf'(t - h) + Pf(t - k) = 0$$

where the time lags h and k are regarded as independent of t. In the approximate theory $f'(t-h)$ and $f(t-k)$ are replaced by $f'(t)-hf''(t)$ and $f(t)-kf'(t)+k^2f''(t)/2$ respectively and then x is used in place of $f(t)$. The resulting equation is

$$(1 - Nh + Pk^2/2)x'' + (Q + N - Pk)x' + Px = 0.$$

When $Q=N=0$ the lag in R gives a negative damping and so oscillation with increasing amplitude may be expected to occur. When $N=0$ and $Q\neq 0$ the negative damping may be overcome by positive damping depending on Q. When $N=0$ both N and Q may tend to overcome the negative damping but if h is large the coefficient of x'' may become negative and completely alter the character of the motion. Minorsky[53] indicates an asymptotic form in which the apparent restoring term is different from Px, but by the multiplication of the equation by a suitable factor this form may be reduced to the previous ones in which only the coefficients of x' and x'' are affected by the lags.

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[49] J. G. Ziegler and N. B. Nichols, *Process lags in automatic-control circuits*, Transactions of the American Society of Mechanical Engineers vol. 65 (1943) pp. 433–444.

[50] H. L. Hazen, *Servo mechanisms*, Journal of the Franklin Institute vol. 218 (1934) pp. 279–331, 543–580.

[51] N. Minorsky, *Control problems*, ibid. vol. 232 (1941) pp. 451–488.

[52] Editorial Staff, *The damping effect of time lag*, The Engineer vol. 163 (1937) p. 439.

[53] N. Minorsky, *Self-excited oscillations in dynamical systems possessing retarded actions*, Transactions of the American Society of Mechanical Engineers vol. 64 (1942) pp. A65–A71, discussion by H. Poritzky, pp. A195–A196.

The indications of the approximate theory need to be checked by exact analysis but they are such as to make it plain that the effects of time lag may be quite serious.

When there is a single time lag which is treated as constant some progress may be made with the aid of the known theory of linear equations of mixed differences. The particular equations discussed by Hartree and his collaborators are

$$u'(x) = f(x) + v(x) - cu(x),$$
$$-v'(x+1) = pu(x) + qu'(x) + ru''(x)$$

where p, q, r and c are real constants. When $f(x)=0$ the free motion is described by means of terms of type $u(x)=K \exp (kx)$ where k is determined by means of the transcendental equation

$$k(k+c) = e^{-k}(p + qk + rk^2).$$

This equation and some related equations are discussed chiefly by graphical methods but for the equation

$$z^2e^z + a_0z + a_1 = 0$$

an approximate solution $z=\log (a_0/b)+ib$, $b=(2n+1/2)\pi+b^{-1}[\log (a_0/b)-a_1/a_0]$, is given for the value of z for a high harmonic on the supposition that b is large compared with a_1/a_0 and the real part of z. This approximation may hold in some cases for the fundamental and if it does it indicates that if the fundamental is positively damped the higher harmonics are more strongly damped.

Transcendental equations of the form $e^z=$ rational function of z are of frequent occurrence. A simple equation of this type $z=a-ce^{-z}$ occurs in economics[54] in the work of Kalecki, Frisch, Holme, James and Belz. It is a generalization of an equation considered by Euler[55] in 1750.

In his discussion of control problems König avoids the assumption of a constant time-lag and obtains an integral equation of Poisson's type,

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[54] M. Kalecki, *A macrodynamic theory of business cycles*, Econometrica vol. 3 (1935) pp. 327–344. R. Frisch and H. Holme, ibid. pp. 225–239. R. W. James and M. H. Belz, ibid. vol. 4 (1936) pp. 157–160.

[55] L. Euler, *Investigatio curvarum quae evolutae sui similes producunt*, Akademiia Nauk vol. 12 (1750) pp. 3–52. See also M. Alle, *Ein beitrag zur Theorie der Evoluten*, Akademie der Wissenschaften Wien (IIa) vol. 113 (1904) pp. 53–70.

$$x(t) + m\int_0^t k(t-T)x(T)dT = x_0(t),$$

where $x(t)$ is the variable quantity to be regulated, $x_0(t)$ is the variation of this quantity when there is no control, m is the factor of amplification and $k(t)$ is a function of type $I'(t)$, where $I(t)$ is the influence function which electrical engineers call the *transfer function.* The function $k(t)$ is generally zero up to time t_0, it then rises gradually in value until it reaches a peak value and then remains practically constant from a time t_1 on. The graph of $k(t)$ generally has a peak but in a simple case worked out by König, $k(t)$ is constant for $t_0<t<t_1$ and is zero for other values of t. In a stable kind of regulation the free motion with $x_0(t)=0$ is damped. When undamped oscillations or growing oscillations can arise the system may be capable of spontaneous oscillation.

König seeks the condition that there may be a solution of type $x(t)=A$ exp $(iwt-ht)$ and obtains the conditions

$$\int k(u)e^{hu}\sin(wu)du = 0, \qquad \int k(u)e^{hu}\cos(wu)du = 0.$$

The limit of stability is then given by $h=0$. The motion due to a transitory disturbance may be found by Poisson's method of successive approximations[56] in which $x(t)$ is expanded in powers of m or it may be found by a method recommended by V. Pareto[57] and the present author[58] in which a relation is found between the generating functions

$$X(z) = \int_0^\infty e^{-zt}x(t)dt, \qquad K(z) = \int_0^\infty e^{-zt}k(t)dt,$$

$$X_0(z) = \int_0^\infty e^{-zt}x_0(t)dt.$$

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[56] S. D. Poisson, *Mémoire sur la théorie du magnétisme en mouvement,* Académie des Sciences, Paris, 1826, 130 pp. (pp. 28–30).

[57] V. Pareto, *Sur les fonctions génératrices d'Abel,* J. Reine Angew. Math. vol. 110 (1892) pp. 29–323.

[58] H. Bateman, *Report on the history and present state of the theory of integral equations,* British Association for the Advancement of Science, 1910, pp. 345–424 (p. 394); *An integral equation occurring in a mathematical theory of retail trade,* Messenger of Mathematics vol. 49 (1920) pp. 1–4.

3.2. Feedback. An early use of feedback to regulate a water clock has been ascribed to James Watt[59] who apparently used a pump to maintain the desired constant level in the reservoir from which the water flows. The rate of flow should be constant if the level of the water in a receiving vessel is to give a correct measure of time. A cascade system of reservoirs which kept this rate very nearly constant for a short time was adopted long ago in the design of a water clock of Canton, China, known as "Hon-woo-et-low" (copper jars dropping water). James Arthur[60] saw this clock in 1897 and was told that it had been in existence for over 3000 years, being known as the clock of the street arch.

A mathematical theory of a cascade system of reservoirs based upon the formula for the discharge of a weir was given by E. Maillet[61] about 1905. The system of differential equations is nonlinear but some interesting conclusions are drawn relating to the existence of a steady state and the manner in which it is approached. The problem of stability of the steady state is considered and some attention is given also to the case in which water is fed into the reservoirs from an outside source. When in addition feedback is introduced there are many mathematical problems to be solved. Maillet's analysis is of some mathematical interest as it led him to researches on almost periodic functions.

Feedback has been much used in recent years in systems employing vacuum tubes and amplifiers. In his description of stabilized feedback amplifiers H. S. Black[62] says: "By building an amplifier whose gain is deliberately made, say, 40 decibels higher than necessary (10000 fold excess on energy basis) and then feeding the output back on the input in such a way as to throw away the excess gain, it has been found possible to effect extraordinary improvement in constancy of amplification and freedom from nonlinearity."

In the simplified mathematical theory which has been developed by the electrical engineers[63] the effects of inertia or induction are neglected and the electrical system is supposed to be built up from simple delay elements and elements with a constant type of amplification over a limited range. The result of feedback is thus represented by an equation

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[59] See the discussion by Field of the paper by J. Woods, *Exhibition and description of the chronometric governor, invented by Messrs E. W. and C. W. Siemens*, Minutes and Proceedings of the Institute of Civil Engineers, London, vol. 5 (1846) pp. 255–265.

[60] James Arthur, *Time and its measurement*, Windsor, Chicago, 1909.

[61] E. Maillet, *Sur la vidage des systèmes de réservoirs*, C. R. Acad. Sci. Paris vol. 140 (1905) pp. 712–714; *Sur les équations différentielles et les systèmes de réservoirs*, ibid. vol. 147 (1908) pp. 966–968; *Sur les systèmes de réservoirs*, ibid. vol. 149 (1909) pp. 105–107. See also Bull. Soc. Math. France vol. 33 (1905) pp. 129–145; Annales des Ponts et Chaussées (1906) pp. 110–149; J. École Polytech. (2) vol. 13 (1909) pp. 27–56; J. Math. Pures Appl. (6) vol. 9 (1913) pp. 171–231.

[62] H. S. Black, *Stabilized feedback amplifiers*, Bell System Technical Journal vol. 13 (1934) pp. 1–18.

$$x = -Qx$$

where x denotes the amplification factor which is the product $m_1 m_2 \cdots m_n$ of a number of individual amplification factors and Q is the product of a number of differential operators of type

$$q = 1/(1 + \tau p), \qquad p = d/dt.$$

The differential equation for x is

$$[(1 + \tau_1 p)(1 + \tau_2 p) \cdots (1 + \tau_n p) + \mu]x = 0$$

and when a particular solution is of form $x = ae^{zt}$, z satisfies the algebraic equation obtained by replacing p by z. A graphical method of finding the condition for stability has been given by H. Nyquist.[64]

It is clear from an algebraic standpoint that there is only one condition because μ enters into only one of Hurwitz's determinants and the others are automatically positive on account of the time constants τ. For a given set of time constants the system will generally be stable when μ lies below a certain critical value μ_0 and unstable for $\mu \geqq \mu_0$. When $\mu = \mu_0$ there can be one or more oscillations with constant amplitude. In the graphical form of the criterion there is stability when the point $(-1, 0)$ lies outside a certain curve traced out by a radius vector representing the complex quantity μQ when p is replaced by $i\omega$. The graphical method has been discussed by others.[65] The differential equation for x can, of course, be replaced by an integral equation which is of the type considered by König or of a slightly more general type. In this connection it may be worth while to recall the investigations of P. Hertz[66] and G. Herglotz[67] on natural vibrations of an electron. The integral equation considered was then of the form

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[63] D. G. Prinz, *Contributions to the theory of automatic controllers and followers*, Journal of Scientific Instruments vol. 21 (1944) pp. 53–64.

[64] H. Nyquist, *Regeneration theory*, Bell System Technical Journal vol. 11 (1932) pp. 126–147; Annales des Postes, Télégraphes et Téléphones, Paris vol. 23 (1934) pp. 1010–1016. See also K. Kupfmüller, *Über die Dynamik der selbsttdgigen Verstärkungsregler*, Elektrische Nachrichten Technik vol. 5 (1928) pp. 459–467.

[65] R. Feiss, *Bestimmung der Regelungsstäbilitat an Hand des Vektorbildes*, Zeitschrift für der Verein Deutsches Ingenieures vol. 84 (1940) pp. 819–824. E. Peterson, J. G. Kreer and L. A. Ware, *Regeneration theory and experiments*, Proceedings of the Institute of Radio Engineers vol. 22 (1934) pp. 1191–1210, Bell System Technical Journal vol. 13 (1934) pp. 680–700. D. G. Reid, *Necessary conditions for stability (or self oscillation) of electrical circuits*, Wireless Engineers vol. 14 (1937) pp. 588–596. C. A. A. Wass, *Feedback amplifiers*, Nature vol. 150 (1942) pp. 381–382.

$$x(t) = \int_0^\gamma x(t - T)q(T)dT$$

and the problem was to find the complex roots of the equation

$$\int_0^\gamma e^{-pT}q(T)dT = 1.$$

The foregoing theory of feedback is based on linear differential equations with constant coefficients and is only approximate. Actually the resistances and capacities may vary with frequency and may even vary with time. In radio-telephony the voice acts so as to modify the resistance of the oscillatory circuit or the capacity of its condenser. J. R. Carson[68] proposed a differential equation with periodic coefficients as a basis of a theory of modulation and the theory has been worked out more fully by O. Emersleben,[69] W. L. Barrow,[70] and A. Erdélyi.[71] Conditions of stability are obtained with the aid of the theory of integral equations and of asymptotic solutions of linear differential equations. Feedback is not always desirable. In a discussion of receivers and transmitters for demonstrating frequency modulation M. Hobbs[72] says that in order to avoid acoustical feedback it is necessary to locate the signal generator and microphone in one studio and the receivers in another.

When the differential equations of the system are nonlinear the theory of stability or of sustained oscillations is more difficult but there is a large literature on the subject.[73] Nonlinear feedback oscillations have been discussed by G. Hakata and M. Abe.[74]

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[66] P. Hertz, *Die Bewegung eines Elektrons unter dem Einflusse einer stets gerichteten Kraft*, Math. Ann. vol. 56 (1908) pp. 1–86.

[67] G. Herglotz, *Über die Integralgleichungen der Elektronentheorie*, ibid. pp. 87–106.

[68] J. R. Carson, *Notes on the theory of modulation*, Proceedings of the Institute of Radio Engineers vol. 10 (1922) pp. 57–64.

[69] O. Emersleben, *Natural oscillation of circuits containing variable capacities and resistances*, Physikalische Zeitschrift vol. 22 (1921) pp. 393–400.

[70] W. L. Barrow, *Frequency modulation and the effects of a periodic capacity variation in a non-dissipative oscillatory circuit*, Proceedings of the Institute of Radio Engineers vol. 21 (1933) pp. 1182–1202.

[71] A. Erdélyi, *Über die freien Schwingungen in Kondensatorkreisen mit periodisch veranderlicher Kapazitat*, Annalen der Physik (5) vol. 19 (1934) pp. 585–622.

[72] M. Hobbs, *A low-power transmitter for demonstrating F-M receivers*, Electronics vol. 14 (1941) pp. 20–23.

3.3. Transcendental equations in the theory of integral equations. With the abbreviations

$$(xgy) = \int_0^1 \int_0^1 x(s)g(s,\, t)y(t)dsdt, \qquad (uv) = \int_0^1 u(s)v(s)$$

a brief study will be made of the linear integral equations

$$(1)\quad f(s) = \int_0^1 g(s,\, t)F(t)dt + \lambda \int_0^1 h(s,\, t)F(t)dt + \lambda^2 \int_0^1 k(s,\, t)F(t)dt,$$

$$(2)\quad f(s) = F(s) + \lambda \int_0^1 h(s,\, t)F(t)dt + \lambda^2 \int_0^1 k(s,\, t)F(t)dt.$$

When λ is a complex quantity $a+ib$ where a and b are real and $f(s)$ is regarded as independent of λ and real, the solution $F(t)$ will also be a complex quantity $u(t)+iv(t)$ with $u(t)$, $v(t)$ real provided the kernels $g(s,\, t)$, $h(s,\, t)$, $k(s,\, t)$ are real for real values of s and t which lie between 0 and 1. The combination $u(t)-iv(t)$ will be denoted by the symbol $F^*(t)$ and for both equations the properties of the function

$$w(\lambda) = \int_0^1 f(s)F(s)ds, \qquad w(\lambda^*) = \int_0^1 f(s)F^*(s)ds$$

will be studied. If c is a real constant the zeros and poles of the function $w(\lambda)-c$ will be pseudo-negative when the same is true for the zeros and poles of the function $w(\lambda^*)-c$.

In the important case in which $g(s,\, t)=g(t,\, s)$, $h(s,\, t)=h(t,\, s)$, $k(s,\, t)=k(t,\, s)$ it is readily seen that in the two cases

$$(1')\quad \begin{aligned} w(\lambda^*) &= (ugu)+(vgv)+a[(uhu)+(vhv)]+(a^2-b^2)[(uku)+(vkv)] \\ &\quad +ib[(uhu)+(vhv)]+2iab[(uku)+(vkv)], \end{aligned}$$

$$(2')\quad \begin{aligned} w(\lambda^*) &= (uu)+(vv)+a[(uhu)+(vhv)]+(a^2-b^2)[(uku)+(vkv)] \\ &\quad +ib[(uhu)+(vhv)]+2iab[(uku)+(vkv)]. \end{aligned}$$

The right-hand sides of these equations are zero when λ is such that $w(\lambda^*)=0$ and also when λ is such that $F(t)$ exists when $f(s)=0$. In the important case in which the functions h and k are of positive type the integrals (uhu), (vhv), (uku), (vkv) are all positive and so when the imaginary terms on the right are equated to zero it is seen that either a is negative or b is zero. When b is zero the equation obtained by equating the real part of the right-hand side to zero indicates that a is negative in case (2′) and this is true also in case (1′) if $g(s, t)$ is also of positive type. If c is negative it is readily seen that a must be negative when $w(\lambda^*) = c$.

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[73] K. Heegner, *The self-oscillating vacuum tube*, Arkiv för Elektrot. vol. 9 (1920) pp. 127–152.

[74] G. Hakata and M. Abe, *Non-linear differential feedback oscillations*, Nippon Electrical Communication Engineering no. 5 (1939) pp. 526–536.

When $k(s, t) = 0$ and $g(s, t)$, $h(s, t)$ are of positive type it is known that the zeros and poles of the function $w(\lambda)$ are all negative and occur alternately. The situation is analogous to that which occurs in the theorem of Routh, Jouguet and Chipart relating to the even and odd parts of an algebraic equation with pseudo-negative roots and so the function $w(\lambda)$ can be used quite often to construct a transcendental equation with only pseudo-negative roots. As an example of the first theorem we take equation (2) with

$$h(s, t) = pst, \quad k(s, t) = s(1 - t) \quad \text{or} \quad t(1 - s)$$

according as $s \lessgtr t$; the equation for λ is then

$$\coth \lambda = (1/\lambda) - (1/p)$$

when λ is a pole. When the equation is written in the form

$$\operatorname{ch} (\lambda) - (1/\lambda) \operatorname{sh} (\lambda) + (1/p) \operatorname{sh} (\lambda) = E(\lambda) + O(\lambda) = 0$$

it is seen that

$$E(z^{1/2}) = \operatorname{ch} (z^{1/2}) - z^{-1/2} \operatorname{sh} (z^{1/2}), \qquad z^{-1/2}O(z^{1/2}) = z^{-1/2} \operatorname{sh} (z^{1/2}).$$

It is readily seen that the functions on the right are transcendental functions of z with negative zeros which occur alternately.

In the second theorem if $h(s, t) = s(1-t)$ or $t(1-s)$ according as $s \lessgtr t$ it is found that if

$$C(z) = \int_0^1 \operatorname{ch} (zt) f(t) dt, \qquad S(z) = \int_0^1 \operatorname{sh} (zt) f(t) dt$$

then the equation

$$0 = \operatorname{sh} (z) + z \operatorname{sh} (z)(ff) - z \operatorname{sh}^2 (z) S(z) C(z) + \operatorname{sh} (z) \operatorname{ch} (z) [S(z)]^2$$

has only pseudo-negative roots.

4. THE SEPARATION OF VIBRATIONS

4.1. Acoustical filters. The early work of Poisson on the propagation of sound along a branched pipe was followed by the inventions of John Herschel and Quincke for the production of interference of waves by the rejunction of the divided branches of a pipe. The theory based on the idea of velocity potential and simplified boundary conditions was improved by Stewart and others and then replaced by a theory of lumped impedances so that the theory of acoustic filters could be developed along the same lines as the theories of mechanical and electrical filters. A good account of the theory from this standpoint is given in the book of Stewart and Lindsay.

The filtering action of a regularly spaced series of similar sheets of muslin was considered by Rayleigh[75] in the period 1887–1896 and is described in a passage inserted in the 1896 edition of his *Theory of sound.* He states that if a moderate number of such sheets be placed parallel to one another and at such distances apart that the partial reflections agree in phase, then a sensitive flame may be powerfully affected. With the aid of a device for adjusting the interval between two consecutive sheets it is easy to find how this interval depends on the wave length λ when the condition for effective reflexion is satisfied. Rayleigh states that with $a=\lambda/2$ the condition is satisfied for normal incidence but in the actual experiment it is more convenient to use oblique incidence and the calculations necessary for this case are readily made.

In his mathematical investigations[76] Rayleigh considered the transverse vibrations of a stretched string periodically loaded, but the analysis is rather difficult as it depends on the properties of the solutions of differential equations with periodic coefficients and use is made of infinite determinants as in the work of G. W. Hill. The vibrations of this type of string in which the density varies continuously have been studied further by Strutt[77] but more progress has been made in the study of the older problem in which the density of the string varies discontinuously. This case will be considered later for both transverse and longitudinal vibrations. The former case is interesting on account of analogies with optical phenomena, the latter on account of the analogies with acoustical phenomena.

The problem of the loaded string was much studied by the great mathematicians of the eighteenth century and many of their results are given by Routh in his *Rigid dynamics*. Routh also mentions that in April 1875 Lord Kelvin studied the vibrations and waves in a stretched uniform chain of symmetrical gyrostats connected together by universal flexure joints. His thoughts returned to this subject in his Baltimore lectures of 1884.

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[75] Lord Rayleigh, *Iridescent crystals*, Proceedings of the Royal Institute of London vol. 12 (1889) pp. 447–449; Nature vol. 40 (1889) pp. 227–228; *Scientific papers*, vol. 3, pp. 264–266 (see also pp. 1–14, 204–212); *Theory of sound*, vol. 2, p. 311.

[76] Lord Rayleigh, *On the maintenance of vibrations by forces of double frequency, and on the propagation of waves through a medium endowed with a periodic structure*, Philosophical Magazine vol. 24 (1887) pp. 145–159; *On the remarkable phenomenon of crystalline reflexion described by Stokes*, ibid. vol. 26 (1888) pp. 256–265.

[77] M. J. O. Strutt, *Eigenschwingungen einer Saite mit sinusformiger Massenverteilung*, Annalen der Physik vol. 85 (1928) pp. 129–136.

The theory of the loaded string became definitely associated with the theory of mechanical filters in 1898 when Godfrey and Lamb published their researches and when Campbell and Pupin became interested in the properties of the loaded electrical transmission line. Acoustic filters are much used as mufflers for internal combustion engines and as may be seen from the lists of patents in the Journal of the Acoustical Society of America baffles, holes and side branches in the exhaust pipe are among the devices used. Resonating side branches sometimes communicate with one another through partitions of absorbing material. Even a long pipe has a filtering action as the attentuation is higher for sound of some frequencies than for sound of some other ranges of frequency. Problems relating to pipes will be discussed in the section dealing with hydrodynamics in which some consideration will be devoted not only to the elimination of noise but also the reduction of dangerous vibrations in hydraulic pipe lines.

4.2. Passage of sound through a slab. Let ρ, v, and $Z=\rho v$ be the density velocity of sound and radiation resistance of a homogeneous slab of thickness a which is of infinite extent in any direction parallel to the plane faces. Let ρ, v', $Z'=\rho'v'$ be the corresponding quantities for the medium outside the slab. For normal incidence of waves on the face $x=0$ the velocity potentials are

$$\begin{aligned} \phi &= De^{iw(t-x/v')} + Ee^{iw(t+x/v')} && \text{for } x \leqq 0, \\ \phi &= Be^{iw(t-x/v)} + Ce^{iw(t+x/v)} && \text{for } 0 \leqq x \leqq a, \\ \phi &= Ae^{iw(t-x/v')} && \text{for } x \geqq a. \end{aligned}$$

The boundary conditions are

$$\begin{aligned} \rho'(D+E) &= \rho(B+C), & \rho(Be^{-is}+Ce^{is}) &= \rho'Ae^{-is'}, \\ is'(E-D) &= is(C-B), & is(Ce^{is}-Be^{-is}) &= -is'Ae^{-is'} \end{aligned}$$

where $s=wa/v$, $s'=wa/v'$. Thus

$$A/D = 2e^{is}/[2\cos(s) + i(Z/Z'+Z'/Z)\sin(s)]$$

and the coefficient of reflection is

$$r = (Z'/Z - Z/Z')[4 \cot^2 (s) + (Z'/Z - Z/Z')^2]^{-1/2}.$$

This is the formula of Lord Rayleigh.[78] Interesting applications of this formula to the reflection and transmission of sound through partitions have been made by Boyle[69] and Davis.[80] It is clear that $r=0$ when $\sin (s)=0$ and so there are certain critical thicknesses a for which there is no reflection of sound waves of the prescribed frequency $w/2\pi$. Davis regards the formula as applicable to the transmission of sound through light thin panels such as sheets of paper, sailcloth or fibre board. When a is very small there is a tendency for the reduction factor of the energy to vary as the square of the frequency f of the incident sound. With materials as light as paper a term due to air damping is important and there is less variation with f. For heavy panels such as two inch boards or brick walls the reduction factor is less than that given by Rayleigh's formula. Davis has given a formula

$$\text{Reduction factor} = (1/2R)^2[(r + 2R)^2 + (m/w)^2(w^2 - w_0^2)]$$

which indicates that resonances can account for a reduced insulating value. This formula is derived from a differential equation of type

$$m\ddot{\xi} + (r + 2R)\xi + S\xi = 2R\xi_0 e^{iwt}, \qquad w_0 = (S/m)^{1/2}.$$

To account for the behavior of actual panels it seems necessary to assume that there are several modes of vibration with which there can be resonance. This is in accordance with the general theory of the vibration of plates and with experiment as is pointed out by Davis and Littler.[81]

In the work of Boyle on the influence of the thickness of the plate on the transmission of sound, use was made of a high f(135000∼, 300000∼ and 528000∼). It was found that when a was a large multiple of $\lambda/4$ (where λ is the wave length in the plate) there was a maximum of energy reflected and a minimum of energy transmitted. When a was a few integral multiples of $\lambda/2$ there was an almost complete transmission of energy. Thus in the latter case the plate acted as a high pass filter and in the former case as a high frequency rejector. This work has been continued by Boyle and Sproule[82] with the aid of a torsion pendulum and the previous conclusion confirmed.

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[78] Lord Rayleigh, *Theory of sound*, vol. 2, 1896, p. 88.

[79] R. W. Boyle, *Transmission of sonic and ultrasonic waves through partitions*, Nature vol. 121 (1928) pp. 55–56. See also R. W. Boyle and D. K. Froman, Canadian Journal of Research vol. 1 (1929) pp. 405–424.

[80] A. H. Davis, *Transmission of sound through partitions*, Philospohical Magazine (7) vol. 15 (1933) pp. 309–316.

[81] A. H. Davis and T. S. Littler, *The measurement of transmission of sound by partitions of various materials*, Philosophical Magazine (7) vol. 3 (1927) pp. 177–194, vol. 7 (1929) pp. 1050–1062.

4.3. **Multiple partitions.** The passage of sound through several slabs of different materials is of some interest but the analysis is rather complicated. The case of three different media was considered by Brillié and his work is discussed by Stewart and Lindsay. The case of five media two pairs of which are alike in properties and are symmetrically related to a third medium in the middle is of much interest in relation to the sandwich type of radiator and receiver which was used at one time in underwater work. The natural vibrations of such a sandwich are of interest.

The vibrations of a column of gas, one portion of which is at a temperature T_1 and the other at a temperature T_2 have been studied by Lees[83] in connection with some experiments on the vibration of travelling flames made by Coward and Hartwell.[84] Lees considered 3 cases: 1°. Tube closed at both ends. 2°. Tube open at both ends. 3°. Tube closed at one end and open at the other. In the last named case the frequency of vibration is determined by the equation

$$(n_1/F_1) \tan (m_1\pi/2) = -(n_2/F_2) \tan [(m_2 - 1)(\pi/2)]$$

where n_1 is the fundamental frequency when the whole column is at temperature T_1, n_2 is the corresponding frequency when the temperature is T_2, F_1 and F_2 are the moduli of adiabatic elasticity for longitudinal displacements in the column at the two temperatures and

$$m_1 = na/n_1c, \qquad m_2 = nb/n_2c, \qquad a + b = c$$

where a, b are the lengths of the two portions of the tube.

Meyer observes that it is well known that the sound-damping action of a homogeneous wall increases with its thickness provided the exciting frequency is sufficiently higher than the natural frequency of vibration of the wall—a condition that is usually satisfied. When several such walls have air between them the composite wall acts as a mechanical damper because the layers of air form buffers if they are small compared with the wave length. If m denotes the mass of the wall in grams per square centimeter, if l is the length of an air buffer, if v is the velocity of sound and if ρ is the density of the air, the fundamental natural frequency is

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[82] R. W. Boyle and D. O. Sproule, *Transmission of sound energy and thickness of plate transmitter at normal incidence-ultrasonic method*, Canadian Journal of Research vol. 2 (1930) pp. 3–12.

[83] C. H. Lees, *Free periods of a composite elastic column or composite stretched wire*, Proceedings of the Physical Society of London vol. 41 (1929) pp. 204–213.

[84] J. F. Coward and F. J. Hartwell, *Extinction of methane flames by diluent gases*, Journal of the Chemical Society vol. 129 (1926) pp. 1522–1532.

$$f_0 = (v/\pi)(\rho l/m)^{1/2}.$$

This formula has been confirmed over a frequency range from 400∼ to 7000∼. Meyer[85] states that in such composite walls the velocity of transmission decreases as the frequency increases but the damping does not seem to increase very rapidly. Investigations led to the following rules for the design of composite walls. The dimensions should be such that f_0 is less than the practically important range of f's and the cross vibrations in the air buffers should be damped. By following these rules sound can be effectively damped without the use of heavy walls. A four-fold composite wall forty centimeters thick (15 3/4 inches) weighing 50 kilograms per square meter (10.25 pounds per square foot) gave better insulation than a solid brick wall weighing 1000 kg. per sq.m. The effect of composite walls has been discussed by many other writers[86] and the mathematical theory has been elucidated by Constable.[87] For $f>f_0$ insulation is at first decreased as the separation of the walls is increased, but it afterwards increases continuously up to a point at which the separation is approximately $\lambda/4$. After this point it decreases to a second minimum and thereafter minima occur at successive increases of $\lambda/2$ in the separation of the walls. The first minimum at which the insulation can be less than that of one component alone can be attributed to the effect of air coupling.

The properties of a double partition constructed from dissimilar components were examined by Renault[88] and by Constable.[87] The latter found that at f's for which the resonances of the components and the air coupling resonance can be neglected, the insulation to be obtained from a double partition for a given total weight and thickness is greatest when the components are similar.

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[85] E. Meyer, *Die Mehrfachwand als akustisch-mechanische Drosselkette*, Zeitschrift für Technische Physik vol. 16 (1936) pp. 565–566; *Über das Schallschluckvermögen schwingungsfähiger, nichtporöser Stoffe*, Elektrische Nachrichten Technik vol. 13 (1936) pp. 95–102.

[86] E. Wintergast, *Theorie der Schalldurchlassigkeit von einfachen und zusammengesetzten Wanden*, Schalltechnik vol. 4 (1931) pp. 85–91, vol. 5 (1932) pp. 1–8. J. E. R. Constable and G. H. Aston, *The sound insulation of single and complex partitions*, Philosophical Magazine (7) vol. 23 (1937) pp. 161–181. E. Lubcke and A. Eisenberg, *Zur Schallübertragung von dünnen Einfachwänden*, Zeitschrift Technische Physik vol. 18 (1937) pp. 170–174.

[87] J. E. R. Constable, *The acoustical insulation afforded by double partition constructed from similar components*, Philosophical Magazine (7) vol. 18 (1934) pp. 321–343; *Acoustical insulation afforded by double partitions constructed from dissimilar components*, ibid. vol. 26 (1938) pp. 253–259.

[88] L. Renault, *La transmission du son à travers cloisons métalliques*, Revue d'Acoustique vol. 6 (1937) pp. 69–101.

The transmission of sound by a series of equidistant similar partitions has been studied by Hurst[89] whose results are very similar to those of Lamb in his first example of a mechanical filter (§4.5). The method of indicating the regions of attenuation is also similar to that of Lamb. Hurst gives also a theory of transmission through a series of circular panels each of which is set into a rigid wall. In this theory effects of diffraction are taken into account.

4.4. **Results found by energy methods.** The absorption of sound by a porous wall was studied theoretically by Rayleigh with the aid of a theory of the propagation of sound in a capillary tube which will be examined in §6.[89a] There are, however, some results which can be obtained by energy methods which may be mentioned here for comparison with the other results.

4.4a. **The passage of energy through a single absorbing wall.** By means of an extension of the analysis of Buckingham and Eckhardt,[90] Davis[91] has obtained the following equations for two rooms separated by an absorbing wall:

$$4V\dot{I} + vaSI = 4E + vkWI, \qquad 4V_1\dot{I}_1 + va_1S_1I_1 = vkWI.$$

The first equation refers to the room containing the source of sound which is supposed to emit energy at a constant rate E while operative. W denotes the area of the wall used as a partition between the two rooms. V, V_1 are the volumes of the rooms; S, S_1 are the respective total areas of the exposed surfaces of the walls of these rooms (including the partition); a, a_1 are the mean fractions of incident energy lost by the respective rooms at each reflection by absorption or transmission to other rooms; v is the velocity of sound and k is a factor of type a for the partition. In the steady state $\dot{I}=\dot{I}_1=0$ so the maximum values of I and I_1 are J and J_1 respectively, where

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[89] D. G. Hurst, *The transmission of sound by a series of equidistant partitions*, Canadian Journal of Research vol. 12 (1935) pp. 398–407.

[89a] References to §§5 and 6 refer to material which the author hopes will appear after the war.

[90] E. A. Eckhardt, *The acoustics of rooms, reverberation*, Journal of the Franklin Institute vol. 195 (1923) pp. 799–814.

[91] A. H. Davis, *Reverberation equation for two adjacent rooms connected by an incompletely sound-proof partition*, Philosophical Magazine (6) vol. 50 (1925) pp. 75–80. See also C. F. Eyring, *Methods of calculating the average coefficient of sound absorption*, Journal of the Acoustical Society vol. 4 (1933) pp. 178–192.

$$J = \frac{4Ea_1S_1}{v(aSa_1S_1 - k^2W^2)}, \qquad J_1 = JkW/(a_1S_1).$$

At time t after the source has been cut off

$$\frac{I}{J} = \frac{\lambda_2 - \lambda_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_1 - \lambda_0}{\lambda_1 - \lambda_2} e^{-\lambda_2 t},$$

$$\frac{I_1}{J_1} = \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 t}$$

where

$$\left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\} = (v/8)\{N + N_1 \mp [(N_1 - N)^2 + 4k^2W^2/VV_1]^{1/2}\},$$

$$N = aS/V, \qquad N_1 = a_1S_1/V_1, \qquad \lambda_0 = 4V_1\lambda_1\lambda_2/(va_1S_1).$$

4.4b. **The passage of waves through a series of interceptors.** The reflection of light from a pile of plates was discussed in a restricted form by Augustin Fresnel[92] and Franz Neumann, the formulae of the latter being quoted by Wild in 1856. In 1862 Stokes[93] treated the problem in a general way by means of the functional equations

$$r(m + n) = r(m) + r(n)[t(m)]^2/[1 - r(m)r(n)],$$

$$t(m + n) = t(m)t(n)/[1 - r(m)r(n)]$$

in which $r(m)$ denotes the fraction of energy reflected from a pile of m plates, $t(m)$ denotes the fraction of energy transmitted and $a(m) = 1 - r(m) - t(m)$ is the fraction of energy absorbed in the plates. In the derivation of these equations the plates are supposed to be all formed of the same material and to be all of the same thickness. The plates themselves and the interposed sheets of air are supposed to be so thick that the phenomenon of the colours of thin plates does not occur to any appreciable extent. The analysis deals only with intensities of light or other wave motion.

If $r(1) = r$, $t(1) = t$ and quantities a, b are defined by the equations

$$r \operatorname{ch} a + t \operatorname{ch} b = 1, \qquad r \operatorname{sh} a - t \operatorname{sh} b = 0,$$

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[92] A. Fresnel, *Calculs sur les intensités de lumière réfléchi par une, deux et quatre glacés*, Oeuvres complètes, vol. 2, pp. 789–792.

[93] Sir George Stokes, *On the intensity of the light reflected from or transmitted through a pile of plates*, Proc. Roy. Soc. London vol. 11 (1862) pp. 545–556. J. Stirling, *Note on a functional equation treated by Sir George Stokes*, Proc. Roy. Soc. London Ser. A vol. 90 (1914) pp. 237–239.

an appropriate solution is

$$r(n) = \frac{\text{sh } (nb)}{\text{sh } (a + nb)}, \qquad t(n) = \frac{\text{sh } (a)}{\text{sh } (a + nb)}.$$

It should be noticed that

$$4 \text{ sh}^2 (a/2) = (r^{-1/2} - r^{1/2})^2 - t^2/r, \quad 4 \text{ sh}^2 (b/2) = (t^{-1/2} - t^2)^2 - r^2/t.$$

Since $1-r>t$ and $1-t>r$ the quantities a and b defined by these equations are both real. It should be noticed also that

$$t^2 = 1 - 2r \text{ ch } a + r^2, \qquad r^2 = 1 - 2t \text{ ch } b + t^2,$$
$$[t(n)]^2 = 1 - 2r(n) \text{ ch } a + [r(n)]^2,$$
$$[r(n)]^2 = 1 - 2t(n) \text{ ch } (nb) + [t(n)]^2.$$

Hence in order to investigate the effect of frequency on $r(n)$ and $t(n)$ it is useful to have plots in the rt-plane of the two systems of hyperbolas $a =$ constant, $b =$ constant.

4.5. **Mechanical filters.** A general theory of mechanical filters was given by Horace Lamb[94] and Charles Godfrey[95] in 1898. In the analysis of Lamb dynamical systems of any degree of complexity but all exactly alike are supposed to be interpolated at regular intervals along a line which is regarded for simplicity as an infinite string capable of longitudinal vibrations. The position and configuration of any one of these systems is imagined to be determined by the coordinate ξ of the point of the string where it is attached and by means of n other coordinates q_s $(s=1, 2, \cdots, n)$. The kinetic energy T and potential energy V are, moreover, assumed to have the forms

$$2T = \sum_{r=1}^{n} a_r\dot{q}_r^2 + 2\sum_{r=1}^{n} \alpha_r\dot{\xi}\dot{q}_r + P\dot{\xi}^2, \quad 2V = \sum_{r=1}^{n} b_r q_r^2 + 2\sum_{r=1}^{n} \beta\xi q_r + Q\xi^2,$$

so that the equations of motion are

$$a_r\ddot{q}_r + b_r q_r + \alpha_r\ddot{\xi} + \beta_r\xi = 0, \qquad r = 1, 2, \cdots, n,$$

$$P\ddot{\xi} + Q\xi + \sum_{r=1}^{n} (\alpha_r\ddot{q}_r + \beta_r q_r) = X$$

where X is the external force corresponding to the coordinate x representing in Lamb's model the difference of tensions on the two sides of the gap into which the dynamical system is introduced.

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[94] H. Lamb, *On waves in a medium having a periodic discontinuity of structure*, Memoirs of the Manchester Literary and Philosophical Society vol. 42, no. 3, 1898, 20 pp.

[95] C. Godfrey, *Discontinuities of wave-motion along a periodically loaded string*, Philosophical Magazine (5) vol. 45 (1898) pp. 356–363.

When all quantities contain the time factor exp $(ikct)$ the first n equations of motion give the relations

$$q_r = -\frac{k^2c^2\alpha_r - \beta_r}{k^2c^2a_r - b_r}, \qquad r = 1, 2, \cdots, n,$$

and the last equation gives

$$\left\{k^2c^2P - Q - \sum_{r=1}^{n} (k^2c^2\alpha_r - \beta_r)^2/(k^2c^2a_r - b_r)\right\}\xi = -X.$$

The successive dynamical systems may be distinguished by suffixes (D_r) and may be pictured as particles at intervals of length a. If $2\pi/k$ is the wave length for a disturbance with the same time factor on the unloaded string, the motion to the right of D_r is of type

$$\xi = \xi_r \cos(kx) + [\xi_{r+1} - \xi_r \cos(ka)] \sin(kx) \csc(ka)$$

while on the left it is

$$\xi = \xi_r \cos(kx) + [\xi_r \cos(ka) - \xi_{r-1}] \sin(kx) \csc(ka).$$

The tensions of the string on the two sides of D_r are

$$E(d\xi/dx)_+ = kE \csc(ka)[\xi_{r+1} - \xi_r \cos(ka)],$$
$$E(d\xi/dx)_- = kE \csc(ka)[\xi_r \cos(ka) - \xi_{r-1}],$$

so

$$X_r = kE \csc(ka)[\xi_{r+1} - 2\xi_r \cos(ka) + \xi_{r-1}]$$

and there is a difference equation

$$\xi_{r+1} - 2[\cos(ka) - f(k)\sin(ka)]\xi_r + \xi_{r-1} = 0$$

where

$$2kE\cdot f(k) = k^2c^2P - Q - \sum_{r=1}^{n} \frac{(k^2c^2\alpha_r - \beta_r)^2}{k^2c^2a_r - b_r}.$$

The solution of the difference equation has different forms according as the coefficient of $2\xi_r$ does or does not lie between the limits ± 1. The critical values of k are determined by

$$C(ka) \equiv \cos(ka) - f(k)\sin(ka) = \pm 1.$$

The roots of this equation give the ranges of frequency within which there is total reflection or partial transmission.

When $C(ka)$ lies between ± 1 there is a real angle θ such that

$$\cos\theta = C(ak)$$

and then the solution of the difference equation is

$$\xi_r = Pe^{ir\theta} + Qe^{-ir\theta}$$

where P and Q are suitable constants independent of r. The wavelength λ in the loaded medium may be defined to be

$$\lambda = (2\pi a/\theta) = (ka/\theta)\lambda_0$$

where $\lambda_0 = (2\pi/k)$ is the wave length in the unloaded medium. The wave velocity in the loaded medium may also be regarded as given by the equation

$$v = (\lambda kc/2\pi).$$

The group velocity is then

$$w = v - \lambda dv/d\lambda = -(\lambda^2 c/2\pi a)[d(ka)/d\theta](d\theta/d\lambda) = -[c/C'(ak)]\sin\theta.$$

Unless $C'(ak) = 0$ for the critical values of k the group velocity will be zero when k has a critical value and $\sin\theta = 0$. When θ lies between 0 and π the group velocity is positive when $C'(ak)$ is negative. The behavior of v may be found by Lamb's graphical method in which use is made of the curves

$$y = \cot(x/2), \qquad y = -\tan(x/2), \qquad y = f(x/a),$$

and the points of intersection of the last curve with the first two. With many forms of the function f the intervals of partial transmission are represented by intervals beginning respectively at $\pi, 2\pi, 3\pi, \cdots$, and the intervals of total reflection by intervals ending at these points. There may also be an interval of partial transmission between 0 and π.

In Lamb's first example $f(k) = \mu ka/2$, where $\mu = M/\rho a$. The dynamical system then consists of a mass M and ρa is the mass of the portion of the string between two consecutive masses. When the string is infinite in both directions there is transmission only for certain values of k below a certain limit. When ka is large θ is given approximately by $\theta^2 = k^2a^2(1+\mu)$ and the refractive index N is

$$N = (\lambda/\lambda_0) = (1+\mu)^{1/2}.$$

The effect of the loads is to increase the average density of the medium.

In Lamb's second example the mass M is urged towards its mean position by a spring and so

$$f(k) = \frac{\mu}{2}\left(ka - \frac{\sigma^2 a^2}{c^2 ka}\right).$$

The chief difference between this case and the last is that there is an interval of total reflection beginning at $ka=0$ which is separated by an interval of partial transmission from the interval of total reflection which ends at $ka=\pi$. Lamb regards this as remarkable.

In Lamb's third example $f(k)=(\mu ka/2)(1-k^2c^2/\sigma^2)$. The chief difference between this case and the first is that as the frequency increases the intervals of partial transmission become wider and wider instead of shorter and shorter so that the medium is transparent for short waves of high frequency. This example was offered by Lamb to illustrate the transparency of a medium to Röntgen rays, a phenomenon which Stokes[96] had endeavoured to explain in his Wilde lecture of 1897.

In Lamb's model there are a number of equal light rigid circular frames each of which is attached to the string at opposite ends of a diameter and has in its interior a particle M connected with the frame by means of similar springs so that the particle is at the center of the frame when the springs are equally extended. If ξ_s denotes the displacement of the point of the string to which one of the frames is attached and if η_s denotes the displacement of the associated particle, the equation of motion of this particle is

$$M\ddot{\eta}_s + M\sigma^2(\eta_s - \xi_s) = 0$$

while there is a balance of tensions if

$$\begin{aligned} 0 = kE[\xi_{s+1} - \xi_s \cos(ka)] - kE[\xi_s \cos(ka) - \xi_{s-1}] \\ + M\sigma^2(\eta_s - \xi_s)\csc(ka). \end{aligned}$$

These equations give the relation between ξ_{s+1}, ξ_s and ξ_{s-1} of the type mentioned.

Lamb's model furnishes a particular type of high pass filter.

4.6. **Selective reflection.** In Lamb's work the string is unloaded to the left of 0 and on the right has masses M at the points $x=0, a, \cdots, na$ and is unloaded beyond the point na. For $x<0$ it is assumed that

$$\xi = e^{ik(ct-x)} + Ae^{ik(ct+x)}.$$

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[96] Sir George Stokes, *On the nature of the Röntgen rays*, Memoirs of the Manchester Literary and Philosophical Society vol. 41, no. 15, 1897; *Mathematical and physical papers*, vol. 5, pp. 256–277.

For $0<x<na$ it is assumed that

$$\xi_r = Ce^{i(kct-r\theta)} + De^{i(kct+r\theta)}$$

and for $x>na$

$$\xi = Be^{ik(ct-x)}.$$

The kinematical conditions at $x=0$ and $x=na$ are

$$1 + A = C + D, \qquad Be^{-inka} = Ce^{-in\theta} + De^{in\theta}$$

while the dynamical equations are

$$\begin{aligned} 1 - A &= Ci \csc (ka)\left[\cos (ka) - e^{i\theta}\right] + iD \csc (ka)\left[\cos (ka) - e^{-i\theta}\right], \\ Be^{-inka} &= iCe^{-in} \csc (ka)\left[e^{-i\theta} - \cos (ka)\right] \\ &\quad + iDe^{in\theta} \csc (ka)\left[e^{i\theta} - \cos (ka)\right] \end{aligned}$$

and from these equations it is found that

$$B = \frac{\sin\theta \sin (ka) \exp \left[(n+1)ika\right]}{\sin\theta \sin (ka) \cos (n+1)\theta + i[1 - \cos\theta \cos (ka)] \sin (n+1)\theta},$$

$$A = \frac{i(\cos\theta - \cos (ka)) \sin (n+1)\theta e^{ika}}{\sin\theta \sin (ka) \cos (n+1)\theta + i[1 - \cos\theta \cos (ka)] \sin (n+1)\theta}.$$

The intensities of the reflected and transmitted waves are respectively

$$\begin{aligned} R &= \left[(\cos\theta - \cos (ka))^2 \sin^2 (n+1)\theta\right]/H, \\ T &= \left[\sin^2\theta \sin^2 (ka)\right]/H, \end{aligned}$$

where $H = \sin^2\theta \sin^2 (ka) \cos^2 (n+1)\theta + (1 - \cos\theta \cos (ka))^2 \sin^2 (n+1)\theta$. When n is large slight changes in ka and θ will make $\cos (n+1)\theta$ and $\sin (n+1)\theta$ vary considerably. Mean values of R and T for values of λ close to $2\pi/k$ may be found by integration as in Kirchhoff's *Optik* (p. 165) and in Lamb's paper by using the well known formula

$$\int_0^{\pi/2} \frac{d\phi}{\alpha^2 \cos^2\phi + \beta^2 \sin^2\phi} = \pi/2\alpha\beta.$$

Consequently

$$\overline{T} = \left|\frac{\sin\theta \sin (ka)}{1 - \cos\theta \cos (ka)}\right|, \qquad \overline{R} = 1 - \overline{T},$$

$$\overline{T} = \frac{1-q^2}{1+q^2}, \qquad \overline{R} = \frac{2q^2}{1+q^2}$$

where

$$q = \left| \frac{\sin (ka/2 - \theta/2)}{\sin (ka/2 + \theta/2)} \right|.$$

This quantity q is the coefficient of reflection obtained by Lamb in his analysis of the case in which waves travelling towards 0 along the unweighted portion of the string are reflected from the weighted portion on which weights evenly spaced extend to infinity. This case is similar to the problem considered by Godfrey.

An interesting variation of the present problem is one in which the unweighted portions of the string are replaced by weighted portions in which the weights are evenly spaced but at intervals differing in length from the intervals in the intermediate portion of the string. If the intervals in this intermediate portion are greater than in the rest of the string the problem is analogous to that of waves passing from a dense medium into a plate composed of a light substance. This problem is of interest for both longitudinal and transverse vibrations.

4.7. The reactance theorem and its generalizations. In 1886 the late Lord Rayleigh[97] gave a dynamical theorem which in modern terminology is a theorem relating to the driving point impedance in a chain like dynamical system. Applications to electrical networks were considered. In 1908 the theorem was extended by the present author[98] to a linear integral equation

$$u(s) = U(s) - \lambda \int_0^1 k(s, t) U(t) dt$$

in which $k(s, t)$ is a real symmetric kernel. If

$$w(\lambda) = \int_0^1 u(s) U(s) ds$$

the theorem then states that the zeros and poles of the function $w(\lambda)$ are all positive when the kernel is such that for any real nonvanishing function $x(s)$ the double integral

$$(xkx) = \int_0^1 \int_0^1 x(s) k(s, t) x(t) ds dt$$

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[97] Lord Rayleigh, *The reaction upon the driving-point of a system executing forced harmonic oscillations of various periods, with applications to electricity,* Philosophical Magazine (5) vol. 21 (1886) pp. 369–381; *Scientific papers,* vol. 2, pp. 475–485.

[98] H. Bateman, *The reality of the roots of certain transcendental equations occurring in the theory of integral equations,* Trans. Cambridge Philos. Soc. vol. 20 (1908) pp. 371–382; *Notes on integral equations* III, *the homogeneous integral equation of the first kind,* Messenger of Mathematics vol. 39 (1909) pp. 6–19.

is positive. The derivative $w'(\lambda)$ is then positive for all real values of λ, becoming infinite, if at all, only at certain values of λ at which the homogeneous integral equation (with $u=0$) can have a nonvanishing solution.

It is the analogue of this theorem for a system of linear algebraic equations

$$u_r = U_r - \lambda \sum_{s=1}^{n} k_{r,s} U_s, \qquad r = 1, 2, \cdots, n,$$

which is essentially the theorem given by Rayleigh. The real constants $k_{r,s}$ are then supposed to be symmetric ($k_{r,s}=k_{s,r}$) and such that the quadratic form

$$(xkx) = \sum_{r,s=1}^{n} x_r k_{r,s} x_s$$

is positive for all nonvanishing sets of real quantities x_r. The function $w(\lambda)$ is then defined by the equation

$$w(\lambda) = \sum_{r=1}^{n} u_r U_r$$

and its derivative is positive for all real values of λ. This indicates that the zeros and poles of the rational function $w(\lambda)$ occur alternately.

In 1922 G. A. Campbell[99] gave an expression for the driving-point impedance Z of a non-dissipative reactance network

$$Z = R + iX = iM \frac{f(f_2^2 - f^2) \cdots (f_{2n}^2 - f^2)^m}{(f_1^2 - f^2) \cdots (f_{2n-1}^2 - f^2)}$$

in which M is a positive constant, $f_2, f_4, \cdots$ are constants representing resonant frequencies, $f_1, f_3, \cdots$ are constants representing anti-resonant frequencies and the exponent m of the last factor in the numerator is 1 or 0 according as a resonant or anti-resonant frequency is the last member when the frequencies f_r are arranged in order of increasing magnitude

$$0 \leqq f_1 < f_2 < f_3 < f_4 \cdots .$$

This type of arrangement is possible because $dX/df > 0$ and so the zeros and poles of X occur alternately. The theorem may be derived from the previous theorem for a system of algebraic equations by taking all but one of the quantities u_r to be zero. There is a corresponding theorem for the more general set of linear equations

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[99] G. A. Campbell, *Physical theory of the electric wave-filter*, Bell System Technical Journal vol. 1 (1922) pp. 30–31.

$$u_r = \sum_{s=1}^{n} h_{r,s}U_s - \lambda \sum_{s=1}^{n} k_{r,s}U_s, \quad r = 1, 2, \cdots, n,$$

where the coefficients $h_{r,s}$, $k_{r,s}$ are real symmetric quantities such that the quadratic forms (xhx), (xkx) are positive for nonvanishing sets of quantities x_r. There is also a similar theorem for the integral equation of the first kind

$$u(s) = \int_0^1 h(s, t)U(t)dt - \lambda \int_0^1 k(s, t)U(t)dt$$

in which $h(s, t)$, $k(s, t)$ are real symmetric kernels such that (xhx) and (xkx) are positive for any nonvanishing real continuous function $x(z')$. In all these theorems the function $w(\lambda)$ increases continually with λ but if the conditions imposed on the function k or the coefficients $k_{r,s}$ are relaxed so that k is merely real and symmetric then the function $\lambda w(\lambda)$ increases with λ and there are zeros and poles of this function which occur alternately but are not all positive. The theorem that $w(\lambda)$ or $\lambda w(\lambda)$ increases with λ may be proved in many ways but one way is to use partial fractions. A number of theorems may be proved in this way, some of them are well known in the theory of algebraic equations and some have found useful applications in the theory of electric filters.

The reactance theorem has been proved by many writers. Zobel[100] gives a proof by induction while Foster[101] bases a proof on dynamical theorems of Routh, Rayleigh and Webster. Foster also calls attention to a connection with algebraic equations whose roots are all pseudonegative (that is, with negative real parts). In Routh's work, for instance, such an equation $F(z)=0$ is expressed in the form $F(z)=E(z)+O(z)=A(z^2)+zB(z^2)$ where $E(z)$ is an even function of z and $O(z)$ is an odd function of z and it is found that the roots of the equations $A(x)=0$, $B(x)=0$ are all negative and occur alternately. The converse of this theorem has already been considered in section 2.2.

Foster also considers the driving-point impedance of a general network and connects it with a theorem associated with three positive definite quadratic forms which may be generalized as follows:

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[100] O. J. Zobel, *Theory and design of uniform and composite electric wave-filters*, Bell System Technical Journal vol. 2 (1923) pp. 35–36.

[101] R. M. Foster, *A reactance theorem*, ibid. vol. 3 (1924) pp. 259–267; *Theorems regarding the driving-point impedance of two-mesh circuits*, ibid. vol. 3 (1924) pp. 651–685.

Consider the system of linear algebraic equations

$$u_r = \sum_{s=1}^{n} (g_{r,s} + \lambda h_{r,s} + \lambda^2 k_{r,s}) W_s, \quad r = 1, 2, \cdots, n,$$

in which the coefficients $g_{r,s}$, $h_{r,s}$, $k_{r,s}$ are real symmetrical quantities such that $g_{r,s} = g_{s,r}$, $h_{r,s} = h_{s,r}$, $k_{r,s} = k_{s,r}$ and that (xgx), (xhx), (xkx) are positive for any real nonvanishing set of quantities x. Then the zeros and poles of the function $w(\lambda) = \sum_{r=1}^{n} u_r W_r$ are all pseudo-negative.

When λ is a complex quantity $\sigma + i\tau$ and $W_r = U_r + iV_r$, where σ, τ, U_r and V_r are all real, the conjugate complex quantity to W_r, namely $W_r^* = U_r - iV_r$, arises when λ is replaced by $\lambda^* = \sigma - i\tau$ and the zeros of $w(\lambda)$ will be pseudo-negative when the zeros of

$$w^*(\lambda) = \sum_{r=1}^{n} u_r W_r^* = (W^*gW) + \lambda(W^*hW) + \lambda^2(W^*kW)$$

are all pseudo-negative. When, however, the right-hand side is resolved into its real and imaginary parts and each of these equated to zero it is seen at once that σ must be negative and a similar argument is applicable when $u = 0$ and the possible values of λ are, perhaps, poles of $w(\lambda)$. The theorem may be extended to the zeros and poles of the function $w(\lambda) + P$ where P is a positive constant and may also be extended to certain integral equations of type

$$u(s) = \int_0^c [g(s, t) + \lambda h(s, t) + \lambda^2 k(s, t)] W(t) dt$$

where $g(s, t)$, $h(s, t)$, $k(s, t)$ are real symmetric functions such that (xgx), (xhx), (xkx) are positive for any nonvanishing function $x(s)$ continuous in the range $0 \leqq s \leqq c$.

Proofs and extensions of the reactance theorem have been given by Cauer,[102] Baerwald,[103] Epheser and Glubrecht[104] and many other writers. There are also extensions to the case in which the quadratic forms (xgx), (xhx), (xkx) contain an infinite number of variables and to the case in which the range of integration 0 to c is infinite in the last mentioned theorem. Extensions to the case of Hermitian forms are also to be considered.

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[102] W. Cauer, *Die Verwirklichung von Wechselstrom Widerstanden vorgeschrieben Frequenzabhängigkeit*, Dissertation, Technische Hochschule, Berlin, 1926, Arkiv för Elektrot. vol. 17 (1926) pp. 355–388; *Ein Reaktanztheorem*, Sitzungsbericht der Akademie der Wissenschaften, Berlin, 1931, pp. 673–681; *Ein Satz über zwei zusammenhangen Hurwitzsche Polynome*, Sitzungbericht der Berlin Mathematische Gesellschaft vol. 27 (1928) pp. 25–31; *Wechstromschaltingen*, Akademie der Wissenschaften, Leipzig, 1941.

[103] H. G. Baerwald, *Ein einfacher Beweis der Reaktanztheorems*, Elektrische Nachrichten Technik vol. 7 (1930) pp. 331–332.

[104] H. Epheser and H. Glubrecht, *Die Grundlagen der Siebschaltungstheorien*, ibid. vol. 17 (1940) pp. 169–192. See also K. Franz, *Eine Verallgemeinerung des Fosterschen Reaktanztheorems auf beliebige Impidanzen*, Elektrische Nachrichten Technik vol. 20 (1943) pp. 113–115.

4.8. **The sorting of vibrations by means of filters.** It is often important to separate desirable vibrations from the undesirable ones particularly in electric and hydraulic transmission lines, but the problem of separation is important in acoustical and mechanical systems. In acoustics sound waves of a particular frequency or range of frequencies may be needed for experimental work as in the determination of times of reverberation in an auditorium. The elimination of noise is an important requirement in some buildings and laboratories. The absorption of the vibrations produced by engines and heavy machinery provides many mechanical problems.

The filtering properties of a system are generally examined by first finding the behavior of the system in a sinusoidal type of vibration, the so-called steady state, but it is important to know also the somewhat different response of the system to transient disturbances. This second problem is generally much harder than the first. Transient oscillations in electric wave filters were, however, studied by John Carson and Otto Zobel[105] in 1923. The building up of sinusoidal currents in loaded electric lines had been investigated by Carson[106] in 1917 and by Clark[107] and Kupfmüller[108] in 1923. Carson's analysis is based largely on the use of Fourier integrals. For the simpler problem of transients in mechanically loaded lines when use can be made of partial differential equations or of differential difference equations with constant coefficients there is a powerful method of influence functions which has been used effectively by Koppe, Havelock, Schrödinger, Pollaczek and the present writer.[109] Effective use can also be made of the inverse Laplace transformation as suggested by the writer in 1910.[110] In 1921 at the present writer's suggestion,[111] this method was adopted by Carson for work in electrical theory in which the so-called Duhamel integrals and operational methods of Heaviside were being used.

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[105] J. R. Carson and O. J. Zobel, *Transient oscillations in electric wave filters*, Bell System Technical Journal vol. 2 (1923) pp. 1–52.

[106] J. R. Carson, *Theory and calculation of variable electrical systems*, Physical Review (2) vol. 17 (1921) pp. 116–134; *General expansion theorem for the transient oscillations of a connected system*, ibid. vol. 10 (1917) pp. 217–225; *Theory of the transient oscillations of electrical networks and transmission systems*, Transactions of the American Institute of Electrical Engineers vol. 38 (1919) pp. 345–427, discussion by M. I. Pupin and A. H. Cowles, pp. 462–464. See also T. C. Fry, *The solution of circuit problems*, Physical Review (2) vol. 14 (1919) pp. 115–136; J. R. Carson, *The building up of sinusoidal currents in long periodically loaded lines*, Bell System Technical Journal vol. 3 (1924) pp. 558–566; F. Pollaczek, *Theory of the switching-on process of multimesh artificial lines*, Elektrishe Nachrichten Technik vol. 2 (1925) pp. 197–226.

[107] Alva B. Clark, *Telephone transmission over very long cable circuits*, Transactions of the American Institute of Electrical Engineers vol. 42 (1923) pp. 86–97, discussion by J. J. Pilliod.

[108] K. Küpfmüller, *Free oscillation, echo effect, and influence of temperature in telephony over long Pupinised cables*, Telegraphen- und Fernsprech-Technik vol. 12 (1923) pp. 53–60; *Über Beziehungen zwischen Frequenzcharakteristiken und Ausgleichs vorgängen in linearen Systemen*, Elektrische Nachrichten Technik vol. 5 (1928) pp. 18–32, 459.

4.9. Transverse vibrations of an infinite light string uniformly loaded at regular intervals. The advantages of the methods mentioned in the last section may be seen by considering the function

$$x_n(t) = \int_0^t J_{2n}[2c(t-s)]f(s)ds$$

in which the integrand of this Poisson-Duhamel integral involves the influence function for the infinite loaded line. When n is a positive or negative integer it is readily seen that $x_n(t)$ satisfies the homogeneous equation

$$\text{(A)} \qquad x_n''(t) = c^2[x_{n+1}(t) + x_{n-1}(t) - 2x_n(t)]$$

but when $n=0$ it satisfies the nonhomogeneous equation

$$x_0''(t) = f'(t) + c^2(x_1 + x_{-1} - 2x_0)$$

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[109] Some references to the literature are given in a paper by the author, *Some simple differential difference equations and the related functions*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 494–512.

[110] H. Bateman, *The solution of a system of differential equations occurring in the theory of radio-active transformations*, Proc. Cambridge Philos. Soc. vol. 15 (1910) pp. 423–427; *The solution of linear differential equations by means of definite integrals*, Trans. Cambridge Philos. Soc. vol. 21 (1909) pp. 171–196; *Report on the history and present state of the theory of integral equations*, Report of the British Association for the Advancement of Science, 1911, 80 pp.; *An integral equation occurring in a mathematical theory of retail trade*, Messenger of Mathematics vol. 49 (1919–1920) pp. 134–137.

[111] See a letter written by the author to J. R. Carson after the appearance of the latter's paper in Physical Review (1921). The method of the inverse Laplace transformation was recommended also by the author to M. D. Hersey in answer to a letter written during the last war regarding some equations occurring in the theory of recording instruments. The method has been taught by the writer for nearly forty years. Simeon Denis Poisson was probably the first man to use the method for the treatment of differential difference equations in his memoir *Sur la distribution de la Chaleur dans les corps solides*, J. École Polytech. (1) vol. 12 (1923) pp. 1–144, 249–403 (especially pp. 29–34).

and so the motion of the particle numbered 0 is forced while the motion of the other particles is free.

Following the lead of A. E. Heins[112] who has used the method of the inverse Laplace transformation for the classical equation we form (A) the function

$$X_n(z) = \int_0^\infty e^{-zt} x_n(t)dt$$

and a difference equation for $X_n(z)$ can be readily found. In the particular case when $f(t) = \sin(2at)$ the multiplication theorem for integrals of Laplace's type gives the relation

$$X_n(z) = 2a(z^2 + 4a^2)^{-1}(z^2 + 4c^2)^{-1/2}[T(z)]^{|2n|}$$

where

$$2cT(z) = (z^2 + 4c^2)^{1/2} - z.$$

From this expression for $X_n(z)$ the behavior of $x_n(t)$ for large positive values of t may be derived by the method of Mellin and Haar explained in my paper in this Bulletin. Except for a damped initial motion the motion is eventually the steady state of periodic oscillation given by

$$x_n(t) \sim (a^2 - c^2)^{-1/2}[e^{2iat}\{T(2ia)\}^{2n} \pm e^{-2iat}\{T(-2ia)\}^{2n}].$$

When $a > c$ the right-hand side is real when the $+$ sign is used, when $a < c$ the negative sign must be taken to make the right-hand side real. Another interesting example is that in which the pth particle differs in mass from all the others so that

$$\begin{aligned} x_n'' &= c^2(x_{n+1} + x_{n-1} - 2x_n), && n \neq p, \\ x_p'' &= a^2(x_{p+1} + x_{p-1} - 2x_p), && a \neq c,\ p > 0. \end{aligned}$$

A solution for the case in which $x_0(0) = 1$, $x_n(0) = 0$, $n \neq 0$, $x_n'(0) = 0$ is

$$x_n(t) = J_{2n}(2ct) + (1 - c^2/a^2)\int_0^t J_{2n-2p}[2c(t-s)]x_p'(s)ds$$

and

$$X_p(z) = \frac{\{T(z)\}^{|2p|}}{(z^2 + 4c^2)^{1/2} - (1 - c^2/a^2)z}.$$

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[112] A. E. Heins, *On the solution of linear difference differential equations*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 19 (1940) pp. 153–157.

The resulting estimate of $x_p(t)$ for large positive values of t is for $c^2 > 2a^2$

$$x_p(t) \sim \frac{a^2(c^2 - a^2)}{c^2(c^2 - 2a^2)} (1 - 2a^2/c^2)^{-|p|} \exp\left[- \frac{2a^2 t}{(c^2 - 2a^2)^{1/2}} \right].$$

When $c^2 > 2a^2$ the motion ultimately decays according to an exponential law. When $c^2 < 2a^2$ the function $X_p(z)$ is infinite for two imaginary values of z and $x_p(t)$ is ultimately periodic. When $c^2 = 2a^2$ the solution is simply

$$x_p(t) = (1/2ct) J_{|2p|+1}(2ct).$$

CALIFORNIA INSTITUTE OF TECHNOLOGY

BOOK REVIEWS

Mathematical statistics. By S. S. Wilks. Princeton University Press, 1943. 11+284 pp. $3.75.

The aim of this book is to present, for advanced undergraduate or for beginning graduate students, a summary of the mathematical theory of statistics developed during the twenty years prior to April, 1943, with a minimum of excursions into applied mathematical statistical problems. Aside from this book and the author's *Statistical inference* published in 1937, most of the theory covered still exists only in the original contributions widely scattered in scientific literature. A reference list of literature for supplementary reading, which is appended, includes these contributions as well as some text books and several articles more than twenty years old. It is of interest to note that the author himself is responsible for nine of the 123 references.

As a matter of fact, the material summarized is not limited to developments since 1922, since a good many accepted principles and formulas are necessary to lead up to the newer developments or serve as particular illustrations. For the purpose in hand, this method of treatment seems desirable, but it is worth noting that the point of view is not historical and the book does not attempt the difficult task of tracing in detail the growth of understanding of new principles.

Subjects covered are a discussion of distribution functions, including the Stieltjes integral, regression, partial and multiple correlation, the normal distribution, the Pearson system, and the Gram-Charlier series; sampling theory, sampling from a normal population, including a discussion of the χ^2 distribution, the "Student" t-distribution, and Snedecor's F-distribution; the theory of statistical estimation, including confidence intervals and regions, point estimation and maximum likelihood estimates, and tolerance interval estimation; tests of statistical hypotheses; normal regression theory and its application to the analysis of variance, including the case of incomplete layouts; combinatorial statistical theory, including some discussion of the theory of runs, matching theory, independence in contingency tables, and sampling inspection; and an introduction to multivariate statistical analysis.

As would be expected from the author and sponsorship, the book is an important addition to available statements of mathematical statistical principles. The exposition follows accepted standards of generality and precision. The mathematical theorist will find and enjoy the consistent development of advanced mathematical theory based for the most part on concise mathematical formulation of the hypotheses used. Material that does not lend itself to precise mathematical statement is avoided and the steps which are not pertinent to a straightforward mathematical development of the subject are thrown out.

These latter procedures will not appeal to the practicing statistician who must often use material of a less precise sort and support his arguments by a presentation of alternative points of view. The practicing statistician will also regret that the description of the Pearson and Gram-Charlier distribution functions does not include any judgment as to how well these curves work in actual practice, or how much they reflect physical law rather than merely provide equations, one of which will describe fairly closely nearly any particular case. It will also be regretted that no problems are included in the text as published, although it is indicated that when revised and issued in permanent form, certain problems will be inserted at the ends of sections and chapters.

The reviewer would like to suggest a shift in point of view in order to make the book more helpful to applied statisticians, especially those working in the social sciences. It would seem desirable for the final form of the book to go even further than the author promises in assigning a place of importance to problems. Principles, especially in the field of applied mathematics, get their orientation from the problems to which they apply and the historical, as well as the psychological, approach begins with problems. It might also be desirable to adopt a broader view as to what are the problems of the statistician in which mathematical methods may be helpful. In the first paragraph of the introduction, modern statistical methodology is divided into two broad classes: first, routine collection, tabulation, and description of large masses of data, and, second, making predictions or drawing inferences from a given sample of observations about a larger population of potential observations. It is suggested that a three-fold division into description, analysis, and testing of results would more nearly fit the daily work of many practicing statisticians, and that their ordinary processes of analysis and testing might be greatly improved by use of more of the type of statistical reasoning developed in this book if the connecting links between abstract theory and practice were properly forged.

It is conceded that any author has a right to prepare his book to fit the need as he sees it and as he believes he can fill it. The above suggestions are offered in order to explore the possibility of such adjustments in point of view as might enable the advancement of knowledge to proceed without half a dozen layers of books between the theoretical mathematical statistician and the run-of-the-mine working statistician.

As the book stands, it fits the more important findings in the field of advanced mathematical statistical theory into their proper places in a unified treatment providing uniform notation and terminology. Such a treatment will clearly prove of great value to college and university students, and also will facilitate important technical statistical work.

The reviewer wishes to acknowledge gratefully the assistance of Dr. Bradford F. Kimball of the New York State Public Service Commission. It is to be noted, however, that Dr. Kimball does not necessarily agree with all of the conclusions.

R. W. BURGESS

A treatise on the theory of Bessel functions. By G. N. Watson. 2d ed. Cambridge University Press; New York, Macmillan, 1944. 6+804 pp. $15.00.

The first edition of this book was published in 1922 and was reviewed by Professor R. D. Carmichael (Bull. Amer. Math. Soc. vol. 30 (1924) pp. 362–364). At that time the book constituted an almost encyclopedic account of results concerning Bessel functions which had been obtained by various mathematicians during the previous century. Furthermore, it was written in such a manner as to be comprehensible for those readers in possession of the fundamentals of modern analysis.

The present edition makes no attempt to take account of additions to the literature during the period since 1922, and the changes are for the most part limited to correction of minor errors. However, the book still remains a very comprehensive account of the bulk of the literature on the field in question. It will undoubtedly be of great service to workers in pure and applied mathematics who have occasion to make use of the various formulas and theories connected with Bessel functions.

There exists, however, one rather curious gap in the original edition which has not been filled in the current one. The original investigation by Fourier of a problem concerning the flow of heat in a cylinder, which led to the development of an arbitrary function in a series of Bessel functions of order zero, is outlined in §1.5. Reference is there made to Chapter 18 for a discussion of the validity of the expansion. But in order to show that the solution which Fourier gave for his problem is really valid, it is necessary to establish the uniform convergence or the uniform summability of the development in the neighborhood of the origin. In Professor Watson's book the uniform convergence of the development in this neighborhood after multiplication by the factor $x^{1/2}$ is established, but since this factor vanishes at the origin, this result throws no light on the uniform convergence of the development itself in the neighborhood in question.

As far as the reviewer is aware, the only proofs in the literature that Fourier's solution is mathematically valid are to be found in three papers published by him in 1909, 1911, and 1920 (Trans. Amer. Math. Soc. vol. 10 (1909) pp. 391–435; vol. 12 (1911) pp. 181–206; vol. 21 (1920) pp. 107–156). The second paper furnishes a proof based on uniform convergence; the first and third combined furnish a proof based on uniform summability. Professor Watson refers to these papers in his Chapter 18 but fails to point out that results of this sort are essential as a justification of Fourier's solution.

CHARLES N. MOORE

NOTES

Of those attending the recent celebration of the two hundred twentieth anniversary of the founding of the Russian Academy of Sciences were these American members of the Society: J. W. Alexander, Harlow Shapley, M. S. Vallarta.

A note has been received from Hans Freudenthal, privaat-docent at the Municipal University of Amsterdam, stating that he is well and that he has recently seen Professor J. F. Koksma of the Free University of Amsterdam, Professor D. van Dantzig of the Technical University of Delft, and some other mathematicians. He adds that Professor G. Schaake of the University of Leiden died some weeks ago, and that Professor Julius Wolff of the University of Utrecht, M. J. Belinfante of the University of Amsterdam, and Robert Remak, formerly of the University of Berlin and later a refugee in Holland, were deported to Germany or Poland.

The Case School of Applied Science held an all day symposium on mathematical analyzers and their applications in industry on June 6, 1945.

A symposium on mathematical statistics and probability was held at the University of California at Berkeley, August 13–18, 1945.

Associate Professor R. S. Burington of the Case School of Applied Science has received the Meritorious Civilian Award from the United States Navy.

Dr. J. W. Calkin of the Illinois Institute of Technology and Dr. Edwin Hewitt, Operations Analyst, Army Air Forces, have been awarded Guggenheim fellowships for postwar work in science.

Professor L. L. Dines of the Carnegie Institute of Technology has retired with the title professor emeritus.

Dr. D. G. Humm has received an honorary Doctor of Science degree from Bucknell University.

Professor Arthur Korn of the Stevens Institute of Technology has retired with the title professor emeritus.

Professor C. J. Rees of the University of Delaware has received a citation for his work in a civilian capacity with the 14th Air Force Headquarters.

Professor C. F. Thomas of the Case School of Applied Science has retired with the title professor emeritus.

Associate Professor Edith I. Atkin of Illinois State Normal University has retired.

Associate Professor E. F. Beckenbach of the University of Texas has been appointed to an associate professorship at the University of California at Los Angeles.

Associate Professor T. C. Benton of Pennsylvania State College has been promoted to a professorship.

Assistant Professor Herbert Busemann of the Illinois Institute of Technology has been appointed to an assistant professorship at Smith College.

Professor Theodosia T. Callaway of Stephens College has retired.

Associate Professor R. H. Cameron of the Massachusetts Institute of Technology has been appointed to a professorship at the University of Minnesota.

Professor A. R. Congdon of the University of Nebraska has retired.

Dr. Frances R. Dewing of Bennington College has retired.

Assistant Professor H. W. Eves of Syracuse University has been appointed to an assistant professorship at the College of Puget Sound, Tacoma, Washington.

Assistant Professor C. H. Fischer of the University of Michigan has been promoted to an associate professorship.

Assistant Professor A. L. Foster of the University of California at Berkeley has been promoted to an associate professorship.

Assistant Professor R. H. Fox of Syracuse University has been appointed to an assistant professorship at Princeton University.

Assistant Professor Beatrice L. Hagen of Pennsylvania State College has been promoted to an associate professorship.

Dr. J. L. Jones of the University of Akron has retired.

Assistant Professor J. L. Kelley of the University of Notre Dame has been appointed to an assistant professorship at the University of Chicago.

Professor C. G. P. Kuschke of the University of Puerto Rico has retired.

Professor J. H. McDonald of the University of California has retired.

Dr. Margaret P. Martin of Columbia University has been appointed to an assistant professorship in biostatistics at the University of Minnesota.

Professor L. E. Mehlenbacher of Arizona State Teachers College has been appointed to an associate professorship at the University of Detroit.

Dr. W. A. Mersman of the College of Agriculture of the University of California has been promoted to an assistant professorship.

Dr. R. v. Mises of Harvard University has been appointed to a professorship of aerodynamics and applied mathematics.

Associate Professor C. B. Morrey of the University of California has been promoted to a professorship.

Dr. W. K. Morrill of Johns Hopkins University has been promoted to an assistant professorship.

Professor C. C. Morris of Ohio State University has retired.

Miss Dorothy J. Morrow of Bryn Mawr College has been appointed to an assistant professorship at George Washington University.

Associate Professor C. W. Munshower of Colgate University has been promoted to a professorship.

Assistant Professor E. P. Northrop of the University of Chicago has been promoted to an associate professorship.

Dr. Helen B. Owens of Pennsylvania State College has been promoted to an assistant professorship.

Associate Professor H. H. Pixley of Wayne University has been appointed assistant dean of the College of Liberal Arts.

Assistant Professor Helen G. Russell of Wellesley College has been promoted to an associate professorship.

Assistant Professor K. C. Schraut of the University of Dayton has been promoted to an associate professorship.

Dr. R. H. Sorgenfrey of the University of California at Los Angeles has been promoted to an assistant professorship.

Dr. R. R. Stoll of Williams College has been promoted to an assistant professorship and has been granted a year's leave of absence to study at Indiana University as a National Research Fellow.

Dr. L. H. Swinford of the University of California at Berkeley has been promoted to an assistant professorship.

Dr. Alfred Tarski of the University of California at Berkeley has been promoted to an associate professorship.

Dr. A. R. Turquette of Cornell University has been appointed visiting assistant professor of philosophy at the University of Illinois.

Assistant Professor Alexander Weinstein of the University of Toronto has been promoted to an associate professorship.

Dr. Max Wyman of the University of Alberta has been promoted to an assistant professorship.

Professor Antoni Zygmund of Mt. Holyoke College has been appointed to a professorship at the University of Pennsylvania.

The following appointments to instructorships are announced: University of Arizona: Dr. H. S. Kieval; University of Illinois: Dr. J. S. Stubbe; University of Michigan: Dr. George Piranian; North Carolina State College: Mr. C. W. Williams; Rutgers University: Mr. E. P. Churchill, Dr. Katherine E. Hazard; Vassar College: Miss Winifred A. Asprey; Wellesley College: Miss Miriam C. Ayer; University of Wisconsin: Mr. R. V. Andree, Mr. R. A. Good.

Professor P. J. da Cunha of the University of Lisbon died on February 4, 1945.

Professor K. S. K. Iyengar of the University of Mysore, Bangalore, India, died June 23, 1944, at the age of forty-five years. He had been a member of the Society since 1935.

The death of Professor Stefan Mazurkiewicz in Warsaw in July, 1945 has been reported.

President Emeritus W. A. Bratton of Whitman College died in June, 1945. He had been a member of the Society for thirty-three years.

The death of Professor Emeritus H. C. Moreno of Stanford University has been announced. He had been a member of the Society for twenty-nine years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

145. A. T. Brauer: *A problem of additive number theory and its application in electrical engineering.*

A set S of non-negative integers is called a basis of order 2 with regard to addition for the integer n if each non-negative integer $t \leqq n$ is the sum of two elements of S. Denote by $k=k_a(n)$ the smallest number of elements in any such basis for n, and for given k by $n=n_a(k)$ the greatest number for which a basis of k elements exists. Rohrbach [Math. Zeit. vol. 42 (1937) pp. 1–30] proved Schur's conjecture that $k_a(n) = O(n^{1/2})$. Moreover he proved that $.4992\, k^2 > n_a(k) > k^2/4+11k/6-237/12$. Similarly a basis with regard to subtraction is defined. For a basis with regard to addition and subtraction Rohrbach proved that $n_{a,s} \geqq k^2/4+k/2-1$. In this paper, it is proved that for every k a basis with regard to subtraction can be determined for which $n_s \geqq k^2/4 + 7k/6-53/12$. For the construction of a special resistance, it was of interest to determine the exact value of $k_s(30)$. It is proved in this paper that $k_s(30)=10$. Similarly $k_s(n)$ may be determined for every given n. (Received July 19, 1945.)

146. F. L. Brown: *A simplification of the postulates of tri-operational algebra.*

The author retains the postulate $10=1$ (juxtaposition denoting substitution) which is independent of the other assumptions of tri-operational algebra (cf. Reports of a Mathematical Colloquium, nos. 5–6, p. 13) and introduces the postulate $0f=f$ for each f. Then the postulates concerning 0, 1, and $-f$ (that is, $0+f=f$, $1\cdot f=f$, $f+(-f)=0$ for each f) can be replaced by the weaker assumptions $0+j=j$, $1\cdot j=j$, and the existence of an element n such that $j+n=0$ where j is the neutral element with respect to substitution ($jf=fj=f$). The postulate $0f=0$ is independent of the other assumptions. While $0f$ is never equal to 1 or j, there does exist an algebra with five elements for one of which one has $0h=h\neq 0$. One may replace $0f=0$ by $1f=1$ in conjunction with $01=0$. (Received June 23, 1945.)

147. A. H. Copeland and Paul Erdös: *Note on normal numbers.*

D. G. Champernowne proved that if all of the positive integers are expressed in the base ten and arranged in order, the resulting sequence of digits when regarded as an infinite decimal constitutes a number which is normal in the sense of Borel. He conjectured that if the sequence of all integers were replaced by the sequence of primes, the corresponding decimal would be normal. In the present paper it is proved that a normal number can be produced in this manner by any sufficiently dense increasing sub-sequence of the positive integers. The validity of Champernowne's conjecture then follows from the fact that the primes are sufficiently dense. The same property holds for the sequence consisting of every integer which is the sum of two squares. These results are based on a concept of Besicovitch called (ϵ, k) normality. The paper concludes with two additional conjectures. (Received July 10, 1945.)

148. C. J. Everett: *Two representations for real numbers.*

Ideas on iterated functions due to B. H. Bissinger (*A generalization of continued fractions*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 868–876) are applied to *increasing* functions to obtain two quite different representations for real numbers with remarkable algebraic properties. (Received June 13, 1945.)

149. Bjarni Jónsson and Alfred Tarski: *On direct products of algebras.*

Consider algebras A closed under an operation $+$ and having a zero. A closed set $B \subseteq A$ with $0 \in B$ is a *subalgebra* of A. The notion of a (*weak*) *direct product* of two or arbitrarily many subalgebras, $B \times C$ or $\prod_{i \in I} B_i$, is assumed to be known. B is a *factor* of A, $B \in \mathfrak{F}_A$, if $A = B \times C$ for some C. The *center* A^c is the set of all elements $a \in A$ with (i) $a+(x+y)=(a+x)+y=x+(a+y)$ for any x, $y \in A$, (ii) $a+\bar{a}=0$ for some $\bar{a} \in A$. *Theorems*: I. If (i) $\mathfrak{F}_A$ is a Boolean algebra under set inclusion, then (ii) A has the *refinement property*: $A=\prod_{i \in I} B_i=\prod_{j \in J} C_j$ implies $B_i=\prod_{j \in J} D_{ij}$, $C_j=\prod_{i \in I} D_{ij}$ for some subalgebras D_{ij} and all i, j; hence, (iii) A has at most one representation $A=\prod_{i \in I} B_i$ where B_i's are indecomposable. Conversely, I(ii) implies I(i). I(i) holds in all A where $x \in A^c \rightarrow x=0$ (examples: centerless groups, semigroups with $x+y=0 \rightarrow x=y=0$, and lattices); also in algebras whose homomorphic images with two distinct elements are never commutative semigroups (examples: groups coinciding with their commutator groups, and A with a $u \in A$ such that $x \in A \rightarrow x+u=u$). II. If I(i) holds and the union of an increasing sequence of factors is always a factor of A, then A has just one representation I(iii); and conversely. II applies to all finite algebras satisfying I(i); also to algebras whose elements satisfy a finite chain condition. (Received July 27, 1945.)

150. Bjarni Jónsson and Alfred Tarski: *A generalization of Wedderburn's theorem.*

Using the terminology of the preceding abstract it is noted that A^c is always an Abelian group. *Theorem*: Let A be an algebra in which A^c is finite (or, more generally, in which every decreasing sequence of subgroups of A^c is finite, that is, where A^c is a torsion group of finite rank). Then (i) A has the *exchange property*: if $A=B \times C =\prod_{i \in I} D_i$, then $A=B \times \prod_{i \in I} D_i'$ where D_i' is a certain subalgebra of D_i for $i \in I$. Hence, (ii) A has the *weak refinement property*: if $A=\prod_{i \in I} B_i=\prod_{j \in J} C_j$, then, for some subalgebras B_{ij} and C_{ij}, $B_i=\prod_{j \in J} B_{ij}$, $C_j=\prod_{i \in I} C_{ij}$, and $B_{ij} \cong C_{ij}$ for $i \in I$ and $j \in J$. Consequently, (iii) if the factors B_i and C_j in (ii) are indecomposable, then they are pairwise isomorphic in a certain order. *Corollary*: Every finite algebra A (and, more generally, every algebra in which every decreasing sequence of congruence relations is finite) has, up to isomorphism, a unique representation as a direct product of finitely many indecomposable factors. (Received July 27, 1945.)

151. A. N. Milgram: *Cyclotomically saturated polynomials and tri-operational algebra.*

Let $f(x)$ be a polynomial whose coefficients are integers mod p where p is a prime number. Call $f(x)$ cyclotomically saturated if it has the property that for each irreducible polynomial $\phi(x)$, if $[\phi(x)]^\nu | f(x)$, then also $(x^{p^n}-x)^\nu | f(x)$ where n is the degree of $\phi(x)$. In tri-operational algebra (cf. Reports of a Mathematical Colloquium, nos. 5–6, p. 5) Menger raised the question: What polynomials with coefficients over the integers mod p have the property $f(x)|f(g(x))$ for each polynomial $g(x)$? The answer is: $f(x)|f(g(x))$ for each $g(x)$ if and only if $f(x)$ is cyclotomically saturated. (Received June 23, 1945.)

152. J. M. H. Olmsted: *Transfinite rationals.*

As suggested by the treatment of ratios by Eudoxus, two cardinal number pairs, (a, b) and (c, d), are defined to be equivalent if and only if for every pair of cardinal numbers, m and n, ma and nb have the same order relation as mc and nd. Addition, multiplication, division, and ordering are defined among the equivalence classes of cardinal number pairs, the resulting system being a lattice with familiar algebraic laws (for example, multiplication is distributive over addition, joins, and meets). This system is an extension of both the positive rational numbers and the cardinal numbers. Furthermore, it is the smallest extension subject to certain conditions. (Received June 4, 1945.)

153. Gordon Pall: *Hermitian quadratic forms in a quasi-field.*

Let F be a quasi-field, B_1 and B_2 nonsingular hermitian matrices of order $n-1$ in F, and let a be a nonzero scalar. Let there be given a transformation of $\bar{x}_0 a x_0 + \bar{x}' B_1 x$ into $\bar{x}_0 a x_0 + \bar{x}' B_2 x$. Then explicit transformations are constructed which replace B_1 by B_2. This is an extension of a similar result due to Witt for fields. (Received July 23, 1945.)

Analysis

154. E. F. Beckenbach: *On a characteristic property of linear functions.*

Let there be given a class of real functions $\{f(x)\}$, defined and continuous in a closed and bounded interval, such that there is a unique member of the family which, at arbitrary distinct x_1, x_2 in the interval, takes on arbitrary values y_1, y_2 respectively. The class of linear functions is an example. It is shown that a real function $g(x)$, defined and continuous in the interval, is a member of $\{f(x)\}$ if and only if for each x_0 interior to the interval there exists an $h_0 = h_0(x_0)$ with $x_0 \pm h_0$ in the interval such that the member of $\{f(x)\}$ coinciding with $g(x)$ at $x_0 \pm h_0$ coincides with $g(x)$ also at x_0. (Received June 21, 1945.)

155. Stefan Bergman: *Pseudo harmonic vectors and their properties.*

The author applies the operator $\mathfrak{P}(f, \mathfrak{Q}, \mathfrak{T})$ introduced in Bull. Amer. Math. Soc. (vol. 49 (1943) p. 164) to complex functions $f = s^{(1)}(x, y) + is^{(2)}(x, y)$, for which $s_y^{(1)} = s_x^{(2)}$, $s_x^{(1)} = -s_y^{(2)} l(x)$ holds. Here $s_x^{(1)} = (\partial s^{(1)}/\partial x)$, $\cdots$ and $l(x)$ is an analytic function of a real variable x. $\mathfrak{P}(s^{(1)} + is^{(2)}, \mathfrak{Q}, \mathfrak{T})$ yields a three-dimensional vector $\mathfrak{S}(X, Y, Z) = \mathfrak{S}^{(1)} + i\mathfrak{S}^{(2)} = \sum_{k=1}^{3} (S^{(k1)} + iS^{(k2)}) \mathfrak{i}_k$ for which curl $\mathfrak{S}^{(1)} = 0$, $S_X^{(11)}$

$+[S_Y^{(21)}+S_Z^{(31)}]l(x)=0$ and $S_X^{(22)}=l(X)S_Y^{(12)}$, $S_X^{(32)}=l(X)S_Z^{(12)}$, $S_Y^{(32)}=S_Z^{(22)}$, div $\mathfrak{S}^{(2)}=0$ holds. $S_X^{(11)}=(\partial S^{(11)}/\partial X), \cdots$, $d\Omega(\mathfrak{S})=[S^{(11)}dX-S^{(21)}dY-S^{(31)}dZ]+i[l(X)S^{(12)}dX-S^{(22)}dY-S^{(32)}dZ]$ is a complete differential. Let $f=f(x, y; \zeta)$, $\zeta\in\mathfrak{Q}$, be a family of functions whose components satisfy the above system of equations. For every f there exists a subdivision of XYZ space into parts A_k, such that when $\mathfrak{X}=(X_0Y_0Z_0)\in A_k$ the operator $\mathfrak{P}(f, \mathfrak{Q}, \mathfrak{X})$ yields another vector $\mathfrak{S}_k$. However the $\mathfrak{S}_k$ are related to each other. Let $\mathfrak{J}=\sum_{k=1}^{n}\mathfrak{J}_k$, $\mathfrak{J}_k=\mathfrak{J}\cap A_k$, be a closed curve in the XYZ space. Then under suitable assumptions $\sum_{k=1}^{n}\int d\Omega(\mathfrak{S}_k)=(2\pi i)^{-1}\int \text{Re}[f]d\zeta$, where the integral on the left is taken over $\mathfrak{J}_k$, the integral on the right over $\mathfrak{Q}$, and $\text{Re}[f]$ denotes the sum of the "residues" of $f=f(x, y; \zeta)$ in a certain domain. (Received June 6, 1945.)

156. Arnold Dresden: *An extension of the equation of Lagrange, $y=a+xf(y)$. Preliminary report.*

In this paper the equation $F(y)+xG(y)=0$ is considered. Under suitable conditions on the functions F and G, a solution $y=\phi(x)$ exists, which assumes the values $y_0=\phi(0)$, where y_0 is a fixed root of $F(y)=0$, and which possesses derivatives of all orders in the vicinity of $x=0$. The general form of $\phi^{(n)}(0)$ is found in terms of derivatives of the functions F and G. A final explicit formula is conjectured, but not completely proved. Applications to the solution of algebraic equations, in particular to trinomial equations (compare the papers of Mellin), are immediate. The method of attack is entirely elementary, it can be applied also to equations of the form $F(y)+x_1G_1(y)+\cdots+x_kG_k(y)=0$, in which $x_1, \cdots, x_k$ are independent parameters. (Received July 31, 1945.)

157: Evelyn Frank: *On the zeros of polynomials with complex coefficients.*

Let $P(z)=z^n+A_1z^{n-1}+A_2z^{n-2}+\cdots+A_n$, $A_j=p_j+iq_j$, and form $Q(z)=p_1z^{n-1}+iq_2z^{n-2}+p_3z^{n-3}+iq_4z^{n-4}+\cdots$. In general, Q/P has the continued fraction expansion $1/c_1z+b_1+1+1/c_2z+b_2+\cdots+1/c_nz+b_n$, where the c_j are real and the b_j are pure imaginary. If k of the c_j are positive and $(n-k)$ negative, then $P(z)$ has k zeros with negative real parts and $(n-k)$ with positive real parts. This extends a theorem of Hurwitz (*Werke*, vol. 2, p. 533 ff.) with a method analogous to that of Wall (Amer. Math. Monthly vol. 52 (1945) pp. 308–322) for polynomials with real coefficients. Simultaneously, determinants analogous to the Hurwitz determinants are obtained for $P(z)$ with complex coefficients. If the partial denominators do not have the above form, the algorithm can be modified in several ways to obtain the number k. Bounds for the moduli of the zeros of $P(z)$ are found by Wall's method for real coefficients. Simple methods are given for expanding rational functions into continued fractions. These facilitate the use of the above theorem. There are important applications in electrical networks (Cauer, Archiv für Elektrotechnik vol. 17 (1926) pp. 355–388). (Received July 28, 1945.)

158. Josephine M. Mitchell: *Some properties of solutions of partial differential equations given by their series development.*

Let $U=\sum_{m,n=0}^{\infty}D_{mn}z^m\bar{z}^n$ $(D_{mn}=\bar{D}_{nm},\ z=x+iy,\ \bar{z}=x-iy)$ be a real analytic function of the two real variables x and y. Let U satisfy a linear partial differential equation $L(U)\equiv U_{z\bar{z}}+2\ \text{Re}\ [(\sum_{m,n=0}^{\infty}a_{mn}z^m\bar{z}^n)U_z]+(\sum_{m,n=0}^{\infty}c_{mn}z^m\bar{z}^n)U=0$ $(U_z=(\partial U/\partial x-i\partial U/\partial y)/2,\ U_{z\bar{z}}=\Delta U/4)$. The a_{mn} and c_{mn} $(m, n=0, 1, 2, \cdots)$

are given. Then U is completely determined by the subsequence $\{D_{m0}\}$ since the remaining D_{mn} can be calculated by means of the a_{mn} and c_{mn}. Using results on a class of complex solutions of L introduced by Bergman [Trans. Amer. Math. Soc. vol. 57 (1945) pp. 299–331] and certain of his methods, the author gives sufficient conditions in terms of the $\{D_{m0}\}$ and $\{a_{m0}\}$ that U be continuous on the circle C: $x^2+y^2=$ constant. (These conditions depend neither on the remaining a_{mn} nor on the c_{mn}.) Sufficient conditions that U have a jump on C as well as the size of the jump are given in terms of the same subsequences. These results are extended to other closed curves. By means of a theorem of Bergman [Duke Math. J. vol. 11 (1944) pp. 617–649] similar results are obtained for functions which satisfy differential equations of the fourth order. (Received July 28, 1945.)

159. A. F. Moursund: *Non-summability of the conjugate series of the Fourier series.*

In this paper it is shown that at a point x where $\int_0^t |f(x+u)-f(x-u)|\,du=O(t)$, as $t\to 0$, the divergence to $+\infty$ $[-\infty]$ of the Cauchy integral $\int_0^\pi [f(x+u)-f(x-u)] \cot u/2\,du$ is a necessary and sufficient condition for the divergence in the same sense of the conjugate Fourier series $\sum_{n=1}^\infty (b_n \cos nx - a_n \sin nx)$. (Received July 30, 1945.)

160. M. E. Munroe: *A note on weak differentiability of Pettis integrals.*

An example is given of a function defined to the space C; the function is Pettis integrable, but its integral fails to be weakly differentiable on a set of positive measure. A method is given for the specific construction of such a function to the space of continuous functionals over any compact metric space containing a homeomorph of the Cantor set. (Received July 13, 1945.)

161. Raphael Salem and Antoni Zygmund: *Lacunary power series and Peano curves.*

Let $f(z)=\sum_{k=1}^\infty a_k z^{n_k}$ be a lacunary power series of radius of convergence equal to 1, with $n_{k+1}/n_k \geqq \lambda > 1$, and $\sum |a_k| < \infty$. If λ is larger than a certain absolute constant, and if the convergence of $\sum |a_k|$ is slow enough, the boundary values $f(e^{i\theta})$ fill completely a certain square. (Received July 17, 1945.)

162. Robert Schatten: *The cross-space of linear transformations.*

Let $\mathfrak{B}_1$, $\mathfrak{B}_2$ denote two Banach spaces, and $\mathfrak{B}_1^*$, $\mathfrak{B}_2^*$ their conjugate spaces. The direct product $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$ (Trans. Amer. Math. Soc. vol. 53 (1943) pp. 195–217) depending naturally on the norm α determines uniquely an "associate space" $\mathfrak{B}_1^* \otimes_\alpha \mathfrak{B}_2^*$ (α' denotes the norm associated with α) and a conjugate space $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^*$. Let γ denote the greatest crossnorm, and λ the least crossnorm (whose associate is also a crossnorm). Then $\gamma' = \lambda$. Furthermore, $(\mathfrak{B}_1 \otimes_\gamma \mathfrak{B}_2)^*$ is the Banach space of all linear transformations from $\mathfrak{B}_1$ into $\mathfrak{B}_2^*$ (from $\mathfrak{B}_2$ into $\mathfrak{B}_1^*$), while $\mathfrak{B}_1^* \otimes_\lambda \mathfrak{B}_2^*$ is characterized as the Banach space of all linear transformations from $\mathfrak{B}_1$ into $\mathfrak{B}_2^*$ (from $\mathfrak{B}_2$ into $\mathfrak{B}_1^*$), which may be approximated in norm by a sequence of linear transformations with finite-dimensional ranges. It is shown further, that for any reflexive crossnorm $\alpha(\alpha''=\alpha)$, the following statements are equivalent: (1) $\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$ is reflexive, (2) $\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$ is reflexive, (3) $(\mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2)^* = \mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*$ and $(\mathfrak{B}_1^* \otimes_{\alpha'} \mathfrak{B}_2^*)^* = \mathfrak{B}_1 \otimes_\alpha \mathfrak{B}_2$. (Received May 29, 1945.)

163. C. F. Stephens: *Systems of linear difference equations containing a parameter.*

The author considers the system of equations $y_i(x+1)=\sum_{j=1}^{n} h_{ij}(x, \lambda)y_j(x)+g_i(x, \lambda)$, where $h_{ij}(x, \lambda)$ and $g_i(x, \lambda)$ are analytic in λ and continuous in x for $|\lambda| \leqq r$ and $|x| \geqq R$. One considers three cases: (1) $h_{ij}(x, 0)\not\equiv 0$, $g_i(x, \lambda)\equiv 0$; (2) $h_{ij}(x, 0)\not\equiv 0$, $g_i(x, 0)\equiv 0$; (3) $h_{ij}(x, 0)\equiv 0$, $g_i(x, 0)\equiv 0$. In cases (1) and (2) it is shown that there exist many solutions and in case (3) a unique solution. These solution functions are continuous functions of x and analytic functions of λ. The problems considered are shown to be special cases of more general problems treated in the author's doctoral thesis. (Bull. Amer. Math. Soc. abstract 50-5-148). (Received July 12, 1945.)

164. W. J. Trjitzinsky: *Theory of functions of intervals and applications to functions of a complex variable.*

This work generalizes theorems of Ridder and Besicovitch, relating to conditions under which an additive function of intervals is the integral of a suitably defined derivative of this function. These generalizations, together with certain other considerations, based on the author's earlier work in this field, lead to conditions for analyticity, as well as to a number of theorems on representation of certain general (non-analytic) classes of functions of a complex variable in terms of "double Cauchy integrals." (Received July 5, 1945.)

165. S. M. Ulam and John von Neumann: *Random ergodic theorems.*

The strong ergodic theorems are generalized from the case of a strictly determined flow to a process consisting of a combination of deterministic and random processes. The generalization may be illustrated by the following theorem constituting a special case: Let $S(p)$, $T(p)$ be two given but arbitrary measure preserving transformations of a measure space E into itself. Form all the combinations of the transformations: S, T, $S(S)$, $S(T)$, $T(S)$, $T(T)$, $S(S(T))$, $S(T(S))$, $\cdots$. The ergodic limit exists then for almost every point p of E and almost every choice of the infinite sequence obtained by applying S and T in turn at random, for example, $S(p)$, $T(S(p))$, $T(T(S(p)))$, $\cdots$. The metric transitivity of a given pair (or a given finite number) of transformations is studied and established for almost every pair of transformations in various special cases. These include the case where E is a compact connected group and S, T are, say, left translations (by elements s, t of the group E). (Received August 1, 1945.)

166. C. W. Vickery: *Concerning the notion of measure in metric spaces.*

The author has obtained two treatments of measure applicable to any metric space: (1) a generalization of the notion of measure of Jordan; (2) a generalization of the notion of measure of Borel-Lebesgue. Properties of spaces D^{α}, metric spaces of uncountably many dimensions, previously defined by the author (Bull. Amer. Math. Soc. vol. 45 (1939) pp. 456–462) are employed. Using treatment (2) the notion of a generalized Lebesgue-Stieltjes integral, applicable to real-valued functions defined on any metric space, is defined and properties of such an integral are studied. (Received June 21, 1945.)

167. H. S. Wall: *Analytic functions with positive real parts.*

Let $g_1, g_2, g_3, \cdots$ be any complex numbers such that $|g_p-(1/2)|<1/2$,

$p=1, 2, 3, \cdots$, and put $G_p=|g_p|^2/R(g_p)$, $\tau_p=I(g_p)/R(g_p)$. Let W denote the open region exterior to the cut along the real axis from -1 to $-\infty$, and let $(1+z)^{1/2}$ be the branch of the square root which is 1 for $z=0$. The continued fraction $1/1+G_1z/(1-i\tau_1(1+z)^{1/2})+(1-G_1)G_2z/(1-i\tau_2(1+z)^{1/2})+(1-G_2)G_3z/(1-i\tau_3(1+z)^{1/2})+\cdots$ converges uniformly over every bounded closed region in W. The class of functions $F(z)$ which are analytic and have positive real parts in W, and equal 1 for $z=0$, is coextensive with the class of functions $(1+z)^{1/2}f(z)$, where $f(z)$ is the value of a continued fraction of the above form, or of a terminating continued fraction of that form in which the last G_p may equal 1. (Received July 12, 1945.)

168. H. S. Wall: *Theorems on arbitrary J-fractions.*

Let $1/(b_1+z)-a_1^2/(b_2+z)-a_2^2/(b_3+z)-\cdots$ $(a_p\neq 0)$ be an arbitrary J-fraction. Let $x_p=X_p(z)$, $x_p=Y_p(z)$ be the solutions of the system $-a_{p-1}x_{p-1}+(b_p+z)x_p-a_px_{p+1}=0$, $p=1, 2, 3, \cdots$ $(a_0=1)$ under the initial conditions $x_0=-1$, $x_1=0$ and $x_0=0$, $x_1=1$, respectively. The *indeterminate case* or the *determinate case* holds according as both the series $\sum|X_p(0)|^2$, $\sum|Y_p(0)|^2$ converge or at least one diverges, respectively. It is shown that in the indeterminate case, if the J-fraction converges for a single value of z, it converges for every value of z to a meromorphic function. If the J-fraction is *positive definite*, the associated J-matrix has one or infinitely many bounded reciprocals for $I(z)>0$ according as the determinate or the indeterminate case holds, respectively. Let $k_1, k_2, k_3, \cdots$ be numbers different from zero such that $\sum|k_{2p+1}|$ and $\sum|k_{2p+1}(k_2+k_4+\cdots+k_{2p})^2|$ converge. If $\lim_{p=\infty}|k_2+k_4+\cdots+k_{2p}|=\infty$, the continued fraction $1/k_1z+1/k_2+1/k_3z+1/k_4+\cdots$ converges for every z to a meromorphic function or else to the constant ∞. If the above limit does not exist, or is finite, then the continued fraction diverges by oscillation for every z. (Received June 8, 1945.)

APPLIED MATHEMATICS

169. Stefan Bergman: *The integration of equations of fluid dynamics in the three-dimensional case.*

The author describes methods for the determination of potentials of *three-dimensional* flow patterns which are of interest in the theory of turbines. In order to obtain an approximate potential of an axially symmetric flow of a given type defined in the domain D, he determines the complex potential $g(z)$, $z=x+iy$, of a two-dimensional flow in the meridian plane of D, that is, in the region which is the intersection of D with the plane $\phi=$const. (r, θ, ϕ are the polar coordinates). Applying to $g(z)$ the operator introduced in Math. Zeit. vol. 24 (1926) pp. 641–669 he obtains a function which approximates the potential of the desired flow: (See, for example, the above paper, p. 655, where the potential of an axially symmetric flow in a turbine is given.) Using more complicated processes, potentials of general (not necessarily axis symmetric) flows can be obtained. These potential functions are used as first approximations to solutions of nonlinear equations of fluid dynamics. (Received July 30, 1945.)

170. Herbert Jehle: *A new approach to stellar statistics.*

A wave equation of the Schroedinger type, $[(\bar\sigma^2/2)\nabla^2+(\bar\sigma/i)\partial/\partial t-c^2-U]\cdot\{|\psi|\exp(iS/\bar\sigma)\}=0$, where $\bar\sigma$ is a constant, c the velocity of light and U the potential field, is known to admit a hydrodynamical interpretation: It splits into two real equations, $-\sum\nabla_\nu(|\psi|^2\nabla_\nu S)+(\partial/\partial t)|\psi|^2=0$, a continuity equation with density $|\psi|^2$ and streaming velocity $-\nabla_\nu S$, and $(\bar{\sigma}^2/2|\psi|)\nabla^2|\psi|+\partial S/\partial t-c^2-U-\sum(\nabla_\nu S)^2/2=0$, a Hamilton-Jacobi equation or Bernouilli equation with an assumed pressure function $(\bar{\sigma}^2/2|\psi|)\nabla^2|\psi|$. An analysis of the motion of stars in a stellar system, treated as a hydrodynamical problem, shows that the above assumption about pressure function is plausible and fits the observed facts. Discrete effects (like quantum effects) cannot show up in this astronomical theory because observed hydrodynamical quantities in a stellar system are to be understood as averages taken over volumes containing many statistically independent elements (stars or star clusters), and they have to be confronted with a ψ function which is a superposition of many stationary ψ. Because of its mathematical simplicity this theory provides for an approach to problems such as transient solutions of the wave equation, selfconsistent steady and transient fields (that is, Poisson equation and hydrodynamical equations combined). (Received August 1, 1945.)

171. Arturo Rosenbluth and Norbert Wiener: *Mathematics of fibrillation and flutter in the heart.*

The known facts about the continuation and refractory period of a muscle fiber are used to explain the phenomena of flutter and fibrillation in the vertebra heart. A geometrical discussion is given of the flutter problem while the fibrillation problem is reduced to a statistical form. (Received July 20, 1945.)

172. H. E. Salzer: *Table of coefficients for repeated integration with differences.*

For functions tabulated at a uniform interval, formulas for k-fold integration, using advancing or backward differences, are obtained by integrating the Gregory-Newton advancing-difference interpolation formula or the Newton backward-difference formula. The quantities $G_n^{(k)}\equiv(1/n!)\times\int_0^1\cdots\int_0^p\int_0^p p(p-1)\cdots(p-n+1)(dp)^k$ and $H_n^{(k)}\equiv(1/n!)\times\int_0^1\cdots\int_0^p\int_0^p p(p+1)\cdots(p+n-1)(dp)^k$ are the coefficients of the nth advancing and backward differences respectively in the formulas $\int_{x_0}^{x_1}\cdots\int_{x_0}^{x}\int_{x_0}^{x}f(x)(dx)^k$ $=h^k[f(x_0)/k!+\sum_{n=1}^{m}G_n^{(k)}\nabla^n f(x_0)]+Rm=h^k[f(x_0)/k!+\sum_{n=1}^{m}H_n^{(k)}\nabla^n f(x_0)]+R_m'$, where $x_1-x_0=h=$ the tabular interval. Previous tables (A. N. Lowan, H. E. Salzer, Journal of Mathematics and Physics vol. 22 (1943) pp. 49–50, and W. E. Milne, Amer. Math. Monthly vol. 40 (1933) pp. 322–327) furnish exact values for $k=1$, $n=1$ (1) 20 and $k=2$, $n=1$ (1) 7. The present table has exact values for $k=2$, $n=1$ (1) 20 and decimal values for $k=2$ (1) 6, $n=1$ (1) $22-k$. The quantities $G_n^{(2)}$ and $H_n^{(2)}$ are expressed in several ways as functions of $B_\nu^{(n)}(x)$, Bernoulli polynomials of order n and degree ν, where $t_n e^{xt}/(e^t-1)^n=\sum_{\nu=0}^{\infty}t^\nu B_\nu^{(n)}(x)/\nu!$, and were checked in terms of previously tabulated values of $B_\nu^{(n)}(x)$. Also a simple recursion formula for $G_n^{(k)}$ in terms of $G_n^{(k-1)}$ and $G_{n+1}^{(k-1)}$ (and similarly for $H_n^{(k)}$), valid for $k\geqq 2$, was used for computation when $k>2$. (Received July 7, 1945.)

GEOMETRY

173. P. O. Bell: *Power series expansions for the equations of a variety in hyperspace.*

For the study of the local properties of an arbitrary variety V_m in a linear space S_n a moving reference frame $F(x_0, x_1, \cdots, x_n)$ may be selected whose vertex x_0 is a generic point of V_m. The general projective homogeneous coordinates x_i^ν of the points

x_j, $j=0, 1, 2, \cdots, n$, are solutions of a system of $m(n+1)$ partial differential equations of the form $\partial x_i/\partial u^\alpha = C_{i\alpha h}x^h$, $\alpha=1, 2, \cdots, m$, in which h is a dummy index. Local nonhomogeneous coordinates z^i with respect to F of a point X of V_m (near to x_0) are defined by the vector relation $X=z^h x_h$, in which $z^0=1$. The present paper is concerned with the calculation of power series expansions for $n-m$ coordinates z^r, $r=m+1, m+2, \cdots, n$, in terms of the remaining m coordinates of X. On differentiating with respect to u^α the coordinates z^r and their power series representations with undetermined coefficients and replacing the derivatives of z^i by their values obtained from the conditions of immovability of the point X, the author obtains two power series for each derivative of z^r. On equating corresponding coefficients of these series, the author derives recursion-relations among the required coefficients. The ease with which these relations may be applied is exemplified by new determinations of certain power series expansions. (Received May 19, 1945.)

174. N. A. Court: *On a skew quartic associated with a tetrahedron.*

Associate with a tetrahedron (T) the three orthogonal hyperboloids of one sheet which have for their axes the three pairs of opposite edges of (T). The three surfaces belong to the same pencil of quadrics. The hyperboloid (H) determined by the altitudes of (T) also belongs to this pencil. The vertices of (T) and the feet of its altitudes lie on the skew quartic (C_4) common to the surfaces of the pencil of quadrics. The skew quartic which is thus associated with (T) is the locus of points from which the vertices of (T) are projected by an orthocentric group of lines. The tangents to (C_4) at the vertices of (T) are the orthocentric lines of the respective trihedral angles of (T). The osculating planes of (C_4) at the vertices of (T) are the tangent planes at those points to the hyperboloid (H). If the tetrahedron (T) is isosceles, the skew quartic associated with (T) is also the skew quartic associated with the twin tetrahedron of (T): the common circumcenter of the two tetrahedrons is a center of symmetry of the skew quartic. (Received July 25, 1945.)

175. John DeCicco: *Equilong maps of the ∞^3 circles.*

The author obtains theorems in equilong geometry analogous to those considered by Kasner and DeCicco in the paper *Families of curves conformally equivalent to circles* (Trans. Amer. Math. Soc. vol. 49 (1941) pp. 378–391). Upon applying any equilong transformation to the totality of ∞^3 circles, there results a set of ∞^3 curves, which is termed an ω family. The magnilong group is the totality of all line transformations in the plane carrying every ω family into an ω family. A general, affinilong, or magnilong transformation carries at most $2\infty^2$, ∞^2, or ∞^1 curves of a given ω family into curves of the same family. The focal locus of the osculating parabolas of the ∞^1 curves of an ω family which pass through a lineal element E is a cissoid with the cusp at E. The envelope of the directrices of the osculating parabolas is a parabola with vertex at the point P of E and the line E as the axis. The envelope of the ∞^1 circles defining the focal cissoids constructed along a given line L consists of two straight lines symmetrical to L. Finally the envelope of the ∞^1 directorial parabolas constructed along a given line L consists of two straight lines symmetrical to L. (Received June 25, 1945.)

176. Edward Kasner: *Neo-Pythagorean triangles.*

In contrast to right or Pythagorean triangles, defined by $a^2+b^2-c^2=0$, the author introduces the class of triangles defined by $a^2+b^2+c^2=0$, which he terms neo-

Pythagorean. One property of these imaginary or complex triangles is cos A cos B cos $C = -1$. It is shown that the medians are proportional to the opposite sides, the ratio being $3^{1/2}i/2$. The only other triangles with such a proportionality are the equilateral, with the ratio $3^{1/2}/2$. Many other properties are found. (Received June 21, 1945.)

177. Edward Kasner: *Null hexagons.*

The author defines a null hexagon as one in which each of the six sides is of length zero; the slopes are alternately $+i$ and $-i$. There are three main diagonals and six minor diagonals, all distinct from zero. All the interrelations of the nine diagonals are found. In particular, the square of a main diagonal is equal to the sum of the squares of two minor diagonals. Certain real covariant diagrams are found. (Received June 25, 1945.)

178. Edward Kasner and John DeCicco: *Multi-isothermal systems.*

The authors extend to euclidean space R_{2n} of $2n$ dimensions the results already considered in the paper *Bi-isothermal systems* (Bull. Amer. Math. Soc. vol. 51 (1945) pp. 169–174). A multi-isothermal system of ∞^{2n-1} curves in R_{2n} is any system which is pseudo-conformally equivalent to a parallel pencil of ∞^{2n-1} straight lines. Any such system consists of ∞^{2n-2} isothermal families of ∞^1 curves, each such family lying upon a conformal surface. A multi-isothermal system of ∞^1 hypersurfaces S_{2n-1} in R_{2n} is any system which is pseudo-conformally equivalent to a parallel pencil of ∞^1 hyperplanes. Any such system may be defined by placing a multiharmonic function equal to an arbitrary constant. The pseudo-angle between any multi-isothermal system of hypersurfaces and any multi-isothermal system of curves is a multiharmonic function. If a surface is intersected by every multi-isothermal system of hypersurfaces in an isothermal system of curves, then the surface is conformal. Finally if a given system of ∞^1 hypersurfaces is intersected by every conformal surface in an isothermal system of curves, then the given system of ∞^1 hypersurfaces is multi-isothermal. (Received June 25, 1945.)

179. J. W. Lasley: *On the classification of collineations in the plane.*

This paper reduces nonsingular collineations of the plane into itself to a common canonical form, that of Jordan. The basis of the classification is a pair of absolute invariants: the multiplicity of the roots of the characteristic equation and the rank of the characteristic matrix arising from the multiple root. Considerations of incidence of double elements is limited to the proof of a single theorem. In adjusting the reference frame to meet the needs of the several cases a full use of duality is made. The effect of the adjustments of the reference frame upon the analytic representation of the collineations is kept in the foreground. The essential geometric features of the collineation are permitted to emerge unhampered by needless algebraic complications. The essential unity present in the diverse patterns appears in the Jordan canonical form, by means of which one is able readily to distinguish one class of collineations from another. (Received July 19, 1945.)

180. J. W. Lasley: *On the equations of certain osculants.*

Abel Transon in 1841 provided by means of his theory of aberrancy a visualization of the derivatives of the third and fourth orders. Since then Cesàro, Wilczynski, the Mukhopadhyayas and others have done much toward welding isolated bits of osculant theory into a cohesive whole. This paper obtains the equations of all osculating conics of a plane curve when referred to a general reference frame. The method employed makes possible a unification of types of osculants which are seemingly quite diverse. (Received July 19, 1945.)

181. H. P. Pettit: *On the generation of certain algebraic surfaces.*

A surface of order $2mn$ is the locus of the curve of intersection of two cones in which the intersections of a plane of a pencil with a base surface of order m and a base surface of order n are projected from two fixed points. These fixed or base points are mn-fold points on the generated surface, the tangent cones consisting, respectively, of m cones of order n and n cones of order m. The surface contains a plane n-ic as an m-fold curve and a plane m-ic as an n-fold curve. For $m=n=1$ the method is the ordinary projective generation of the ruled quadric. For a particular choice of the base points relative to the base surfaces certain degeneracies take place in the generated surface. In a plane through the base points, the process produces the method of generating plane curves which was discussed by the author in *The projective description of some higher plane curves*, Tôhoku Math. J. vol. 27 (1926). (Received May 26, 1945.)

Logic and Foundations

182. Garrett Birkhoff: *Universal algebra.*

An unpublished result of Bruce Crabtree is extended to show that, if A is any algebra with finitary operations, and G is any subset and S any subalgebra of A, then there is a maximal subalgebra T satisfying $G\cap T\leqq S$. If the lattice of subalgebras of A is distributive, then it must satisfy $X\cap\bigcup Y_\alpha=\bigcup(X\cap Y_\alpha)$. Hence not every complete lattice is the lattice of all subalgebras of a suitable universal algebra. (Received July 5, 1945.)

183. R. M. Robinson: *Finite sequences of classes.*

This note discusses the definition of a finite sequence of classes, in an axiomatic set theory in which "sets" and "classes" are distinguished, only sets being allowable as elements. (Received July 23, 1945.)

Statistics and Probability

184. Isaac Opatowski: *Direct and reverse transitions in Markoff chains.*

The author considers stochastic processes consisting of successive transitions between $n+1$ states $\{i\}_0^n$ according to the law $dP_i/dt=k_iP_{i-1}-k_{i+1}P_i+g_iP_{i+1}-g_{i-1}P_i$, $P_0(0)=1$, $P_i(0)=0$ for $i\geqq 1$, where $P_i(t)$ is the probability that the system be in the state i at the time t if it is at the time $t=0$ in the state 0. The constants k_i and g_i represent respectively the "intensities" of the direct and reverse transitions $(i-1\rightarrow i)$ $(i+1\rightarrow i)$. $k_1>0$, $g_1\geqq 0$ for any i, except $k_0=g_n=0$. It is shown that if $k_{n+1}=g_{n-1}=0$, and consequently $\sum_{i=0}^{i=n}P_i=1$, the process is equivalent, as far as the probability $P_n(t)$ is concerned, to a new process of the same type, between the same number of states consisting, however, of direct transitions only with intensities $\{\bar{k}_i\}_1^n$. The main part of the proof consists in showing that $\{-\bar{k}_i\}_1^n$, which are the poles of the Laplace transform of dP_n/dt, are all real and negative. They are roots of the determinant $\|a_{i,j}\|_n$ where $a_{i,i}=x+k_{i+1}+g_{i-1}$; $a_{i,i+1}=g_i$, $a_{i,i-1}=k_i$ with all the other $a_{i,j}$'s zero. The paper is a part of an article to be published in the December 1945 issue of the Bulletin of Mathematical Biophysics. (Received July 26, 1945.)

TOPOLOGY

185. R. H. Bing: *Collections filling up a simple plane web.*

It is shown that the compact continuum W is a simple plane web provided there exist an upper semicontinuous collection of mutually exclusive continua filling up W and another such collection H filling up W such that if g and h are elements of G and H respectively, then the common part of g and h exists and is totally disconnected. It is proved that if W has a bounded complementary domain and each of the collections G and H is a dendron with respect to its elements, then each is a non-equicontinuous collection. (Received June 25, 1945.)

186. R. H. Bing: *Generalizations of two theorems of Janiszewski.*

Janiszewski proved that if H and K are continua neither of which cuts the plane, then the sum of H and K cuts the plane only if the common part of H and K is not connected. The present paper gives generalizations of this result by considering more general sets than continua. It is shown that if H and K are two sets of which neither cuts the point A from the point B, $H \cdot K$ is connected, $H - H \cdot K$ is compact and $H - H \cdot K$ and $K - H \cdot K$ are mutually separated, then $H + K$ does not cut A from B. Also, if H and K are connected sets one of which is compact, H is countinuumwise connected and $\bar{H} \cdot K + H \cdot \bar{K}$ is the sum of two mutually separated sets each containing a point of H, then $H + K$ cuts the plane. (Received July 23, 1945.)

187. J. C. Oxtoby: *Invariant measures in groups which are not locally compact.*

In any complete separable metric group which is dense in itself it is possible to construct a left-invariant measure, defined for all Borel sets and zero for points; but such a measure cannot be locally finite, nor can every compact set have finite measure, unless the group is locally compact. This result is proved independently of Haar's theorem, and the measure constructed may have properties quite unlike Haar's measure. Other constructions, based on the idea of extending a Haar measure from a subgroup, or introducing a new topology, are considered and their limitations discussed. The results throw light on Weil's converse of Haar's theorem by giving examples of measures that fail to satisfy Weil's postulates, and by showing that there is a class of groups in which no Borel measure can satisfy them. (Received July 23, 1945.)

188. Moses Richardson: *On weakly ordered systems.* Preliminary report.

A *weakly ordered system* is a system of elements $a, b, \cdots$ with a binary relation $\succ$ such that (1) $a \succ b$ implies $a \neq b$, and (2) $a \succ b$ implies b not greater than a. Transitivity is not assumed. Such a system can be represented by an oriented 1-complex or linear graph in an obvious way. J. von Neumann and O. Morgenstern (*Theory of games and economic behavior*, Princeton, 1944) prove the existence of "solutions" for the case of a strictly acyclic system, and give a method of constructing such solutions. The main result of the present note is the existence and construction, by a different method, of solutions in the non-acyclic case provided all unoriented cycles are even. (Received August 1, 1945.)

189. A. D. Wallace: *Another fixed point theorem.*

The author adheres in part to the notation of a recent paper (*A fixed point theorem*, Bull. Amer. Math. Soc. vol. 51 (1945) p. 414 et seq.). It will not be supposed that P is a topological space and (i) will be replaced by (i_0): Each Δ-simple set has a maximal element in P. Assume that (i_0), (ii), (iii) and (iv) hold in P. Theorem: If T is a one-to-one transformation of P onto itself such that both T and T^{-1} preserve the relation Δ then there is an $f\neq e$ such that both $f\Delta Tf$ and $Tf\Delta f$ hold. (Received July 9, 1945.)

190. A. D. Wallace: *Dimensional types.*

Let H, S be topological spaces. The space H is said to be of dimensional type S if for each closed set X in H and mapping $f:X\rightarrow S$ there is an extension $\bar{f}:H\rightarrow S$. When S is an n-sphere and H is separable metric then H is of dimensional type S if and only if it is of dimension at most n, a result of Hurewicz. Let H be normal and let S be an ANR. If H is the union of a countable family of closed sets of dimensional type S then H is of dimensional type S. The finite case of this theorem follows with no assumptions on H and S except that they are topological spaces. (Received July 9, 1945.)

191. G. T. Whyburn: *Boundary alternation of monotone mappings.*

Let R be a bounded plane region with connected boundary A and let $f(\bar{R})=B$ be any monotone mapping. In this paper it is shown that if (a) for each $x\in A$, $f^{-1}f(x)$ intersects every crosscut in R both of whose regions have x on their boundaries, f is non-alternating on A. Further, if B is a dendrite and $R\cdot f^{-1}(y)$ is connected for each $y\in B$, then (a) is satisfied and hence the same conclusion holds. Also in case A is locally connected, it is noted that this same conclusion follows provided (b) the mapping $fg(E)=B$ is monotone, where E is a 2-cell and $g(E)=\bar{R}$ is the extended inverse of the relative distance mapping on R. In this latter case it is shown that conditions (a) and (b) are equivalent. (Received July 16, 1945.)

192. G. T. Whyburn: *On monotone retractability into simple arcs.*

In recent work on the area of surfaces Radó has had occasion to use the following properties as applied to locally connected continua A: (π): Every simple arc in A is a monotone retract of A; (Π): Every monotone image of A has property (π). Radó has noted that (Π) implies (π) and that the sphere and 2-cell each have (Π). In this paper it is shown that (1) property (Π) is equivalent to unicoherence for locally connected continua in general, (2) property (π) is equivalent to unicoherence for plane locally connected continua, and (3) every closed 2-dimensional connected manifold has property (π). (Received July 16, 1945.)

193. G. S. Young: *The introduction of local connectivity by change of topology.*

Let G be a collection of subsets of a space, S. A point P will be said to be a G-limit point of a subset M of S if every neighborhood of P contains an element of G which intersects P and $M-P$. Under certain conditions on G (for example, if G is the collection of all arcs of S), in the topology for S thus defined, S is locally connected and still has a topology much like before. Thus, if S were originally metric, it would remain metric. This concept permits use of known properties of non-compact continuous curves in certain problems concerning non-locally connected spaces. Several examples of this technique are given, one of which is the following: Let M be a set which contains no arc, let D be a dendrite, and let G be a minimal collection of arcs such that each point of M is joined to D by an arc of G. Then if $M+D+G^*$ contains no simple closed curve, it has the fixed-point property. In order to obtain this last, a study is made of the properties of a type of generalized dendrite. (Received July 30, 1945.)

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ON SURFACES OF CLASS K_1

J. W. T. YOUNGS

The purpose of this note is to make a comment bearing upon the remarkable results of Radó on the semi-continuity of double integrals in parametric form.[1]

The essence of the situation, without at first attempting precision, is this. A continuous surface is of class K_1 if and only if it has a representation for which the classical double integral area formula has meaning. (It is understood that the integration is in the sense of Lebesgue.) This class of continuous surfaces is variously employed in Radó's paper. The primary object of this note is to show that *every surface is of class* K_1.

In putting things more precisely it is both convenient and economical to treat the matter in its true light; namely, as a corollary to Radó's paper. Thus the notation and terminology here follow that of Radó, and numbers in parentheses refer to the appropriate paragraphs in his paper.

To preserve a certain measure of continuity a few of the salient concepts are here reviewed.

A *continuous surface* S (1.21) is, by definition, an equivalence class of *triples of continuous functions* (1.6). If the definition of the equivalence relation is strengthened by the addition of an *orientation* requirement, then any one of the resulting equivalence classes is known as an *oriented continuous surface* ${}_oS$ (1.23). In each case any triple in the equivalence class is known as a *representation* of the surface.

The notation (T, B) is used generically to denote a *continuous triple of functions.*

$$T:\quad x^i(u^1, u^2), \qquad (u^1, u^2) \in B,\ i = 1, 2, 3,$$

where B is the closure of some Jordan region in the plane. The abbreviated notation

$$T:\quad x(u), \qquad u \in B,$$

is also employed (1.6).

If a continuous triple (T, B) is such that the six partial derivatives exist almost everywhere in B^0, then the Jacobians are denoted by

$$X^1(u^1, u^2) = \frac{\partial(x^2, x^3)}{\partial(u^1, u^2)}, \qquad X^2(u^1, u^2) = \frac{\partial(x^3, x^1)}{\partial(u^1, u^2)},$$

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[1] Tibor Radó, *On the semi-continuity of double integrals in parametric form*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 336–361.

$$X^3(u^1, u^2) = \frac{\partial(x^1, x^2)}{\partial(u^1, u^2)}$$

and, of course, also exist almost everywhere in B^0.

If, in addition, the Jacobians are *summable* then the *continuous triple* (T, B) is said to be of *class* K_1 (1.8). (It is to be noted that this class K_1 refers to continuous triples and *not* to continuous surfaces.)

A continuous surface S (*oriented continuous surface* ${}_oS$) is said to be of *class* K_1 in the event it has a representation which is of *class* K_1 (1.26).

The whole matter can be viewed from another point of view. In the classical theory of area, the term "area" is applied to what is here called a continuous triple (T, B), and the well known expression

$$\iint_{B^0} [(X^1)^2 + (X^2)^2 + (X^3)^2]^{1/2} du^1 du^2$$

is employed. It is obvious that the classical discussion applies only to those continuous triples for which the radical $[(X^1)^2+(X^2)^2+(X^3)^2]^{1/2}$ exists, and is in some sense integrable. In the event the *integration* is understood to be that of *Lebesgue*, however, the classical area formula above can be applied to *all continuous triples of class* K_1. Indeed, the continuous triples of class K_1 are precisely those for which the expression

$$\|X\| = [(X^1)^2 + (X^2)^2 + (X^3)^2]^{1/2}$$

is summable over B^0.

If we represent the classical area formula by $I(T, B, \|X\|)$, the statement can be made in the following form:

A continuous triple (T, B) *is of class* K_1 *if and only if* $I(T, B, \|X\|)$ *exists.*

It is now possible to state the theorem.

THEOREM. *Every continuous surface, whether oriented or not, is of class* K_1. *Alternately, every continuous surface* S (*oriented continuous surface* ${}_oS$) *has a representation* (T, B) *such that* $I(T, B, \|X\|)$ *exists.*

PROOF. Suppose S (${}_oS$) is a continuous surface (oriented continuous surface) with a representation

$$\bar{T}: \quad \bar{x}(\bar{u}), \qquad \bar{u} \in \bar{B}.$$

There is clearly no lack of generality in assuming that $\bar{B}$ is the closed unit square $0 \leqq \bar{u}^1 \leqq 1$, $0 \leqq \bar{u}^2 \leqq 1$.

To prove the theorem a representation

$$T: \quad x(u), \qquad u \in B,$$

of S (${}_oS$) must be found which is of class K_1.

Suppose $\gamma(t)$ is the familiar Cantor function. The properties of this function here employed are: $d\gamma/dt=0$ on the open set of unit length which is complementary to the Cantor set, $\gamma(0)=0$, $\gamma(1)=1$.

Consider the triple (T, B) where

$$B: \quad 0 \leqq u^1 \leqq 1, \qquad 0 \leqq u^2 \leqq 1.$$

$$T: \quad x^i(u^1, u^2) = \bar{x}^i(\gamma(u^1), \gamma(u^2)), \qquad i = 1, 2, 3.$$

If we define the transformation $\bar{u}=\mu(u)$ by the couple

$$\bar{u}^1 = \gamma(u^1), \qquad \bar{u}^2 = \gamma(u^2)$$

the triple can also be defined in condensed form by

$$T: \quad x(u) = \bar{x}(\mu(u)), \qquad u \in B.$$

It is easy to see that

$$\frac{\partial x^i}{\partial u^1} = 0 = \frac{\partial x^i}{\partial u^2}, \qquad i = 1, 2, 3,$$

almost everywhere in B^0. Therefore the Jacobian

$$X^i(u^1, u^2) = 0, \qquad i = 1, 2, 3,$$

almost everywhere in B^0. Thus $I(T, B, \|X\|)$ exists, and in point of fact is equal to zero.

It remains but to show that (T, B) is, indeed, a representation of S (${}_oS$). In other words it must be shown that for any $\epsilon>0$ there is a topological transformation

$$\bar{u} = \tau_\epsilon(u)$$

such that the *distance*

$$\|\bar{x}(\tau_\epsilon(u)) - x(u)\| = \|\bar{x}(\tau_\epsilon(u)) - \bar{x}(\mu(u))\| < \epsilon, \qquad u \in B.$$

The required transformation $\bar{u}=\tau_\epsilon(u)$ is selected in the following fashion:

Since $\bar{x}(\bar{u})$ is a continuous triple there is a $\delta>0$ such that the distance $\|\bar{x}(\bar{u})-\bar{x}(\bar{v})\|<\epsilon$ for $\|\bar{u}-\bar{v}\|<\delta$.

Let $\gamma^*(t)$ be a continuous and strictly monotone function such that

$$\gamma^*(0) = 0, \quad \gamma^*(1) = 1, \quad \|\gamma^*(t) - \gamma(t)\| < \delta/2 \text{ for } 0 \leqq t \leqq 1.$$

Now consider the transformation $\bar{u}=\tau_\epsilon(u)$ defined by the couple

$$\bar{u}^1 = \gamma^*(u^1), \qquad \bar{u}^2 = \gamma^*(u^2).$$

This transformation is topological, since γ^* is strictly monotone in addition to being continuous, and

$$\|\tau_\epsilon(u) - \mu(u)\| < \delta, \qquad u \in B.$$

Hence

$$\|\bar{x}(\tau_\epsilon(u)) - \bar{x}(\mu(u))\| < \epsilon, \qquad u \in B.$$

On observing that τ_ϵ is *sense preserving* (1.18) the proof is complete.

It has been noted that the special representation (T, B) has the property that the Jacobians vanish almost everywhere in B^0. This proves the following corollary.

COROLLARY. *Every surface S $({}_oS)$ has a representation (T, B) for which B is the closed unit square, and, in addition, $I(T, B, \|X\|) = 0$.*

In a sense this shows the complete unreliability of the classical area formula if it is applied at random to any representation of a continuous surface (oriented continuous surface) for which the formula has meaning—there is always a representation yielding the value zero. As a matter of fact, two of the most elegant theorems of Radó concern a criterion by which it is possible to select precisely those representations of a continuous surface (oriented continuous surface), if any, to which the classical formula can be applied so as to obtain the Lebesgue area (3.14, 3.18). They are here condensed to read as follows:

THEOREM (RADÓ). *If S $({}_oS)$ is a continuous surface (oriented continuous surface) of class K_1, and (T, B) is a representation of S $({}_oS)$ of class K_1, then the Lebesgue area*

$$L(S) \geqq I(T, B, \|X\|) \qquad (L({}_oS) \geqq I(T, B, \|X\|)).$$

The sign of equality holds if and only if (T, B) is of class K_2 (1.19).

In virtue of the remarks in this note the statement can now be somewhat strengthened.

THEOREM (RADÓ). *Any continuous surface S (oriented continuous surface ${}_oS$) has a representation (T, B) of class K_1 and*

$$L(S) \geqq I(T, B)\|X\|) \qquad (L({}_oS) \geqq I(T, B, \|X\|)).$$

The sign of equality holds if and only if (T, B) is of class K_2.

Several questions arise as a natural consequence of the fact that every surface has a representation for which the classical area is zero. Though these questions will have to await somewhat quieter times for personal investigation they are presented here in the hopes that they will interest others.

QUESTION 1. *Are those representations* (T, B) *of* S $({}_oS)$ *for which* $I(T, B, \|X\|)=0$ *of the second category in the collection of all representations of* S $({}_oS)$ *which are of class* K_1?

For any continuous surface S (oriented continuous surface ${}_oS$) let: $I^*(S)=\sup I(T, B, \|X\|)$ for all representations (T, B) of S which are of class K_1.

$(I^*({}_oS)=\sup I(T, B, \|X\|)$ for all representations (T, B) of ${}_oS$ which are of class K_1.)

The theorem of Radó guarantees that

$$L(S) \geqq I^*(S) \qquad (L({}_oS) \geqq I^*({}_oS)).$$

QUESTION 2. *Does* $I^*(S)$ $(I^*({}_oS))$ *share an important property of the Lebesgue integral; explicitly, is* $I^*(S)$ $(I^*({}_oS))$ *a lower semi-continuous function of* S $({}_oS)$?

QUESTION 3. *Is* $I^*(S)=L(S)$ $(I^*({}_oS)=L({}_oS))$?

An affirmative answer to the last question would be of compelling interest.

PURDUE UNIVERSITY AND
THE EIGHTH AIR FORCE

COLLECTIONS FILLING UP A SIMPLE PLANE WEB

R. H. BING

A web has been defined [2][1] by R. L. Moore. A compact plane continuum W is said [1] to be a simple plane web if there exist an upper semicontinuous collection G of mutually exclusive continua filling up W and another such collection H also filling up W such that (1) G is a dendron with respect to its elements and so is H and (2) if g and h are elements of G and H respectively, the common part of g and h exists and is totally disconnected. It has been shown [1] that we have an equivalent definition if we substitute for (1) in the above definition the condition that (1′) G is a dendron with respect to its elements and H is an arc with respect to its elements. The present paper shows that an equivalent definition is obtained if condition (1) is omitted.

This paper gives conditions under which the collections filling up a simple plane web are *non-equicontinuous*. A collection of mutually exclusive continua is said to be *equicontinuous* if for every positive number ϵ, there is a positive number δ_ϵ such that if P and Q are points of an element g of G at a distance apart of less than δ_ϵ, there is an arc from P to Q in g of diameter less than ϵ. If a collection is not equicontinuous, it is said to be *non-equicontinuous*. If there exists a point P belonging to the limiting set of G such that every subcollection of G whose limiting set contains P is non-equicontinuous, then G is said to be *hereditarily non-equicontinuous* at P.

I wish to thank Professor R. L. Moore for his valuable suggestions in the preparation of this paper.

A simple plane web is a continuous curve [3]. In establishing some of the following theorems, we shall make use of the fact [1, Theorem 1] that a necessary and sufficient condition that a continuous curve be a simple plane web is that it remain connected and locally connected on the omission of any countable set of points.

THEOREM 1. *A necessary and sufficient condition that a compact plane continuum W be a simple plane web is that there exist an upper semicontinuous collection G of mutually exclusive continua filling up W and another such collection H filling up W such that if g and h are elements of G and H respectively, the common part of g and h exists and is totally disconnected.*

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[1] Numbers in brackets refer to the references cited at the end of the paper.

PROOF. The necessity of the above condition follows from the definition of a simple plane web. To demonstrate its sufficiency, we shall show that if Z is a countable subset of W, $W-Z$ is connected and locally connected. The argument is similar to that used in proving the necessity of Theorem 1 of [1].

Let M be the sum of all elements of G that do not intersect Z and let N be the sum of all elements of H that do not intersect Z. As $M+N$ is connected and dense in W, $W-Z$ is connected.

Assume that $W-Z$ is not locally connected. Let d be a domain containing a point P of $W-Z$ such that any domain d' containing P contains points of an infinitude of components of $(W-Z)\cdot d$. An infinitude of such components contain points of $M\cdot d'$. Let C_1, C_2, C_3 be circles in d such that C_1 encloses P, C_2 encloses C_1, and C_3 encloses C_2 but no element of G or H. Then C_1 encloses a sequence of points $P_1, P_2, \cdots$ of M such that (1) no two of them belong to the same component of $(W-Z)\cdot d$ and (2) if g_n is the component containing P_n of the common part of d and the element of G containing P_n, then $g_1, g_2, \cdots$ has a sequential limiting set. There exists a sequence of points $Q_1, Q_2, \cdots$ of N converging to a point of C_2 such that Q_n is a point of a continuum in g_m $(m \geqq n)$ irreducible from C_1 to C_3. Let h_n be the component containing Q_n of the common part of d and the element of H containing Q_n. It follows [2, pp. 391–392] that some nondegenerate continuum F is a subset of the common part of the limiting set of $g_1, g_2, \cdots$ and the limiting set of $h_1, h_2, \cdots$. As G and H are upper semicontinuous collections, F is a subset of the common part of an element of G and an element of H. This is contrary to the hypothesis of this theorem.

THEOREM 2. *If x is the closure of a connected set which is the sum of a countable number of points and complementary domains of a simple plane web W, the common part of W and x is a continuous curve.*[2]

PROOF. It has been shown [1, Theorem 6] that x is locally connected. As W is a continuous curve, for any positive number ϵ there are only a finite number of complementary domains of W of diameter more than ϵ. Since $W\cdot x$ is equal to x minus the sum of a set of complementary domains of W, then $W\cdot x$ is a continuous curve.

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[2] Indeed, $W\cdot x$ may be shown to be *hereditarily* a continuous curve. Furthermore, it may be shown that in order that the compact point set K should be the boundary of the closure of a connected point set which is the sum of a countable number of points and complementary domains of some simple plane web W it is necessary and sufficient that K should be a continuous curve such that (1) every nondegenerate cyclic element of K is a simple closed curve, (2) K does not have uncountably many cut points, and (3) no cyclic element of K encloses a point of K.

THEOREM 3. *If x is the closure of a connected set which is the sum of a countable number of points and complementary domains of a simple plane web W and R is a nondegenerate connected subset of the common part of W and x, then R is arcwise connected.*[3]

PROOF. Let A and B be two points of R and let AB be an arc in $W \cdot x$. If AB contains a point P not of R, there is an arc AOB in $W \cdot x - P$. There is a simple closed curve which is the sum of arcs $A'PB'$ and $A'QB'$ of AB and AOB respectively. It has been shown [1, Theorem 8] that each point of $W \cdot x$ is a limit point of the connected set $W - W \cdot x$. Therefore, no subset of $W \cdot x$ contains a simple closed curve plus an arc having only its end points on the simple closed curve. Using this fact and the fact that there is an arc from A to B in $W \cdot x$ minus a finite number of points not belonging to R, we find that $A'QB'$ is a subset of R. By replacing $A'PB'$ and other subarcs of AB by arcs which are subsets of R, we may obtain an arc from A to B in R.

THEOREM 4. *If x is the closure of a connected set which is the sum of a countable number of points and complementary domains of a simple plane web W and AB is an arc lying in the common part of W and x, then AB contains an arc lying on the boundary of some complementary domain of W.*

PROOF. Since AB is [1, Theorem 7] the sum of a countable number of points and closures of individual complementary domains of W, the boundary of some one of these domains contains a subarc of AB.

THEOREM 5. *Let G be an upper semicontinuous collection of mutually exclusive continua filling up a simple plane web W and let H be another such collection filling up W such that* (1) *H is a dendron with respect to its elements and* (2) *if g and h are elements of G and H respectively, the common part of g and h exists and is totally disconnected. Let g_1 and g_2 be two elements of G belonging to x_1 and x_2 respectively where x_i* $(i=1, 2)$ *is the closure of a connected set which is the sum of a countable number of points and complementary domains of W. Then g_1+g_2 is a subset of the boundary of a complementary domain of W.*

PROOF. By Theorems 3 and 4 there is a complementary domain d of W such that g_1 and the boundary of d have an arc in common. The boundary of d intersects two elements h_1 and h_2 of H and as g_2 intersects each of them, each element of H that separates h_1 from h_2 in W also disconnects both g_2 and $W \cdot x_2$.

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[3] In fact, R is *locally* arcwise connected.

Let Z be a countable set of points such that Z plus the sum of a set of complementary domains of W is a connected set whose closure is x_2. Then each element of H that separates h_1 from h_2 intersects Z or contains a point of x_2 belonging to the boundary of d. As Z is countable and as there are uncountably many elements of H each of which separates h_1 from h_2 in W, the boundary of d contains uncountably many points of g_2.

Assume that g_i ($i=1$ or 2) is not a subset of the boundary of d. By Theorems 3 and 4 there is a complementary domain d' of W other than d such that some arc is a subset of the common part of g_i and the boundary of d'. Both g_1 and g_2 intersect the boundary of d' and by Theorem 3 there are two mutually exclusive arcs in $W \cdot x_1$ and $W \cdot x_2$ respectively such that each has an end point on the boundary of d and an end point on the boundary of d'. This is impossible, for no subset of $W \cdot x_1 + W \cdot x_2$ plus the boundaries of d and d' is a simple closed curve plus an arc having only its end points on the simple closed curve.

THEOREM 6. *Suppose W is a simple plane web with two complementary domains. Suppose that G is an upper semicontinuous collection of mutually exclusive continua filling up W, that H is another such collection also filling up W, that G is a dendron with respect to its elements and so is H and finally that if g and h are elements of G and H respectively, then the common part of g and h exists and is totally disconnected. Then there is an element of G which does not separate W and which is not a subset of the closure of any connected set which is the sum of a countable number of points and complementary domains of W.*

PROOF. Those elements of G that do not separate W are called *end elements* of G. Assume that each end element of G is a subset of the closure of a connected set which is the sum of a countable number of points and complementary domains of W. Then by Theorem 5, if g_1 and g_2 are end elements of G, there is a complementary domain d of W whose boundary contains g_1 and g_2.

Suppose that g_3 is an end element of G other than g_1 or g_2. As the boundaries of two complementary domains of a simple plane web have no more than a point in common, it follows by Theorem 5 that g_3 is a subset of the boundary of d. However, if it were a subset of the boundary of d, no two elements of H could intersect each of the sets g_1, g_2, and g_3.

Suppose that g_1 and g_2 are the only end elements of G. If d' is a complementary domain of W other than d, its boundary is a subset of an element of H, for no three mutually exclusive continua in W could intersect the boundary of d' as well as both g_1 and g_2. Likewise, the boundary of d' is a subset of an element of G, for no three mutually exclusive continua in W could intersect both the boundary of d and the boundary of d'. This is impossible as no element of G intersects an element of H in a nondegenerate continuum.

THEOREM 7. *Let W be a simple plane web with two complementary domains. Let G be an upper semicontinuous collection of mutually exclusive continua filling up W and let H be another such collection filling up W such that* (1) *G is a dendron with respect to its elements and so is H and* (2) *if g and h are elements of G and H respectively, the common part of g and h exists and is totally disconnected. Then the collection G is non-equicontinuous.*

PROOF. Let X be a set such that x is an element of X only if x is maximal with respect to being the closure of a connected set which is the sum of a countable number of points and complementary domains of W. By Theorem 6 there is an end element g_e of G which is not a subset of an element of X. There is [1, Theorem 8] an arc AB in W which contains no point of an element of X and which has only its end points in common with g_e.

Let d be one of the complementary domains of g_e+AB which has AB as a subset of its boundary. If F is a point of W in d and E is a point of W belonging to neither d, AB, nor g_e, then there is an arc EF in W which contains no point of AB. Let C be the first point of the closure of d on EF in the order from E to F. If ρ is one-half the distance from C to AB, we shall show that for any positive number ϵ there exist points P and Q belonging to an element g of G such that the distance from P to Q is less than ϵ and the diameter of any subcontinuum of g containing P and Q is of diameter more than ρ.

There exist points P_1 and P_2 such that each can be joined to C by an arc in W of diameter less than the minimum of ρ and $\epsilon/2$ and such that P_1 belongs to d and P_2 belongs to neither d nor g_e. As g_e is an end element of G, there is an element g of G which separates g_e from P_1+P_2 in W. There exist on g points P and Q at a distance from C of less than the minimum of ρ and $\epsilon/2$ such that P belongs to d and Q does not. Every subcontinuum of g that contains P and Q intersects AB and therefore is of diameter more than ρ.

THEOREM 8. *Let W be a simple plane web with two complementary domains. Let G be an upper semicontinuous collection of mutually exclusive continua filling up W and let H be another such collection filling up W such that* (1) *G is an arc with respect to its elements,* (2) *H is a* *dendron with respect to its elements, and* (3) *if g and h are elements of G and H respectively, the common part of g and h exists and is totally disconnected. Then W contains a point at which G is hereditarily non-equicontinuous.*

PROOF. Obtain g_e, AB, C, p, and g as in Theorem 7. Of every countable sequence of different elements of G having a subset of g as a limiting set, all but a finite number separate g from g_e. Hence G is hereditarily non-equicontinuous at C.

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THE UNIVERSITY OF TEXAS

DIMENSIONAL TYPES

A. D. WALLACE

Let H and S be topological spaces. We say that H is of *dimensional type* S (symbol: D_S) if for each closed set X and mapping $f:X\rightarrow S$ there exists an extension $\bar{f}:H\rightarrow S$.

It is clear that (from a result due to Hurewicz [1, p. 83]) when H is separable metric and S is an n-sphere, then H can be of dimensional type S if and only if dim $H\leqq n$. For simplicity we write D_n for D_S when S is an n-sphere. It is, of course, possible to define dim H as the least integer n for which H is of type D_n even when H is not separable metric. But this seems to be open to objection except in certain cases (cf. (d) below).

It is at once clear that we have:

(a) *If H is of type D_S then so also is any closed subset.*

(b) *If the closed sets H_1 and H_2 are of type D_S then so also is the set H_1+H_2.*

As a matter of notation we may suppose that $H=H_1+H_2$. Let $f:X\rightarrow S$. Several cases may arise of which we shall consider only the

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one in which $X\cdot H_1\cdot H_2$ is not empty. By (a) the mapping $g=f|X\cdot H_1\cdot H_2$ may be extended to a mapping $\bar{g}:H_1\cdot H_2\rightarrow S$. Let $g_i=\bar{g}$ on $H_1\cdot H_2$ and $g_i=f$ on $X\cdot H_i$, $i=1, 2$. Then $g_i:(H_1\cdot H_2+X\cdot H_i)\rightarrow S$ and so by assumption may be extended to a mapping $\bar{f}_i:H_i\rightarrow S$. Since $\bar{f}_1=\bar{f}_2$ on $H_1\cdot H_2$ we may combine these mappings into a transformation $\bar{f}$ carrying H into S.

With the aid of (b) and additional hypotheses we can extend (b) to

(c) *Let H be a normal space and S an absolute neighborhood retract. If $H=H_1+H_2+\cdots$ where each H_i is closed and of dimensional type S then H is of dimensional type S.*

From (b) there is no loss of generality in supposing that $H_1\subset H_2\subset H_3\cdots$. Let f map the closed set X into S. We may assume that X meets H_1. Let $f=f_1$ so that (since S is an ANR) there is an extension $\bar{f}_1$ of f_1 mapping a neighborhood U_1 of X into S. From the normality of H it follows that there exists a neighborhood V_1 of X whose closure is contained in U_1. Let $X_1=\bar{V}_1$ and set $g_1=\bar{f}_1|H_1\cdot X_1$. By assumption g_1 admits an extension $\bar{g}_1:H_1\rightarrow S$. Let $f_2=\bar{g}_1$ on H_1 and $f_2=\bar{f}_1$ on X_1 so that f_2 is a mapping of H_1+X_1 into S. We may extend f_2 to $\bar{f}_2: U_2\rightarrow S$, where U_2 is a neighborhood of H_1+X_1. Let V_2 be neighborhood of H_1+X_1 for which $X_2=\bar{V}_1\subset U_2$. Suppose that n exceeds 2 and assume that $\bar{f}_n:U_n\rightarrow S$ is an extension of $f_r:(H_{n-1}+\bar{V}_{n-1})\rightarrow S$ and that $X_n=\bar{V}_n\subset U_n$ where V_n is a neighborhood of $H_{n-1}+\bar{V}_{n-1}$. Let $g_n=\bar{f}_n|H_n\cdot X_n$ and then denote by $\bar{g}_n$ an extension of g_n to H_n. Define f_{n+1} as $\bar{g}_n$ on H_n and put $f_{n+1}=\bar{f}_n|X_n$ so that f_{n+1} maps H_n+X_n into S. Extend f_{n+1} to a transformation $\bar{f}_{n+1}$ of U_{n+1} into S where U_{n+1} is an open set containing H_n+X_n. Let V_{n+1} be a neighborhood of this set with $X_{n+1}=\bar{V}_{n+1}\subset U_n$.

It follows that:

(i) $V_1\subset V_2\subset V_3\subset\cdots$.

(ii) $\bar{f}_{n+1}$ is an extension of $\bar{f}_n$ defined on V_{n+1}, $\bar{f}_1$ being an extension of f.

(iii) $H=\sum_{n=1}^{\infty}V_n$.

If $x\in H$ then, for some n, $x\in V_n$ and we may write $\bar{f}(x)=\bar{f}_n(x)=\bar{f}_{n+1}(x)=\cdots$. That $\bar{f}$ is continuous follows from the fact that the sets V_n are open.

Other generalizations of the classical sum-theorem have been given by P. Alexandroff, E. Čech, and other mathematicians.

To validate the definition suggested earlier it seems necessary to restrict the class of spaces to which it is intended to apply. A space H will be said to have *property* V provided that for any two closed sets X_1, X_2 there exist closed sets H_1, H_2 such that $H=H_1+H_2$, $H_1\cdot H_2\cdot(X_1+X_2)=X_1\cdot X_2$ and $X_i\subset H_i$. This is a well known property of metric spaces but we have no reference to its formulation in the literature as an axiom.

(d) *If H is of type D_n and has property V then H is of type D_{n+1}.*

Following a line of argument used by Hurewicz [2, p. 144] let f map the closed set X into S_{n+1}. We may assume that S_{n+1} is given by the equation $y_1^2+ \cdots +y_{n+2}^2=1$ and let A_1, A_2 denote the subsets of S_{n+1} given by $(y_{n+2}\geqq 0)$, $(y_{n+2}\leqq 0)$. Then A_1 and A_2 are $(n+1)$-cells which meet in the $S_n:(y_1^2+ \cdots +y_{n+1}^2=1)\cdot(y_{n+2}=0)$. We may clearly suppose that $f(X)$ meets both A_1 and A_2 so that $X_i=f^{-1}(A_i\cdot f(X))$ is not empty. Since H has property V we have $H=H_1+H_2$ where $X_i\subset H_i=\overline{H}_i$, and $H_1\cdot H_2\cdot X=X_1\cdot X_2$. We consider only the case where this latter set is not vacuous. Denote by g the mapping f restricted to $X_1\cdot X_2$ so that g has values lying in S_n. By (a), $H_1\cdot H_2$ is of type D_n and so we may extend g to a mapping $\bar{g}:H_1\cdot H_2 \to S_n$. Then (by Tietze's extension theorem) we may extend $\bar{g}$ to a mapping g_i of H_i into A_i. Since $g_1=g_2$ on $H_1\cdot H_2$ we combine these mappings to secure an extension $\bar{f}$ of f taking H into S_{n+1}.

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1. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941.
2. W. Hurewicz, *Über Abbildungen topologischen Räume auf die n-dimensionale Sphäre*, Fund. Math. vol. 24 (1935) pp. 144–150.

THE UNIVERSITY OF PENNSYLVANIA

ON THE EQUATIONS OF MOTION IN A RIEMANN SPACE

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Introduction. The object of this note is to give a geometric interpretation to the conditions of integrability of Killing's equations[1] and to the similar equations for collineations[2] in an affinely connected space. This is done by considering the more general problem of mapping a space upon itself, the mapping preserving some differential invariant. We shall be concerned with purely local properties so that the functions involved are assumed to have continuous derivatives of any necessary order and the groups are assumed to be infinitesimal.

1. **Infinitesimal mapping of V_n upon itself.** By an infinitesimal mapping of V_n upon itself we understand a correspondence generated by a vector field $\xi^i(x)$; that is, to a point $P(x)$ corresponds a point $P(\bar{x})$ where

$$\bar{x}^i = x^i + \xi^i(x)\delta t. \tag{1.1}$$

In order to see the effect of such a mapping on a differential invariant, we consider—merely to be specific—a tensor field of components $T^i{}_j(x)$. Such a tensor field is mapped into one whose components are

$$\overline{T}^i{}_j(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^\alpha}\cdot\frac{\partial x^\beta}{\partial \bar{x}^j} T^\alpha{}_\beta(x),$$

which for the mapping (1.1) gives

$$\overline{T}^i{}_j(\bar{x}) = T^i{}_j(x) + \left(\frac{\partial \xi^i}{\partial x^\alpha} T^\alpha{}_j - \frac{\partial \xi^\beta}{\partial x^j} T^i{}_\beta\right)\delta t. \tag{1.2}$$

By the variation of $T^i{}_j(x)$ we shall understand $\lim_{\delta t\to 0}(T^i{}_j(\bar{x}) - \overline{T}^i{}_j(\bar{x}))/\delta t$ and shall denote it by $\delta T^i{}_j(x)/\delta t$. Then (1.2) gives

$$\frac{\delta T^i{}_j(x)}{\delta t} = \xi^\alpha \frac{\partial T^i{}_j}{\partial x^\alpha} - \frac{\partial \xi^i}{\partial x^\alpha} T^\alpha{}_j + \frac{\partial \xi^\alpha}{\partial x^j} T^i{}_\alpha, \tag{1.3}$$

with a corresponding expression for any scalar or tensor field. If a tensor field is to remain invariant, it is necessary and sufficient that its variation be zero. From (1.3) it is obvious that a zero tensor field

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[1] L. P. Eisenhart, *Riemannian geometry*, chap. 6.

[2] L. P. Eisenhart, *Non-Riemannian geometry*, chap. 3; our $R^i{}_{jkl}$ is the negative of that used by Eisenhart.

is invariant under an arbitrary mapping; it is also evident that the variation is a differential invariant. Hence if the space V_n is affinely connected, partial differentiation may be replaced by covariant differentiation.

2. **Affinely connected** V_n. Denoting covariant differentiation by $T_{,k}$ we have from (1.3)

$$\left(\frac{\delta T^i{}_j}{\delta t}\right)_{,k} = \xi^\alpha T^i{}_{j,\alpha,k} + \xi^\alpha{}_{,k} T^i{}_{j,\alpha} + \xi^\alpha{}_{,j,k} T^i{}_{\alpha,k} - \xi^i{}_{,\alpha,k} T^\alpha{}_j - \xi^i{}_{,\alpha} T^\alpha{}_{j,k} + \xi^\alpha{}_{,j} T^i{}_{\alpha,k}$$

and

$$\frac{\delta(T^i{}_{j,k})}{\delta t} = \xi^\alpha T^i{}_{j,k,\alpha} + \xi^\alpha{}_{,j} T^i{}_{\alpha,k} + \xi^\alpha{}_{,k} T^i{}_{j,\alpha} - \xi^i{}_{,\alpha} T^\alpha{}_{j,k};$$

subtracting and using Ricci's identities we get

$$\left(\frac{\delta T^i{}_j}{\delta t}\right)_{,k} - \frac{\delta(T^i{}_{j,k})}{\delta t} = T^i{}_\alpha(\xi^\alpha{}_{,j,k} + \xi^\beta R^\alpha{}_{jk\beta}) - T^\alpha{}_j(\xi^i{}_{,\alpha,k} + \xi^\beta R^i{}_{\alpha k\beta}), \tag{2.1}$$

where R is the curvature tensor of the V_n. We recall[2] that

$$\xi^\alpha{}_{,j,k} + \xi^\beta R^\alpha{}_{jk\beta} = 0 \tag{2.2}$$

are precisely the differential equations of an affine collineation. Hence we have the following theorem.

THEOREM 1. *A necessary and sufficient condition that a mapping must satisfy so that variation and covariant differentiation be commutative is that the mapping be an affine collineation.*

The conditions of integrability of (2.2) may be written[2] as

$$\begin{aligned} \xi^\alpha R^i{}_{jkl,\alpha} - R^\alpha{}_{jkl}\xi^i{}_{,\alpha} + R^i{}_{\alpha kl}\xi^\alpha{}_{,j} + R^i{}_{j\alpha l}\xi^\alpha{}_{,k} + R^i{}_{jk\alpha}\xi^\alpha{}_{,l} &= 0, \\ \xi^\alpha R^i{}_{jkl,m,\alpha} - R^\alpha{}_{jkl,m}\xi^i{}_{,\alpha} + R^i{}_{\alpha kl,m}\xi^\alpha{}_{,j} + \cdots &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &, \end{aligned}$$

and from (1.3) it follows that they may be written as

$$\delta R^i{}_{jkl}/\delta t = 0, \; \delta R^i{}_{jkl,m}/\delta t = 0, \cdots . \tag{2.3}$$

Hence we have the following theorem.

THEOREM 2. *A necessary and sufficient condition that an affinely connected space must satisfy to admit a group of collineations is that it admit* *a group of mappings leaving the curvature tensor and its successive covariant derivatives invariant.*

From the theory of differential equations of the form (2.2) and from (2.3) it is evident that

THEOREM 3. *The group of affine collineations is a subgroup of mappings preserving the curvature tensor and its successive covariant derivatives.*

That the two groups are not necessarily identical can be seen from the case of an affinely flat space. Since $R^i{}_{jkl}=0$ in this case the group of mappings is the group of analytic transformations while the group of collineations is (equivalent to) the group of linear transformations. On the other hand if $\mathfrak{G}_r$, the group of collineations, is of generic rank r, then, as can be shown, the group of mappings is also $\mathfrak{G}_r$.

3. **Motion in a Riemann space.** If in a metric space, with metric tensor g_{ij}, we require that the mapping preserve this tensor, we get according to (1.3)

$$g_{i\alpha}\xi^\alpha{}_{,j}+g_{\alpha j}\xi^\alpha{}_{,i}=0, \tag{3.1}$$

which are the equations of Killing for an infinitesimal motion. If they hold then (2.2) and (2.3) necessarily follow.[1] We consider a space of constant (nonzero) curvature. In this case

$$R^i{}_{jkl}=c(\delta^i{}_k g_{jl}-\delta^i{}_l g_{jk})$$

and since $R^i{}_{jkl,m}=0$ in this case, (2.3) reduce to

$$\delta^i{}_k(g_{\alpha l}\xi^\alpha{}_{,j}+g_{j\alpha}\xi^\alpha{}_{,l})-\delta^i{}_l(g_{\alpha k}\xi^\alpha{}_{,j}+g_{j\alpha}\xi^\alpha{}_{,k})=0 \tag{3.2}$$

which by contraction give (3.1). Hence we have the following theorem.

THEOREM 4. *In a space of constant (nonzero) curvature a mapping preserving curvature is a motion.*

Since the group of motions in any case is a subgroup of the group of collineations it follows as a corollary of Theorem 4 that in a space of constant nonzero curvature the group of motions and the group of affine collineations are the same. A little more than this can be easily proved; this corollary is true for an Einstein space V_n $(n>2)$ which is not flat.

4. **Motions as a subgroup of collineations.** Suppose that a Riemannian V_n admits a group $\mathfrak{G}_r$ of affine collineations $(r\leqq n^2+n)$. This

means that (2.3) admit r linearly independent solutions $\xi^i{}_{(\sigma)}$, $\sigma=1, 2, \cdots, r$. Let the variation of g_{ij} corresponding to $\xi^i{}_{(\sigma)}$ be denoted by $h^{(\sigma)}{}_{ij}$ (cf. (3.1)). Then the question is whether there exist r constants $c^{(\sigma)}$ such that $c^{(\sigma)}\xi^i{}_{(\sigma)}$ is a motion; that is, such that

$$c^{(\sigma)}h^{(\sigma)}{}_{ij} = 0. \tag{4.1}$$

Obviously a necessary condition is that the matrix of $n(n+1)/2$ rows and r columns whose elements are $h^{(\sigma)}{}_{ij}$ be of rank less than r. We shall show that this condition is also sufficient. For suppose this rank is $s<r$; then (4.1) admit $r-s$ linearly independent solutions $\phi^{(\sigma)}{}_{\alpha}(x)$, $\alpha=1, 2, \cdots, r-s$, so that

$$\phi^{(\sigma)}{}_{\alpha}h^{(\sigma)}{}_{ij} = 0 \tag{4.2}$$

are identities in x. Hence

$$\phi^{(\sigma)}{}_{\alpha,k}h^{(\sigma)}{}_{ij} + \phi^{(\sigma)}{}_{\alpha}h^{(\sigma)}{}_{ij,k} = 0$$

and because of (2.2), $h^{(\sigma)}{}_{ij,k}=0$. Hence $\phi^{(\sigma)}{}_{\alpha,k}$ is also a solution of (4.1) and hence

$$\phi^{(\sigma)}{}_{\alpha,k} = A^{\beta}{}_{\alpha k}\phi^{(\sigma)}{}_{\beta}. \tag{4.3}$$

If there is to be a solution $c^{(\sigma)}$ then a set of $r-s$ functions p^{α} must exist such that

$$c^{(\sigma)} = p^{\alpha}\phi^{(\sigma)}{}_{\alpha},$$

from which it follows that

$$p^{\beta}{}_{,k}\phi^{(\sigma)}{}_{\beta} + p^{\alpha}A^{\beta}{}_{\alpha k}\phi^{(\sigma)}{}_{\beta} = 0.$$

Hence, since $\phi^{(\sigma)}{}_{\alpha}$ are linearly independent we must have

$$p^{\beta}{}_{,k} + p^{\alpha}A^{\beta}{}_{\alpha k} = 0. \tag{4.4}$$

From (4.3) and the linear independence of $\phi^{(\sigma)}{}_{\beta}$ it follows that

$$A^{\gamma}{}_{\alpha k,l} - A^{\gamma}{}_{\alpha l,k} + A^{\beta}{}_{\alpha k}A^{\gamma}{}_{\beta l} - A^{\beta}{}_{\alpha l}A^{\gamma}{}_{\beta k} = 0$$

and these are precisely the conditions of complete integrability of the differential equations (4.4). Hence (4.4) admit a set of $r-s$ linearly independent solutions; thus we have the following theorem.

THEOREM 5. *If a Riemann space admits a group $\mathfrak{G}_r$ of affine collineations a necessary and sufficient condition that it admit a group of motions is that the matrix $\|h^{(\sigma)}{}_{ij}\|$ be of rank less than r. If this rank is s, then the group of notions is of order $r-s$.*

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UNION CURVES AND UNION CURVATURE

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Introduction. A rectilinear congruence in ordinary three-dimensional Euclidean space may be defined by specifying the direction of a unique line at each point of a given surface. A *union curve*[1] on the surface relative to a given congruence has the property that its osculating plane at each point of the curve contains the line of the congruence through the point. It is well known that the union curves relative to the congruence of normals to a surface are the geodesic curves on the surface. The principal aim of this paper is to generalize for union curves certain known results concerning geodesic curves.

The analytical representation of the congruence in §1 is followed in §2 by the derivation of the differential equations of the union curves referred to an arbitrary system of coordinates on the surface. From the definition of the *union curvature vector* in §3, it is seen that a union curve on a surface may be defined as a curve for which the union curvature vector is a null vector at every point of the curve. Finally, there appears in §4 a geometric interpretation of the union curvature of a curve on a surface which agrees with the definition of geodesic curvature of the curve for the particular case of the congruence of normals to the surface.

The notation of Eisenhart[2] will be employed for the most part, although $\Gamma^{\alpha}_{\beta\gamma}$ will be used here as the Christoffel symbol of the second kind. Greek indices will always take the range 1, 2, and Latin indices the range 1, 2, 3. The summation convention of the tensor analysis will be observed.

1. **Analytical representation of the congruence.** Let the surface S be defined analytically with reference to an orthogonal cartesian system of coordinates by

$$x^i = x^i(u^1, u^2) \qquad (i = 1, 2, 3), \tag{1}$$

where the functions x^i and their partial derivatives to the second order are understood to be continuous at any point P on S. Let the line l of the congruence at P have direction cosines given by

$$\lambda^i = \lambda^i(u^1, u^2), \qquad \lambda^i\lambda^i = 1, \tag{2}$$

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[1] Sperry, *Properties of a certain projectively defined two-parameter family of curves on a general surface*, Amer. J. Math. vol. 40 (1918) p. 213.

[2] Eisenhart, *Differential geometry*, Princeton University Press, 1940.

where the functions λ^i are continuous at P. If X^i denote the direction cosines of the normal to S at P, the direction cosines of l at P may be written in the form

$$(3) \qquad \lambda^i = p^\alpha x^i_{,\alpha} + qX^i \qquad (q > 0),$$

where, for convenience, the notation of the covariant derivative $x^i_{,\alpha}$ of x^i with respect to the first fundamental tensor ($g_{\alpha\beta} \equiv x^i_{,\alpha} x^i_{,\beta}$) of S is used instead of $\partial x^i/\partial u^\alpha$. By virtue of the second of equations (2), there results

$$(4) \qquad \lambda^i \lambda^i \equiv (p^\alpha x^i_{,\alpha} + qX^i)(p^\beta x^i_{,\beta} + qX^i) \equiv g_{\alpha\beta} p^\alpha p^\beta + q^2 = 1.$$

If θ is the angle between l at P and the normal to S at P, it is seen from equation (3) that

$$(5) \qquad \cos\theta = \lambda^i X^i = q,$$

and from (4) that the magnitude of the vector with contravariant components p^α is $\sin\theta$. It may be noticed also that if C is any curve on S through P represented by $u^\alpha = u^\alpha(s)$, where s denotes arc length, then the cosine of the angle ϕ between l and the direction of the tangent to C at P is given by

$$(6) \qquad \cos\phi = \lambda^i \frac{dx^i}{ds} = (p^\alpha x^i_{,\alpha} + qX^i) x^i_{,\beta} \frac{du^\beta}{ds} = g_{\alpha\beta} p^\alpha \frac{du^\beta}{ds}.$$

2. **Differential equations of the union curves.** The osculating plane to the curve C: $u^\alpha = u^\alpha(s)$ on S at P has the determinantal equation

$$(7) \qquad \delta^{123}_{ijk}(\bar{x}^i - x^i) \frac{dx^j}{ds} \frac{d^2x^k}{ds^2} = 0,$$

the $\bar{x}^i$ being current coordinates. On writing

$$(8) \qquad \frac{dx^j}{ds} = x^j_{,\sigma} u'^\sigma, \qquad \frac{d^2x^k}{ds^2} = x^k_{,\tau} u''^\tau + \frac{\partial^2 x^k}{\partial u^\alpha \partial u^\beta} u'^\alpha u'^\beta,$$

where the primes indicate differentiation with respect to s, and, in turn, the Gauss equations of the surface S, namely,

$$(9) \qquad \frac{\partial^2 x^k}{\partial u^\alpha \partial u^\beta} = \Gamma^\gamma_{\alpha\beta} x^k_{,\gamma} + d_{\alpha\beta} X^k,$$

we see that equation (7) takes the form

$$(10) \qquad \delta^{123}_{ijk}(\bar{x}^i - x^i)(u'^\sigma x^j_{,\sigma})(\rho^\tau x^k_{,\tau} + d_{\alpha\beta} u'^\alpha u'^\beta X^k) = 0,$$

where ρ^τ, the components of the curvature vector[3] of C at P, are given by

$$(11) \qquad \rho^\tau \equiv u''^\tau + \Gamma^\tau_{\alpha\beta} u'^\alpha u'^\beta.$$

If the osculating plane to C at P contains line l, the coordinates $x^i + t\lambda^i \equiv x^i + t(p^\gamma x^i_{,\gamma} + qX^i)$ must satisfy equation (10) for all t. Thus, the condition

$$(12) \qquad \delta^{123}_{ijk}(p^\gamma x^i_{,\gamma} + qX^i)(u'^\sigma x^j_{,\sigma})(\rho^\tau x^k_{,\tau} + d_{\alpha\beta} u'^\alpha u'^\beta X^k) = 0$$

must obtain. Use of the facts that $\delta^{123}_{ijk} x^i_{,\gamma} x^j_{,\sigma} x^k_{,\tau} \equiv 0$ (because γ, σ, τ cannot all be different) and $\delta^{123}_{ijk} X^i x^j_{,\sigma} X^k \equiv 0$, together with a change of umbral indices, allows equation (12) to take the form

$$(13) \qquad (\delta^{123}_{ijk} X^i x^j_{,\sigma} x^k_{,\tau})(p^\sigma u'^\tau d_{\alpha\beta} u'^\alpha u'^\beta + q u'^\sigma \rho^\tau) = 0.$$

It is to be noticed here that $d_{\alpha\beta} u'^\alpha u'^\beta$ is the normal curvature K_n of the curve C with direction u'^α $(\alpha = 1, 2)$ on the surface S.

Summing on σ and τ in equation (13), and neglecting the nonzero determinant $\delta^{123}_{ijk} X^i x^j_{,1} x^k_{,2}$, we obtain

$$(14) \qquad e_{\sigma\tau}(p^\sigma u'^\tau K_n + q u'^\sigma \rho^\tau) = 0,$$

where[4] $e_{12} = 1$, $e_{21} = -1$, $e_{11} = e_{22} = 0$.

Equation (14) is the differential equation of second order of the union curves on the surface S through any point P on S, the parametric curves being any whatever. Lane[5] has given the differential equation of union curves on a metric surface for the case in which the lines of curvature on the surface are taken as parametric.

From equation (14) one may conclude that if $q = 0$ (which means that l lies in the tangent plane to S at P), then the only union curves are those in the directions given by $p^1 du^2 - p^2 du^1 = 0$ and by $K_n \equiv d_{\alpha\beta} u'^\alpha u'^\beta = 0$, the asymptotic directions. Suppose, henceforth, that $q \neq 0$, and let l^σ be written for p^σ / q. Then, if σ and τ be interchanged in the second term of equation (14), the differential equation of the union curves on S through P becomes

$$(15) \qquad e_{\sigma\tau} u'^\sigma (\rho^\tau - K_n l^\tau) = 0.$$

If the components ρ^τ of the curvature vector of the curve are zero, the curve is a geodesic. From equation (15), it can be seen that *the*

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[3] Eisenhart, loc. cit. p. 187.

[4] Eisenhart, loc. cit. p. 135.

[5] Lane, *Projective differential geometry of curves and surfaces*, University of Chicago Press, 1931, p. 240.

geodesics and union curves on a surface coincide in the three directions given by $p^2du^1 - p^1du^2 = 0$ *and the directions of the asymptotic curves on the surface.* Moreover, it may be seen from equation (15) that if the congruence is normal to the surface ($l^\sigma = 0$), the union curves are geodesic curves on S.

3. **Union curvature of a curve on a surface.** Equation (15) is a single differential equation of the second order. It will be shown to be equivalent to a pair of differential equations of the second order.

The curvature vector with components ρ^τ is orthogonal to the direction u'^α of C on S. Hence[6]

$$g_{\alpha\beta}\rho^\alpha u'^\beta = 0. \tag{16}$$

If equation (16), multiplied by u'^1, and equation (15), multiplied by $g_{2\beta}u'^\beta$, are subtracted, there results, by use of $g_{\alpha\beta}u'^\alpha u'^\beta = 1$, the first of the following differential equations of the union curves on S,

$$\begin{aligned} \eta^1 &\equiv \rho^1 - K_n g_{2\beta}u'^\beta \cdot e_{\sigma\tau}l^\sigma u'^\tau = 0, \\ \eta^2 &\equiv \rho^2 + K_n g_{1\beta}u'^\beta \cdot e_{\sigma\tau}l^\sigma u'^\tau = 0, \end{aligned} \tag{17}$$

and the second equation is obtained in a similar manner. The vector with components η^α given by (17) lies in the tangent plane to S at P. It may be called the *union curvature* vector of the curve C on S at P. It may be concluded from equations (17) that the *union curvature vector is a zero vector at each point of a union curve.*

The geodesic curvature[7] K_g of the curve C at P is given by

$$K_g \equiv \epsilon_{\alpha\beta}u'^\alpha\rho^\beta, \tag{18}$$

where $\epsilon_{\alpha\beta} = g^{1/2}e_{\alpha\beta}$. Therefore, it appears appropriate to define the *union curvature* K_u of C at P by

$$K_u \equiv \epsilon_{\alpha\beta}u'^\alpha\eta^\beta, \tag{19}$$

which may be written, by use of equations (16) and (17), in the form

$$K_u \equiv \epsilon_{\sigma\tau}u'^\sigma(\rho^\tau - K_n l^\tau) = K_g - K_n\epsilon_{\sigma\tau}u'^\sigma l^\tau. \tag{20}$$

It may be observed from equation (20) that the geodesic curvature along a union curve ($K_u \equiv 0$) is given by

$$K_g = K_n\epsilon_{\sigma\tau}u'^\sigma l^\tau. \tag{21}$$

From equation (15) it is seen that the coordinate curves on the surface are union curves if, and only if, the congruence is specified by

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[6] Eisenhart, loc. cit. p. 171.

[7] Eisenhart, loc. cit. p. 187.

$$(22)\qquad \overset{1}{l} = \Gamma^{1}_{22}/d_{22}, \qquad \overset{2}{l} = \Gamma^{2}_{11}/d_{11}.$$

For this particular congruence, the geodesic curvature of any union curve on S at P is given by

$$(23)\qquad K_g = g^{1/2}\left(\frac{\Gamma^{2}_{11}}{d_{11}}u'^1 - \frac{\Gamma^{1}_{22}}{d_{22}}u'^2\right)d_{\alpha\beta}u'^\alpha u'^\beta.$$

If K_{g_1}, K_{g_2} denote the geodesic curvatures of the curves through P represented by $du^2=0$, $du^1=0$, it can be seen from equation (23) that the sum of the geodesic curvatures of the coordinate curves on a surface is given by

$$(24)\qquad K_{g_1} + K_{g_2} = g^{1/2}[\Gamma^{2}_{11}(u'^1)^3 - \Gamma^{1}_{22}(u'^2)^3].$$

When the asymptotic curves are taken as parametric, the directions of Segre[8] at the point P on the surface are given by $\Gamma^{2}_{11}(u'^1)^3 - \Gamma^{1}_{22}(u'^2)^3 = 0$. Hence, it can be concluded that *the directions of Segre are those in which the geodesic curvatures of the asymptotic curves differ in sign.* This may be compared with the result[9] that the *torsions* of the asymptotic curves through *any* point differ in sign.

4. **Geometric interpretation of union curvature.** The geodesic curvature of a curve C at P on the surface is the curvature of the curve obtained by projecting the curve C orthogonally onto the tangent plane to the surface at P. Let the curve C be projected onto the tangent plane to the surface at P in the direction of the line l of the congruence. It will be shown that the curvature of the plane curve C' thus obtained is given by the expression in equation (20).

Let the cylindrical surface of projection of curve C onto the tangent plane at P in the direction of line l at P be denoted by S'. If $1/R$ is the normal curvature of C as a curve on S' at P, and α is the angle between the principal normal to C at P and the normal to the plane of l and the tangent line to C at P, then, by the theorem of Meusnier,

$$(25)\qquad e/R = \rho^{-1}\cos\alpha,$$

where $e = \pm 1$, and $1/\rho$ is the curvature of C at P. Further, if $1/r$ is the curvature of the curve C' at P, and if β is the angle between the tangent plane to S at P and the plane through the tangent line to C' normal to the plane of l and this tangent line, then, again by the theorem of Meusnier,

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[8] Lane, loc. cit. p. 77; also Springer, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 901–906.

[9] Eisenhart, loc. cit. p. 248.

$$e/R = r^{-1}\cos\beta. \tag{26}$$

Therefore, by equations (25) and (26)

$$e/r = \cos\alpha/(\rho\cos\beta). \tag{27}$$

Substitution of the analytical expressions for cos α and cos β into equation (27) yields, after some simplification,

$$e/r = \epsilon_{\sigma\tau}u'^{\sigma}(\rho^{\tau} - K_n l^{\tau}),$$

which is the union curvature K_u given in equation (20). Therefore, *the union curvature of a curve C at a point P on a surface S relative to a given congruence is the curvature of the curve obtained by projecting C onto the tangent plane to S at P in the direction of the line l of the congruence at P.*

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A NEW CHARACTERISTIC PROPERTY OF MINIMAL SURFACES

EDWARD KASNER AND JOHN DeCICCO

1. Isothermal families on a surface Σ defined by the Monge equation $z=f(x, y)$. We shall present a new characteristic property of minimal surfaces (Theorem 3). We also discover an extensive class of surfaces (Theorem 1), including minimal surfaces and surfaces of revolution.[1] It is shown that the minimal surfaces are the only ones such that every set of parallel planes intersects the surface in an isothermal family (except for the obvious case of a sphere).

In our preceding work,[2] we have obtained an extension of the theorem of Lie concerning isothermal families in the plane. We have found that a family of curves: $g(x, y)=\text{const.}$, on a surface Σ where any point is defined by general curvilinear coordinates (x, y), is isothermal if and only if the angle $\theta=\theta(x, y)$ between the family and the parametric curves $x=\text{const.}$ satisfies a certain partial differential equation of the second order in (x, y), and therefore if and only if $g(x, y)$ satisfies a certain partial differential equation of the third order in (x, y). Of course, if (x, y) are isothermal parameters on Σ, then our condition reduces to Lie's theorem which states that θ is a harmonic function of (x, y) if and only if the given family of curves is isothermal.

We have applied our result to the case in which the surface Σ is given by the Monge equation: $z=f(x, y)$, where (x, y, z) denote cartesian coordinates of space. The condition is

$$
\begin{aligned}
&(1+f_y^2)\frac{\partial^2\theta}{\partial x^2}-2f_xf_y\frac{\partial^2\theta}{\partial x\partial y}+(1+f_x^2)\frac{\partial^2\theta}{\partial y^2}\\
&\qquad-(1+f_x^2+f_y^2)^{-1}[(1+f_y^2)f_{xx}-2f_xf_yf_{xy}\\
(1)\qquad&\qquad+(1+f_x^2)f_{yy}]\left[f_x\frac{\partial\theta}{\partial x}+f_y\frac{\partial\theta}{\partial y}\right]+(1+f_x^2+f_y^2)^{1/2}\\
&\qquad\frac{\partial}{\partial y}\left\{\frac{f_x[(1+f_y^2)f_{xx}-2f_xf_yf_{xy}+(1+f_x^2)f_{yy}]}{(1+f_y^2)(1+f_x^2+f_y^2)}\right\}=0,
\end{aligned}
$$

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[1] The work of the present paper is concerned not only with real euclidean space but also with the complex euclidean space.

[2] Kasner and DeCicco, *An extension of Lie's theorem on isothermal families*, Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 44–50.

where $\theta=\theta(x, y)$ is the angle between the family of curves defined by the differential equation $dy/dx=p=p(x, y)$ and the parallel plane sections $x=\text{const.}$ The angle θ is given by the formula

$$\theta = \operatorname{arc\,cot} \frac{p(1+f_y^2)+f_xf_y}{(1+f_x^2+f_y^2)^{1/2}}. \tag{2}$$

The two preceding formulas possess some symmetry but not total symmetry since the surface Σ is defined by $z=f(x, y)$ and θ is the inclination to the parallel plane sections $x=\text{const.}$ These formulas may be put into totally symmetric form by letting θ represent the angle between the given family of curves defined by the differential equation of first order $dy/dx=p=p(x, y)$ and the parallel plane sections $z=\text{const.}$ Then formula (1) assumes the form

$$\begin{aligned}
&(1+f_y^2)\frac{\partial^2\theta}{\partial x^2} - 2f_xf_y\frac{\partial^2\theta}{\partial x\partial y} + (1+f_x^2)\frac{\partial^2\theta}{\partial y^2} \\
&\quad - (1+f_x^2+f_y^2)^{-1}[(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy}] \\
&\qquad \cdot\left[f_x\frac{\partial\theta}{\partial x} + f_y\frac{\partial\theta}{\partial y}\right] + (1+f_x^2+f_y^2)^{1/2}\left[f_y\frac{\partial}{\partial x} - f_x\frac{\partial}{\partial y}\right] \\
&\qquad \cdot\left\{\frac{(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy}}{(f_x^2+f_y^2)(1+f_x^2+f_y^2)}\right\} = 0.
\end{aligned} \tag{3}$$

The angle θ is given by the expression

$$\theta = \operatorname{arc\,tan} \frac{(f_x+pf_y)(1+f_x^2+f_y^2)^{1/2}}{f_y - pf_x}. \tag{4}$$

2. **Application to minimal surfaces and spheres.** Minimal surfaces may be defined as those for which the mean curvature is identically zero. Thus the surface Σ defined by the Monge equation, $z=f(x, y)$, is minimal if and only if f satisfies the partial differential equation of the second order

$$(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy} = 0. \tag{5}$$

By means of (1) or (3), and (5), it follows that a family of curves on the minimal surface Σ is isothermal if and only if the angle θ between the given family and any one of the parallel plane sections $x=\text{const.}$, $y=\text{const.}$, or $z=\text{const.}$ satisfies the partial differential equation of second order

$$(1+f_y^2)\frac{\partial^2\theta}{\partial x^2}-2f_xf_y\frac{\partial^2\theta}{\partial x\partial y}+(1+f_x^2)\frac{\partial^2\theta}{\partial y^2}=0. \tag{6}$$

If the condition (1) or (3) is independent of the first order partial derivatives of θ, then Σ must be a minimal surface. If in addition the coefficients of the partial derivatives of θ in (6) are proportional to real constants, then Σ is a plane. For θ to be a harmonic function of (x, y), we find that Σ must be a plane parallel to the (x, y)-plane.

Let us apply the result (3) to the sphere $x^2+y^2+z^2=r^2$. A family of curves defined by the differential equation $dy/dx=p=p(x, y)$ on this sphere is isothermal if and only if the angle θ between the family and the circles of latitude $z=$const. satisfies the partial differential equation of the second order

$$(r^2-x^2)\frac{\partial^2\theta}{\partial x^2}-2xy\frac{\partial^2\theta}{\partial x\partial y}+(r^2-y^2)\frac{\partial^2\theta}{\partial y^2}-2\left(x\frac{\partial\theta}{\partial x}+y\frac{\partial\theta}{\partial y}\right)=0. \tag{7}$$

The angle θ is given by

$$\theta=\arctan\frac{r(x+py)}{(y-px)(r^2-x^2-y^2)^{1/2}}. \tag{8}$$

If Σ is a minimal surface or a sphere, then every system of parallel plane sections is isothermal.

3. **Surfaces with an isothermal system of parallel plane sections.** By equation (3), it is observed that the system of parallel plane sections $z=$const. is isothermal if and only if

$$\frac{(1+f_y^2)f_{xx}-2f_xf_yf_{xy}+(1+f_x^2)f_{yy}}{(f_x^2+f_y^2)(1+f_x^2+f_y^2)}=\phi(f). \tag{9}$$

Let K_m denote the mean curvature of the surface Σ and let γ denote the angle between the normal of Σ and the common normal of any system of parallel plane sections. From (9), we deduce the following:

THEOREM 1. *A given system of parallel plane sections of a surface Σ is isothermal if and only if the product K_m cot γ csc γ is the same at all points of any one of the plane sections.*

Upon the plane sections of Theorem 1 we impose the additional condition that K_m be the same at all points of any one of these sections. In the first place K_m may be identically zero, in which case Σ must be a minimal surface. Next let $K_m\neq 0$. Then γ must be the same at all points of any one of these plane sections. Upon letting these plane sections be $z=\text{const.}$, then $\gamma=\gamma(f)$. Thus $\tan^2\gamma=f_x^2+f_y^2=\psi(f)$. We find from this that

$$(10) \qquad f_x f_{xx} + f_y f_{xy} = \psi_f f_x/2, \qquad f_x f_{xy} + f_y f_{yy} = \psi_f f_y/2.$$

Multiplying these by f_x and f_y respectively and adding the resulting products to the equivalent form of equation (9), obtained by clearing of fractions, we find

$$(11) \qquad (1+\psi)(f_{xx}+f_{yy}) = \phi\psi(1+\psi) + \psi\psi_f/2.$$

From this it follows that $(f_{xx}+f_{yy})$ must be a function of f alone since $\psi\neq -1$. Since $f_x^2+f_y^2=\psi(f)$, these results mean that the orthogonal projections of the parallel plane sections upon the (x, y)-plane must be a parallel and an isothermal family. These projections are then either concentric circles or parallel straight lines. From this we deduce that Σ must be either a surface of revolution with axis perpendicular to the (x, y)-plane or a cylinder with elements parallel to the (x, y)-plane. We have thus completed the proof of the following proposition.

THEOREM 2. *If the mean curvature K_m is the same along all the points of any one of a given isothermal system of parallel plane sections, then the surface Σ is either minimal, or it is a surface of revolution with axis perpendicular to the parallel plane sections, or it is a cylinder with elements parallel to the parallel plane sections.*

For the discussion in the remainder of our paper, it will be found convenient to introduce the quantity Δ defined by the expression

$$(12) \qquad \Delta = \frac{(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy}}{1+f_x^2+f_y^2}.$$

The mean curvature K_m of the surface Σ is given by the formula $K_m=\Delta\cos\gamma$, where γ represents the angle between the normal to Σ and the z-axis. Since K_m and γ are invariant under any motion in the (x, y)-plane, it is found that Δ remains unchanged under any motion in the (x, y)-plane.

For the surface Σ defined by the Monge equation $z=f(x, y)$, it is found from (1), (3), and (12) that the system of parallel plane sections $x=\text{const.}$ is isothermal if and only if $\Delta f_x/(1+f_y^2)=\phi(x)$; or the system of parallel plane sections $y=\text{const.}$ is isothermal if and only if $\Delta f_y/(1+f_x^2)=\phi(y)$; or the system of parallel plane sections $z=\text{const.}$ is isothermal if and only if $\Delta/(f_x^2+f_y^2)=\phi(f)$.

4. Surfaces for which every system of parallel plane sections is isothermal. As indicated in §2, if Σ is any minimal surface (including the plane) or any sphere, then every system of parallel plane sections is isothermal. *We shall show that this property is characteristic for the class of minimal surfaces and spheres.* The following result is obtained.

THEOREM 3. *If a surface Σ possesses more than four distinct isothermal systems of parallel plane sections, all the planes of which are parallel to a given rectilinear direction L, then Σ is a minimal surface or a sphere, and therefore every system of parallel plane sections is isothermal.*

The proof will occupy the next few paragraphs.

Let us choose the given rectilinear direction L as that of the z-axis. Then any system of parallel plane sections of the surface Σ defined by the Monge equation $z=f(x, y)$, all of which are parallel to the given rectilinear direction L, is given by the equation $y=mx+\text{const.}$

In the (x, y)-plane, let us apply the rotation

$$(13)\qquad (1+m^2)^{1/2}X = -mx+y, \qquad (1+m^2)^{1/2}Y = x+my.$$

The system of parallel plane sections $y=mx+\text{const.}$ in the new coordinate system (X, Y, z) is given by $X=\text{const.}$, and the surface Σ: $z=f(x, y)$ is defined by the equation $z=f(X, Y)$.

Thus if $y=mx+\text{const.}$ is an isothermal system of parallel plane sections of $z=f(x, y)$, then $X=\text{const.}$ is an isothermal system of parallel plane sections of $z=f(X, Y)$. Therefore it follows by (1) that $\Delta f_X/(1+f_Y^2)$, where in the definition (12) of Δ the small letters (x, y) are replaced by the capital letters (X, Y), is a function of X alone. That is, we must have

$$(14)\qquad f_X\Delta_Y = \Delta\left[-f_{XY}+\frac{2f_Xf_Yf_{YY}}{1+f_Y^2}\right].$$

By means of (13), we can express the partial derivatives of f and Δ with respect to X and Y in terms of those with respect to x and y. Of course Δ remains unchanged under the rotation (13). Eliminating the capital letters (X, Y) from (14) by means of (13), we obtain the result

$$(15)\qquad \begin{aligned}&[(1+f_y^2)m^2+2f_xf_ym+(1+f_x^2)][(\Delta f_{xy}+f_x\Delta_y)m^2\\ &\quad-\{(f_y\Delta_y-f_x\Delta_x)+\Delta(f_{yy}-f_{xx})\}m-(f_y\Delta_x+\Delta f_{xy})]\\ &\quad-2\Delta[f_xf_ym^2+(f_x^2-f_y^2)m-f_xf_y][f_{yy}m^2+2f_{xy}m+f_{xx}]=0.\end{aligned}$$

This is an equation of the fourth degree in m. Therefore, for any surface Σ, corresponding to any rectilinear direction L, there are at most four distinct isothermal systems of parallel plane sections, all of which are parallel to L.

Let a surface Σ possess more than four such distinct isothermal systems, all parallel to the given rectilinear direction L. Choosing L as the z-axis, it is seen that the preceding equation must be an identity in m. Hence the coefficients of the various powers of m must be zero.

In the first place if $\Delta=0$, the equation (15) is identically zero, and the surface Σ is a minimal surface. Henceforth we may assume $\Delta\neq 0$.

Upon setting the coefficient of m^4 and the term independent of m equal to zero, we find

$$(16)\quad f_x\Delta_y = \Delta\left[-f_{xy}+\frac{2f_xf_yf_{yy}}{1+f_y^2}\right],\quad f_y\Delta_x = \Delta\left[-f_{xy}+\frac{2f_xf_yf_{xx}}{1+f_x^2}\right].$$

These two equations state that $x=\text{const.}$ and $y=\text{const.}$ are both isothermal systems of parallel plane sections.

Neither f_x nor f_y can be identically zero. Let us suppose that $f_y=0$. Since $\Delta\neq 0$, we find $f_{xx}\neq 0$. Also $\Delta_y=0$. Under these conditions, the identity (15) in m leads to the conditions $f_x\Delta_x+\Delta f_{xx}=0$, $\Delta f_x^2 f_{xx}=0$, the last of which is impossible. Hence f_y can not be identically zero. A similar argument can be presented to show that f_x can not be identically zero.

From the above discussion, it is seen that we need to consider only the case where $\Delta\neq 0$, $f_x\neq 0$, and $f_y\neq 0$. Also in the imaginary domain, it is seen that $f_x^2\neq -1$ and $f_y^2\neq -1$, for otherwise these would give imaginary minimal surfaces.

For the remainder of our proof, we shall find it convenient to introduce the quantities S and T defined by

$$(17)\qquad S=\frac{f_{xx}}{1+f_x^2}-\frac{f_{xy}}{f_xf_y},\qquad T=\frac{f_{yy}}{1+f_y^2}-\frac{f_{xy}}{f_xf_y}.$$

We shall proceed to prove that, if $\Delta\neq 0$, the identity (15) will lead to the equations $S=T=0$, which are the partial differential equations defining any sphere in space.

Upon eliminating the partial derivatives of Δ from (15) by means of (16) and then setting the remaining coefficients of m in (15) equal to zero, we obtain

$$(18)\quad \begin{aligned} &(1+3f_x^2+f_y^2+3f_x^2f_y^2)S+(-1-2f_x^2-2f_y^2-f_y^4+2f_x^2f_y^2)T=0,\\ &\qquad\qquad\qquad (1-2f_x^2+f_y^2)S+(-1-f_x^2+2f_y^2)T=0,\\ &(1+2f_x^2+2f_y^2+f_x^4-2f_x^2f_y^2)S\\ &\qquad\qquad\qquad +(-1-f_x^2-3f_y^2-3f_x^2f_y^2)T=0. \end{aligned}$$

Multiplying the middle of the above equations by $-(1+f_x^2)$ and adding the result to the first of the preceding equations, we find

$$(19)\qquad (4f_x^2 + 2f_x^4 + 2f_x^2 f_y^2)S + (f_x^4 - f_y^4 - 4f_y^2)T = 0.$$

Again multiplying the middle of equation (18) by $-(1+f_y^2)$ and adding the result to the last of equations (18), we obtain

$$(20)\qquad (f_x^4 - f_y^4 + 4f_x^2)S - (4f_y^2 + 2f_x^2 f_y^2 + 2f_y^4)T = 0.$$

Subtracting (20) from (19), we discover that

$$(21)\qquad (f_x^2 + f_y^2)^2(S - T) = 0.$$

We shall prove that $S = T$. For otherwise we can get only the imaginary case of $f_y^2 = -f_x^2 \neq 0$. Substituting this into (18), we find

$$(22)\qquad \begin{aligned} (1 + 2f_x^2 - 3f_x^4)S + (-1 - 3f_x^4)T &= 0, \\ (1 - 3f_x^2)S + (-1 - 3f_x^2)T &= 0, \\ (1 + 3f_x^4)S + (-1 - 2f_x^2 + 3f_x^4)T &= 0. \end{aligned}$$

Adding the first and third equations and dividing the result by 2, we find $(1+f_x^2)(S-T)=0$. This is impossible since $f_x^2 \neq -1$ and $S \neq T$. This proves that $S = T$.

Next we shall show that $S = T = 0$. If that is not the case, the equations (18) lead us to the fact that $f_y = f_x = 0$. Since this is impossible, it follows that $S = T = 0$. Hence if the surface Σ is not a minimal surface, its equation $z = f(x, y)$ must satisfy the simultaneous partial differential equations

$$(23)\qquad \frac{f_{xx}}{1 + f_x^2} = \frac{f_{xy}}{f_x f_y} = \frac{f_{yy}}{1 + f_y^2}.$$

These are the partial differential equations of a sphere in general position. Thus the proof of Theorem 3 is complete.

5. **A property of the sphere.** The geodesics of a sphere are all great circles which are plane curves. We inquire if there are any other surfaces whose geodesics are all plane curves of nonzero curvature. In fact we prove the following precise result.

THEOREM 4. *If a surface Σ possesses more than $2\infty^1$ plane nonrectilinear geodesics, then Σ is a sphere.*

The proof of this result is as follows. The ∞^2 geodesics of the surface Σ defined by the Monge equation $z = f(x, y)$ are defined by the differential equation

$$(24)\qquad (1+f_x^2+f_y^2)y'' = (-f_y+y'f_x)(f_{xx}+2y'f_{xy}+y'^2f_{yy}),$$

whereas the ∞^3 plane curves of Σ are defined by the differential equation[3]

$$(25)\qquad \begin{aligned}&(f_{xx}+2y'f_{xy}+y'^2f_{yy})y'''\\&\quad = (f_{xxx}+3y'f_{xxy}+3y'^2f_{xyy}+y'^3f_{yyy})y''+3(f_{xy}+y'f_{yy})y''^2.\end{aligned}$$

Upon substituting (24) into (25), it is found that the geodesics which are plane curves must satisfy either the equation

$$(26)\qquad y'^2f_{yy}+2y'f_{xy}+f_{xx}=0$$

or the equation

$$(27)\qquad \begin{aligned}y'^2[(1+f_y^2)f_{xy}-f_xf_yf_{yy}]+y'[(1+f_y^2)f_{xx}-(1+f_x^2)f_{yy}]\\+[f_xf_yf_{xx}-(1+f_x^2)f_{xy}]=0.\end{aligned}$$

Those geodesics which satisfy (26) must be straight lines. The surfaces which have exactly the $2\infty^1$ straight lines defined by (26) are the ruled surfaces. If the roots of y' in (26) are equal, then only ∞^1 straight lines are in Σ and Σ is developable. Of course if Σ has more than $2\infty^1$ straight geodesics, it must be a plane.

If the geodesics are plane curves but not straight lines, they must satisfy the differential equation (27). Thus any surface Σ has at most $2\infty^1$ plane non-rectilinear geodesics.

If Σ has more than $2\infty^1$ such plane geodesics, then (27) must be an identity in y'. Setting the various coefficient of y' equal to zero, we obtain the partial differential equations (23) for a sphere. This completes the proof of our Theorem 4.[4]

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[3] Kasner, *Dynamical trajectories and the ∞^3 plane sections of a surface*, Proc. Nat. Acad. Sci. U.S.A. vol. 17 (1931) pp. 370–376.

[4] Compare this Theorem 4 with the near-collineation problem. See Kasner, *The characterization of collineations*, Bull. Amer. Math. Soc. vol. 9 (1903) pp. 545–546.

THE STRONG SUMMABILITY OF DOUBLE FOURIER SERIES

HAI-TSIN HSÜ

1. **Introduction.** Corresponding to the well known theorem of Fejér-Lebesgue, we have for the double Fourier series the following proposition:

If $f\log^+|f|$ is Lebesgue integrable on the square $(-\pi\leqq x\leqq\pi, -\pi\leqq y\leqq\pi)$, then the Fejér mean $\sigma_{m,n}(x, y)$ of $f(x, y)$ tends to $f(x, y)$ almost everywhere as m and n independently increase indefinitely. Moreover, for every increasing function $\phi(t)$ satisfying the conditions

$$\phi(0) = 0, \qquad \liminf_{t\to\infty} \frac{\phi(t)}{t\log t} = 0,$$

there is a function $f(x, y)$ such that $\phi(|f|)$ is integrable and that $\sigma_{m,n}(x, y)$ does not converge almost everywhere.[1]

The latter half of this theorem shows that the analogue, in double Fourier series, of the Fejér-Lebesgue theorem is not a trivial extension of that of a function of a single variable.

The purpose of the present note is to discuss the strong summability[2] of double Fourier series. A double series $\sum a_{mn}$ is said to be strongly summable with the positive index k if there exists a constant s such that the expression

$$\frac{1}{(m+1)(n+1)}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}|s_{\mu,\nu}-s|^k \tag{1.1}$$

has the double limit zero as m and n increase without limit, where

$$s_{m,n} = \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} a_{\mu\nu}.$$

It is easily seen from Hölder's equality that the summability says more for larger k.

Suppose now that $f(x, y)$ is integrable in the Lebesgue sense over

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[1] B. Jessen, J. Marcinkiewicz and A. Zygmund [5]. The first example of a function $f(x, y)\in L$ with Fejér mean divergent everywhere was given by A. Zygmund; see S. Saks [8]. Numbers in brackets refer to the Bibliography at the end of the paper.

[2] A notion first introduced in Fourier series by G. H. Hardy and J. E. Littlewood [1]. For subsequent researches, see Hardy and Littlewood [2, 3], J. Marcinkiewicz [6] and A. Zygmund [12].

the square Q $(-\pi, -\pi; \pi, \pi)$ and is doubly periodic with period 2π in each variable. The Fourier series of $f(x, y)$ is

$$\sum_{m,n=0}^{\infty} \lambda_{m,n}[a_{m,n} \cos mx \cos ny + b_{m,n} \sin mx \cos ny + c_{m,n} \cos mx \sin ny + d_{m,n} \sin mx \sin ny], \tag{1.2}$$

where

$$\lambda_{m,n} = \begin{cases} 1/4 & \text{for } m = n = 0; \\ 1/2 & \text{for } m = 0, n > 0 \text{ or } m > 0, n = 0; \\ 0 & \text{for } m > 0, n > 0; \end{cases}$$

and

$$a_{m,n} = \frac{1}{\pi^2}\int_Q\int f(x, y) \cos mx \cos ny dx dy,$$

and so on.

On writing

$$4\phi(u, v) \equiv \phi_{x,y}(u, v) = f(x + u, y + v) + f(x + u, y - v) + f(x - u, y + v) + f(x - u, y - v) - 4s,$$

and

$$\Phi_{x,y}^{(p)}(u, v) = \int_0^u \int_0^v |\phi(\xi, \eta)|^p d\xi d\eta \qquad (p \geqq 1)$$

the theorems obtained in this paper are as follows:

THEOREM I. *If $f(x, y) \in L^p$, $p > 1$, then the double Fourier series* (1.2) *is strongly summable to s for every positive index k whenever*[3]

$$\Phi_{x,y}^{(p)}(u, v) = o(uv). \tag{1.3}$$

THEOREM II. *If $f(x, y) \in L^p$, $p > 1$, then the Fourier series of $f(x, y)$ is strongly summable almost everywhere to $f(x, y)$ for every positive index k.*

The question whether the hypothesis in Theorem II may be replaced by $f \log^+ |f| \in L$ is unsettled in this note. Corresponding questions in Fourier series of a single variable have been answered affirmatively by Marcinkiewicz [6] and Zygmund [12]. Indeed, the theorem holds under the weaker hypothesis $f \in L$. We content ourselves with establishing the following theorem.

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[3] We use the symbol $o(uv)$ to denote a function of u and v such that $\lim_{u,v\to 0} o(uv)/uv = 0$.

THEOREM III. *If* $f(x, y) \log^+ |f(x, y)| \in L$ *and* $\sigma_{m,n}(x, y)$ *denotes the* (m, n)*th Fejér sum of the Fourier series of* $f(x, y)$, *then the relation*

$$\lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | \sigma_{\mu,\nu}(x, y) - f(x, y) |^k = 0$$

holds true almost everywhere, where $k>0$.

2. **Lemmas.** Before proving our theorems, we prove a number of lemmas:

LEMMA 1. *If* $f(x, y) \in L^p$, $p>1$, *then*

$$\lim_{h,k\to 0} \frac{1}{hk} \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} f(x, y)dxdy = f(x_0, y_0)$$

at almost every point (x_0, y_0).

This theorem is due to Zygmund [11]. Compare also [5] and [9].

LEMMA 2. *If* $f(x, y) \in L^p$, $p>1$, *then at almost every point* (x, y),

$$\int_0^h \int_0^k | f(x \pm u, y \pm v) - f(x, y) |^p dudv = o(hk)$$

as $h, k \to 0$.

PROOF. Let α be a rational number, and E_α the set of points (x, y) such that

$$\frac{1}{hk} \int_0^h \int_0^k | f(x \pm u, y \pm v) - \alpha |^p dudv$$

does not tend to $|f(x, y) - \alpha|^p$ as $h, k \to 0$. In virtue of Lemma 1, E_α is of measure zero, and so also is the sum E of all E_α. Let (x, y) be not a point of E and let β be a rational number, then, by Minkowski's inequality,

$$\begin{aligned}\left\{\frac{1}{hk} \int_0^h \int_0^k | f(x \pm u, y \pm v) - f(x, y) |^p dudv\right\}^{1/p} \\ \leqq \left\{\frac{1}{hk} \int_0^h \int_0^k | f(x \pm u, y \pm v) - \beta |^p dudv\right\}^{1/p} \\ + \left\{\frac{1}{hk} \int_0^h \int_0^k | \beta - f(x, y) |^p dudv\right\}^{1/p},\end{aligned}$$

which tends to $2|f(x, y) - \beta|$ as $h, k \to 0$. As $\beta \to f(x, y)$, the result follows.

LEMMA 3. *Let $f(x, y) \in L^p$, $1 < p \leqq 2$, $1/p + 1/q = 1$, and let the Fourier series of $f(x, y)$ be given in the complex form*:

$$f(x, y) \sim \sum_{\mu=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} c_{\mu,\nu} e^{i(\mu x+\nu y)},$$

then

$$\left\{\sum_{\mu=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} | c_{\mu,\nu} |^q\right\}^{1/q} \leqq \left\{\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} | f(x, y) |^p dx dy\right\}^{1/p}. \tag{2.1}$$

This is a double series analogue of the Young-Hausdorff theorem, and may be proved by the method of M. Riesz[4] with an obvious modification.

We also require the following formula of integration by parts:

$$\begin{aligned}\int_{a_1}^{a_2} du \int_{b_1}^{b_2} \rho \psi'(u)\psi''(v) du dv \\ = \rho_1(a_2, b_2)\psi(a_2, b_2) - \int_{a_1}^{a_2} \rho_1(u, b_2)\psi_u(u, b_2) du \\ - \int_{b_1}^{b_2} \rho_1(a_2, v)\psi_v(a_2, v) dv + \int_{a_1}^{a_2} du \int_{b_1}^{b_2} \rho_1 \psi_{uv} dv,\end{aligned} \tag{2.2}$$

where

$$\psi(u, v) = \psi'(u)\psi''(v), \qquad \rho_1(u, v) = \int_{a_1}^{u} d\sigma \int_{b_1}^{v} \rho(\sigma, t) dt.$$

This formula is valid if ρ is integrable on $(a_1, b_1; a_2, b_2)$, ψ' is absolutely continuous on (a_1, a_2), and ψ'' is absolutely continuous on (b_1, b_2).

3. **Proof of Theorem I.** Without loss of generality, we may assume that $x=0$, $y=0$. So that

$$s_{m,n} = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} f(u, v) \frac{\sin (m + 1/2)u}{\sin u/2} \frac{\sin (n + 1/2)v}{\sin v/2} du dv.$$

We have to deduce

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | s_{\mu,\nu} - s |^k = o(uv) \tag{3.1}$$

from (1.3).

Write

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[4] M. Riesz [7], see also A. Zygmund [13].

$$\pi^2(s_{\mu,\nu} - s) = \int_0^\pi \int_0^\pi \phi(u, v) \frac{\sin (m + 1/2)u}{\sin u/2} \frac{\sin (n + 1/2)v}{\sin v/2} dudv$$

$$= \int_0^\pi \int_0^\pi \phi(u, v) \Big(\sin \mu u \cot \frac{u}{2} \sin \nu v + \sin \mu u \cot \frac{u}{2} \cos \nu v$$

$$+ \cos \mu u \sin \nu v \cot \frac{v}{2} + \cos \mu u \cos \nu v \Big) dudv$$

$$= I_1(\mu, \nu) + I_2(\mu, \nu) + I_3(\mu, \nu) + I_4(\mu, \nu),$$

and for $0 < \mu \leqq m$, $0 < \nu \leqq n$,

$$I_i(\mu, \nu) = \int_0^{m^{-1}} \int_0^{n^{-1}} + \int_0^{m^{-1}} \int_{n^{-1}}^\pi + \int_{m^{-1}}^\pi \int_0^{n^{-1}} + \int_{m^{-1}}^\pi \int_{n^{-1}}^\pi$$

$$= I_{i1}(\mu, \nu; m, n) + I_{i2}(\mu, \nu; m, n)$$

$$+ I_{i3}(\mu, \nu; m, n) + I_{i4}(\mu, \nu; m, n),$$

where $i = 1, 2, 3$. For brevity, we also write $I_{ij}(\mu, \nu)$ for $I_{ij}(\mu, \nu; m, n)$. Accordingly,

$$\pi^2(s_{\mu,\nu} - s) = \sum_{i=1}^{3} \sum_{j=1}^{4} I_{ij}(\mu, \nu) + I_4(\mu, \nu).$$

It follows from Minkowski's inequality that

$$\pi^2 \left\{ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | s_{\mu,\nu} - s |^k \right\}^{1/k} \leqq \sum_{i=1}^{3} \sum_{j=1}^{4} \left\{ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | I_{ij}(\mu, \nu) |^k \right\}^{1/k} \tag{3.2}$$

$$+ \left\{ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | I_4(\mu, \nu) |^k \right\}^{1/k}.$$

In the first place, by the analogue of the Riemann-Lebesgue theorem[5] $I_4(\mu, \nu)$ tends to zero as $\mu, \nu \to \infty$. Hence

$$\left\{ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | I_4(\mu, \nu) |^k \right\}^{1/k} = o(mn)^{1/k}. \tag{3.3}$$

Secondly, let us consider the integrals I_{11}, I_{21} and I_{31}. Write

$$K(u, v) \equiv K(u, v; \mu, \nu) = \sin \mu u \cot u/2 \sin \nu v \cot v/2,$$

then for $0 < u \leqq \pi$ and $0 < v \leqq \pi$ there is a constant A such that

$$uv \max (| K |, \mu^{-1} | K_u |, \nu^{-1} | K_v |, \mu^{-1}\nu^{-1} | K_{uv} |) \leqq A. \tag{3.4}$$

We also write

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[5] W. H. Young [10, p. 138].

$$\Phi(u, v) = \int_0^u \int_0^v \phi(\xi, \eta) d\xi d\eta,$$

which is $o(uv)$ by (1.3). Then on applying (2.2),

$$\begin{aligned} I_{11}(\mu, \nu) &= \int_0^{m^{-1}} \int_0^{n^{-1}} \phi(u, v) K(u, v; \mu, \nu) du dv \\ &= \Phi(m^{-1}, n^{-1}) K(m^{-1}, n^{-1}) \\ &\quad - \int_0^{m^{-1}} \Phi(u, n^{-1}) K_u(u, n^{-1}) du \\ &\quad - \int_0^{n^{-1}} \Phi(m^{-1}, v) K_v(m^{-1}, v) dv \\ &\quad + \int_0^{m^{-1}} du \int_0^{n^{-1}} \Phi(u, v) K_{uv} dv. \end{aligned} \tag{3.5}$$

Since $0<\mu \leqq m$, $0<\nu \leqq n$, it is easily seen from (3.4) and (3.5) that

$$I_{11}(\mu, \nu) = o(1). \tag{3.6}$$

In a similar manner, we can prove $I_{21}(\mu, \nu)=o(1)$, $I_{31}(\mu, \nu)=o(1)$. Hence we obtain

$$\left\{ \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | I_{i1}(\mu, \nu) |^k \right\}^{1/k} = o(mn)^{1/k} \qquad (i = 1, 2, 3). \tag{3.7}$$

Thirdly, we consider the integrals I_{14}, I_{24} and I_{34}. We have

$$\begin{aligned} I_{14}(\mu, \nu) &= \int_{m^{-1}}^{\pi} \int_{n^{-1}}^{\pi} \phi(u, v) \sin \mu u \cot \frac{u}{2} \sin \nu v \cot \frac{v}{2} du dv \\ &= \int_{m^{-1}}^{\pi} \sin \mu u \cot \frac{u}{2} du \\ &\quad \cdot \int_{n^{-1}}^{\pi} \cot \frac{v}{2} \left(\frac{\partial}{\partial v} \int_0^v \sin \nu y \phi(u, y) dy \right) dv \\ &= \int_{m^{-1}}^{\pi} \sin \mu u \cot \frac{u}{2} du \left(- \cot \frac{1}{2n} \int_0^{n^{-1}} \sin \nu y \phi(u, y) dy \right. \\ &\quad \left. + \frac{1}{2} \int_{n^{-1}}^{\pi} \csc^2 \frac{v}{2} dv \int_0^v \sin \nu y \phi(u, y) dy \right) \\ &= I'_{14} + I''_{14}, \end{aligned} \tag{3.8}$$

say, where I'_{14} is equal to

$$- \cot\frac{1}{2n}\int_{m^{-1}}^{\pi}\cot\frac{u}{2}\left(\frac{d}{du}\int_{0}^{u}\int_{0}^{n^{-1}}\sin\mu x\sin\nu y\phi(x, y)dxdy\right)du$$

$$= \cot\frac{1}{2m}\cot\frac{1}{2n}\int_{0}^{m^{-1}}\int_{0}^{n^{-1}}\sin\mu x\sin\nu y\phi(x, y)dxdy$$

$$- \frac{1}{2}\cot\frac{1}{2n}\int_{m^{-1}}^{\pi}\csc^{2}\frac{u}{2}\,du\int_{0}^{u}\int_{0}^{n^{-1}}\sin\mu x\sin\nu y\phi(x, y)dxdy.$$

Let $c_{\mu,\nu}(\alpha, \beta)$ denote the (μ, ν)th Fourier coefficient of the odd-odd function $\chi(x, y)$ which is equal to $\phi(x, y)$ in the rectangle $(0, \alpha; 0, \beta)$ and to zero elsewhere. Then we may write

$$\begin{aligned} I'_{14} = {} & \frac{\pi^{2}}{4}\cot\frac{1}{2m}\cot\frac{1}{2n}c_{\mu,\nu}\left(\frac{1}{m}, \frac{1}{n}\right) \\ & - \frac{\pi^{2}}{8}\cot\frac{1}{2n}\int_{m^{-1}}^{\pi}\csc^{2}\frac{u}{2}c_{\mu,\nu}\left(u, \frac{1}{n}\right)du, \end{aligned} \tag{3.9}$$

and I''_{14} may be written as

$$\begin{aligned} & \frac{1}{2}\int_{m^{-1}}^{\pi}\cot\frac{u}{2}\,du \\ & \qquad \cdot\int_{n^{-1}}^{\pi}\csc^{2}\frac{v}{2}\left(\frac{\partial}{\partial u}\int_{0}^{u}\int_{0}^{v}\sin\mu x\sin\nu y\phi(x, y)dxdy\right)dv \\ & = \frac{\pi^{2}}{8}\int_{m^{-1}}^{\pi}\cot\frac{u}{2}\left(\frac{\partial}{\partial u}\int_{n^{-1}}^{\pi}\csc^{2}\frac{v}{2}c_{\mu,\nu}(u, v)dv\right)du \\ & = - \frac{\pi^{2}}{8}\cot\frac{1}{2m}\int_{n^{-1}}^{\pi}\csc^{2}\frac{v}{2}c_{\mu,\nu}\left(\frac{1}{m}, v\right)dv \\ & \quad + \frac{\pi^{2}}{16}\int_{m^{-1}}^{\pi}\int_{n^{-1}}^{\pi}\csc^{2}\frac{u}{2}\csc^{2}\frac{v}{2}c_{\mu,\nu}(u, v)dudv. \end{aligned} \tag{3.10}$$

It follows from (3.8), (3.9) and (3.10) that

$$\begin{aligned} & \left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\left|I_{14}(\mu, \nu)\right|^{k}\right)^{1/k} \\ & \quad \leqq A\cot\frac{1}{2m}\cot\frac{1}{2n}\left(\sum_{\mu=\nu}^{m}\sum_{\nu=0}^{n}\left|c_{\mu,\nu}\left(\frac{1}{m}, \frac{1}{n}\right)\right|^{k}\right)^{1/k} \\ & \quad\quad + A\cot\frac{1}{2n}\int_{m^{-1}}^{\pi}\csc^{2}\frac{u}{2}\left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}\left|c_{\mu,\nu}\left(u, \frac{1}{n}\right)\right|^{k}\right)^{1/k}du \end{aligned}$$

$$+ A \cot \frac{1}{2m} \int_{n^{-1}}^{\pi} \csc^2 \frac{v}{2} \left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \left| c_{\mu,\nu}\left(\frac{1}{m}, v\right) \right|^k \right)^{1/k} dv$$

$$+ A \cot \frac{1}{2m} \cot \frac{1}{2n} \int_{m^{-1}}^{\pi} \int_{n^{-1}}^{\pi} \csc^2 \frac{u}{2} \csc^2 \frac{v}{2}$$

$$\cdot \left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \left| c_{\mu,\nu}(u, v) \right|^k \right)^{1/k} dudv.$$

Now we assume, without loss of generality, that $k > 2$, $k' = k/(k-1) < p$, so that by Lemma 3,

$$\left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \left| c_{\mu,\nu}(u, v) \right|^k \right)^{1/k} \leqq \left(\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| \chi(x, y) \right|^{k'} dxdy \right)^{1/k'}$$

$$= \left(\frac{1}{4\pi^2} \int_{-u}^{u} \int_{-v}^{v} \left| \phi(x, y) \right|^{k'} dxdy \right)^{1/k'}$$

$$= o(uv)^{1/k'},$$

since the condition (1.3) is satisfied a fortiori when p is replaced by the smaller index k'. Therefore

$$\left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \left| I_{14}(\mu, \nu) \right|^k \right)^{1/k} \leqq Amno(mn)^{-1/k'} + An \int_{m^{-1}}^{\pi} \left(\frac{u}{n} \right)^{1/k'} \frac{du}{u^2}$$

$$+ Am \int_{n^{-1}}^{\pi} \left(\frac{v}{m} \right)^{1/k'} \frac{dv}{v^2}$$

$$+ A \int_{m^{-1}}^{\pi} \int_{n^{-1}}^{\pi} \frac{(uv)^{1/k'}}{u^2 v^2} dudv = o(mn)^{1/k}.$$

The integral $I_{24}(\mu, \nu)$ is equal to

$$\int_{m^{-1}}^{\pi} \cot \frac{u}{2} \left(\frac{d}{du} \int_{0}^{u} dx \int_{n^{-1}}^{\pi} \phi(x, y) \sin \mu x \cos \nu y dy \right) du$$

$$= \frac{\pi^2}{4} \cot \frac{1}{2m} \left[c'_{\mu,\nu}\left(\frac{1}{m}, \pi\right) - c'_{\mu,\nu}\left(\frac{1}{m}, \frac{1}{n}\right) \right]$$

$$- \frac{\pi^2}{8} \int_{m^{-1}}^{\pi} \csc^2 \frac{u}{2} \left[c'_{\mu,\nu}(u, \pi) - c'_{\mu,\nu}\left(u, \frac{1}{n}\right) \right] du,$$

where the $c'_{\mu,\nu}(\alpha, \beta)$ $(\mu, \nu = 0, 1, 2, \cdots)$ denote the Fourier coefficients of the odd-even function $\chi'(x, y)$ which is equal to $\phi(x, y)$ in the rectangle $(0, \alpha; 0, \beta)$ and to zero elsewhere. In virtue of Minkowski's inequality and Lemma 3, it is easily seen that

$$\left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \left| I_{24}(\mu, \nu) \right|^k \right)^{1/k} = o(mn)^{1/k}.$$

The integral I_{34} can be treated in the same manner as I_{14}. We omit the details. Collecting the above results, we obtain

$$\left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \left| I_{i4}(\mu, \nu) \right|^k \right)^{1/k} = o(mn)^{1/k} \qquad (i = 1, 2, 3). \tag{3.11}$$

Fourthly, we estimate the integrals I_{12}, I_{22} and I_{32}. We have

$$\begin{aligned} I_{12}(\mu, \nu) &= \int_0^{m^{-1}} \int_{n^{-1}}^{\pi} \phi(u, v) \sin \mu u \cot \frac{u}{2} \sin \nu v \cot \frac{v}{2} \, du dv \\ &= \int_0^{m^{-1}} \cot \frac{u}{2} \sin \mu u du \\ &\quad \cdot \int_{n^{-1}}^{\pi} \cot \frac{v}{2} \left[\frac{\partial}{\partial v} \int_0^{v} \phi(u, y) \sin \nu y dy \right] dv \\ &= \int_0^{m^{-1}} \cot \frac{u}{2} \sin \mu u \left\{ -\frac{\pi}{2} \cot \frac{1}{2n} c_\nu\left(u, \frac{1}{n}\right) \right. \\ &\quad \left. + \frac{\pi}{4} \int_{n^{-1}}^{\pi} \csc^2 \frac{v}{2} c_\nu(u, v) dv \right\} du, \end{aligned} \tag{3.12}$$

where $c_\nu(\alpha, \beta)$ denotes the νth Fourier coefficient of the odd function $\psi(u, \beta)$ which is equal to $\phi(u, v)$ for $0 \leqq v \leqq \beta$ and to zero for $\beta < v < \pi$. It follows from Young-Hausdorff's inequality that

$$\begin{aligned} \left(\sum_{\nu=1}^{n} \left| c_\nu(u, v) \right|^k \right)^{1/k} &\leqq \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \left| \psi(u, y) \right|^{k'} dy \right)^{1/k'} \\ &= \left(\frac{1}{\pi} \int_{-v}^{v} \left| \phi(u, y) \right|^{k'} dy \right)^{1/k'}, \end{aligned}$$

so that

$$\begin{aligned} &\left(\sum_{\nu=0}^{n} \left| I_{12}(\mu, \nu) \right|^k \right)^{1/k} \\ &\leqq A \cot \frac{1}{2n} \int_0^{m^{-1}} \mu \left(\int_{-n^{-1}}^{n^{-1}} \left| \phi(u, y) \right|^{k'} dy \right)^{1/k'} du \\ &\quad + A \int_0^{m^{-1}} \mu du \int_{n^{-1}}^{\pi} \csc^2 \frac{v}{2} \left(\int_{-v}^{v} \left| \phi(u, y) \right|^{k'} dy \right)^{1/k'} dv. \end{aligned} \tag{3.13}$$

Hölder's inequality gives

$$\int_0^{m^{-1}}\left(\int_{-v}^{v}|\phi(u,y)|^{k'}dy\right)^{1/k'}du$$
$$\leqq m^{1/k'-1}\left(\int_0^{m^{-1}}\int_{-v}^{v}|\phi(u,y)|^{k'}dydu\right)^{1/k'}=m^{1/k'-1}o\left(\frac{v}{m}\right)^{1/k'}.$$

Hence (3.13) is reduced to

$$\left(\sum_{\nu=0}^{n}|I_{12}(\mu,\nu)|^{k}\right)^{1/k}$$
$$\leqq \mu\cot\frac{1}{2n}m^{1/k'-1}o(mn)^{-1/k'}+\mu m^{1/k'-1}\int_{n^{-1}}^{\pi}v^{-2}o\left(\frac{v}{m}\right)^{1/k'}dv$$
$$=o(n)^{1/k}.$$

Thus we obtain $(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}|I_{12}(\mu,\nu)|^{k})^{1/k}=o(mn)^{1/k}$. The integrals I_{22} and I_{32} may be treated in a similar manner as above. The following relations are thus established:

$$\left(\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}|I_{i2}(\mu,\nu)|^{k}\right)^{1/k}=o(mn)^{1/k}\qquad(i=1,2,3).\tag{3.14}$$

Finally, we have to consider the integrals I_{13}, I_{23} and I_{33}. The discussion of I_{13} is the same as I_{12}, and the integral I_{23} has been treated implicitly in the discussion of I_{24}. It remains therefore only to deal with I_{33}. Regard the integrals

$$I_{33}(\mu,\nu)=\int_{m^{-1}}^{\pi}\cos\mu u du\int_0^{n^{-1}}\phi(u,v)\sin\nu v\cot\frac{v}{2}dv$$
$$(\mu=0,1,2,\cdots)$$

as the Fourier coefficients of the function of u which is equal to

$$\int_0^{n^{-1}}\phi(u,v)\sin\nu v\cot\frac{v}{2}dv\qquad\text{for } m^{-1}\leqq u\leqq\pi,$$

and to zero for $-\pi\leqq u<m^{-1}$, then by Hausdorff's inequality,

$$\left\{\sum_{\mu=0}^{m}|I_{33}(\mu,\nu)|^{k}\right\}^{1/k}$$
$$\leqq\left(\frac{1}{4\pi^2}\int_{m^{-1}}^{\pi}\left|\int_0^{n^{-1}}\sin\nu v\cot\frac{v}{2}\phi(u,v)dv\right|^{k'}du\right)^{1/k}\tag{3.15}$$
$$\leqq A_\nu\left(\int_{m^{-1}}^{\pi}\left(\int_0^{n^{-1}}|\phi(u,v)|dv\right)^{k'}du\right)^{1/k'}.$$

It follows from Hölder's inequality that

$$\int_0^{n^{-1}} |\phi(u, v)|\, dv \leqq n^{1/k'-1}\left(\int_0^{n^{-1}} |\phi(u, v)|^{k'}dv\right)^{1/k'}$$

so that

$$\nu n^{1/k'-1}\left(\int_{m^{-1}}^{\pi} du \int_0^{n^{-1}} |\phi(u, v)|^{k'}dv\right)^{1/k'} = \nu n^{1/k'-1}O(n^{-1/k'}) = O(1).$$

Hence from (3.15) it results that $\{\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}|I_{33}(\mu, \nu)|^{k}\}^{1/k}=o(mn)^{1/k}$. The following relations are thus proved:

$$\left\{\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} |I_{i3}(\mu, \nu)|^{k}\right\}^{1/k} = o(mn)^{1/k} \qquad (i = 1, 2, 3). \tag{3.16}$$

Collecting the results (3.2), (3.3), (3.7), (3.11), (3.14), and (3.16) we obtain (3.1). Theorem I is thus proved.

4. **Proof of Theorem II.** On account of Theorem I, it suffices to show that the condition (1.3) is satisfied almost everywhere when $s=f(x, y)$. Observing

$$\begin{aligned} 4|\phi_{x,y}(u, v)| \leqq\ & |f(x + u, y + v) - f(x, y)| \\ & + |f(x + u, y - v) - f(x, y)| \\ & + |f(x - u, y + v) - f(x, y)| \\ & + |f(x - u, y - v) - f(x, y)|, \end{aligned}$$

and employing Minkowski's inequality, we immediately obtain the desired result from Lemma 2.

5. **Proof of Theorem III.** The proof depends upon the following two lemmas:

LEMMA 4. *Theorem* III *holds good when* $f(x, y)$ *is bounded.*

Since a bounded function belongs to L^p, $p>1$, the lemma follows from Theorem II.

LEMMA 5. *Let* $h(x)$ *be a function such that* $h \log^+ |h| \in L\ (-\pi, \pi)$. *Let* $\beta_m=\beta_m(x, h)$ $(m=0, 1, 2, \cdots)$ *be the Fejér sums of the Fourier series of* $h(x)$, *and* $\beta^*(x)=\sup_m |\beta_m(x)|$, *then*

$$\int_{-\pi}^{\pi}\beta^*(x)dx \leqq A\int_{-\pi}^{\pi} |h| \log^+ |h|\, dx + B,$$

where A *and* B *are absolute constants.*

This lemma is due to Hardy and Littlewood [4]. See also [13, p. 248].

Before proving the theorem, we extend Lemma 5 to the case of two variables. Let, for fixed y,

$$g(x, y) = \sup_m \beta_m(x; |f|).$$

Integrating this equation with respect to y, we obtain

$$\tag{5.1} \begin{aligned} &\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(x, y)dxdy \\ &\qquad \leqq 2\pi A\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x, y)| \log^+ |f(x, y)| dxdy + 2\pi B. \end{aligned}$$

Writing $K_n(x)$ for the Fejér kernel, we have

$$\sigma_{m,n}(x, y; f) = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u, v)K_m(x - u)K_n(y - v)dudv.$$

It follows that

$$|\sigma_{m,n}(x, y; f)| \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} K_n(y - v)g(x, v)dv.$$

In virtue of Lebesgue's theorem, the last expression tends to $g(x, y)$ at almost every point (x, y). Therefore the relation

$$\sigma^*(x, y; f) = \limsup_{m, n\to\infty} |\sigma_{m,n}(x, y)| \leqq g(x, y)$$

holds good almost everywhere. Combining this result with (5.1) we obtain

$$\tag{5.2} \begin{aligned} &\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sigma^*(x, y; f)dxdy \\ &\qquad \leqq 2\pi A\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x, y)| \log^+ |f(x, y)| dxdy + 2\pi B. \end{aligned}$$

Let λ be a positive constant. Substituting λf for f in (5.2), we obtain

$$\tag{5.3} \begin{aligned} &\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sigma^*(x, y; f)dxdy \\ &\qquad \leqq 2\pi A\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x, y)| \log^+ |f(x, y)| dxdy + 2\pi \frac{B}{\lambda}\cdot \end{aligned}$$

Let ϵ be a positive number; we take λ so large that $2\pi B/\lambda < \epsilon/2$. Let

$$f(x, y) = f'(x, y) + f''(x, y)$$

be such that f' is bounded;

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} | f''(x, y) | \, dxdy < \epsilon, \tag{5.4}$$

$$2\pi A \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} | f''(x, y) | \; \log^{+} | \lambda f''(x, y) | \, dxdy + 2\pi \frac{B}{\lambda} \cdot \tag{5.5}$$

Applying the inequality (5.3) to the function $f''(x, y)$, we obtain

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sigma^*(x, y; f'')dxdy < \epsilon$$

by observing (5.5). Combining this relation with (5.4), we see that the set $E(\epsilon)$ of points (x, y) such that either $|f''(x, y)| > \epsilon^{1/2}$ or $\sigma^*(x, y; f) > \epsilon^{1/2}$ is of plane measure less than $2\epsilon^{1/2}$. Now let $\sigma'_{\mu,\nu}$ and $\sigma''_{\mu,\nu}$ denote respectively the (μ, ν)th Fejér sums of the Fourier series of f' and f'', then

$$\left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | \sigma_{\mu,\nu} - f |^{k} \right) \leqq \left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | \sigma'_{\mu,\nu} - f |^{k} \right)^{1/k}$$
$$+ \left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | \sigma''_{\mu,\nu} - f'' |^{k} \right)^{1/k}.$$

The first term on the right-hand side is $o(mn)^{1/k}$ almost everywhere, by Lemma 4. And

$$\left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | \sigma''_{\mu,\nu} - f'' |^{k} \right)^{1/k}$$
$$\leqq \left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | \sigma''_{\mu,\nu} |^{k} \right)^{1/k} + \left(\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | f'' |^{k} \right)^{1/k}$$
$$\leqq [(m + 1)(n + 1)]^{1/k}(\sigma^*(x, y; f'') + | f'' |).$$

Hence, outside the set $E(\epsilon)$,

$$\limsup_{m,n\to\infty} \left\{ \frac{1}{(m + 1)(n + 1)} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} | \sigma_{\mu,\nu} - f |^{k} \right\}^{1/k}$$
$$\leqq \sigma^*(x, y; f'') + | f'' | \leqq 2\epsilon^{1/2}.$$

Since ϵ is arbitrary, the theorem follows.

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THE CONTIGUOUS FUNCTION RELATIONS FOR ${}_pF_q$ WITH APPLICATIONS TO BATEMAN'S $J_n^{u,v}$ AND RICE'S $H_n(\zeta, p, v)$

EARL D. RAINVILLE

1. Introduction. If in the generalized[1] hypergeometric function

$$
\begin{aligned}
{}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; x) &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n} \frac{x^n}{n!}; \\
(\alpha)_n = \alpha(\alpha+1) \cdots (\alpha+n-1), &\quad (\alpha)_0 = 1,
\end{aligned}
\tag{1}
$$

one, and only one, of the parameters is increased or decreased by unity, the resultant function is said to be contiguous to the ${}_pF_q$ in (1). We restrict ourselves to the case $p \leqq q+1$, in order to insure a nonzero radius of convergence for the series in (1), and note that no β is permitted to be either zero or a negative integer.

For the ordinary hypergometric function ${}_2F_1$ Gauss[2] obtained fifteen relations each expressing ${}_2F_1$ linearly in terms of two of its six contiguous functions and with coefficients polynomials at most linear in x. Instead of the fifteen relations, it is often convenient to use a set of five linearly independent ones chosen from among them. The other relations all follow from the basic set.

Throughout this study the parameters stay fixed and the work is concerned only with the function ${}_pF_q$ and its contiguous functions. Hence, we are able to use an abbreviated notation illustrated by the following:

$$
\begin{aligned}
F &= {}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; x), \\
F(\alpha_1+) &= {}_pF_q(\alpha_1+1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; x), \\
F(\beta_1-) &= {}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1-1, \beta_2, \cdots, \beta_q; x).
\end{aligned}
$$

There are, of course, $(2p+2q)$ functions contiguous to F. Corresponding to Gauss' five independent relations in the case of ${}_2F_1$ there is for F a set of $(2p+q)$ linearly independent relations, which we shall obtain. The canonical form into which we put this basic set may be described as follows:

First, there are $(p+q-1)$ relations each containing F and two of its contiguous functions. These will be called the simple relations. Each

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[1] For an extensive treatment see W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tract No. 32, 1935.

[2] Gauss, *Werke*, vol. 3, p. 130.

simple relation connects F, $F(\alpha_1+)$, and $F(\alpha_k+)$ for $k=2, 3, \cdots, p$, or it connects F, $F(\alpha_1+)$, and $F(\beta_j-)$ for $j=1, 2, \cdots, q$. The simple relations are immediate extensions of two of Gauss' five relations and are not novel in any way.

Second, there are $(p+1)$ less simple relations each containing F and $(q+1)$ of its contiguous functions. In our canonical form we shall select these so that one of them connects F, $F(\alpha_1+)$, and all the functions $F(\beta_j+)$ for $j=1, 2, \cdots, q$. Each of the other p relations will contain F, all the functions $F(\beta_j+)$; $j=1, 2, \cdots, q$, and one of the functions $F(\alpha_k-)$ for $k=1, 2, \cdots, p$. The less simple relations are generalizations of three of Gauss' five relations but differ from them in one essential aspect in that each relation contains F and $(q+1)$ of its contiguous functions. For Gauss' case $q+1=2$ and the less simple relations contain the same number of contiguous functions as do the simple ones.

Since we shall actually exhibit the $(2p+q)$ relations, it will be evident upon looking at them that, just as in the case of the ordinary hypergeometric function, the coefficients are polynomials at most linear in x.

As examples of the use of the contiguous function relations, we shall obtain recurrence relations (not all previously known) for Bateman's[3] $J_n^{u,v}$ and for Rice's[4] $H_n(\zeta, p, v)$.

2. The number and type of relations. It is not difficult to determine, by a standard procedure[5] used in the case of the ordinary hypergeometric function, precisely how many contiguous function relations to expect and how many contiguous functions should appear in each relation. That procedure appears, however, to be ill-adapted to the actual determination of the coefficients. Hence, granted in advance the number and type of relations to be obtained, we shall get them by a simpler process.

3. Notations and preliminary formulas. It is convenient to use the following notations:

$$\prod_{s=1,(k)}^{m} A_s = \prod_{s=1}^{k-1} A_s \cdot \prod_{s=k+1}^{m} A_s, \tag{2}$$

a symbol denoting a product with a particular factor deleted,

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[3] H. Bateman, *Two systems of polynomials for the solution of Laplace's integral equation*, Duke Math. J. vol. 2 (1936) pp. 569–577.

[4] S. O. Rice, *Some properties of* ${}_3F_2(-n, n+1, \zeta; 1, p; v)$, Duke Math. J. vol. 6 (1940) pp. 108–119.

[5] E. G. C. Poole, *Linear differential equations*, Oxford, 1936, pp. 92–93.

$$U_j = \frac{\prod_{s=1}^{p}(\alpha_s - \beta_j)}{\beta_j \prod_{s=1,(j)}^{q}(\beta_s - \beta_j)}, \tag{3}$$

$$W_{j,k} = \frac{\prod_{s=1,(k)}^{p}(\alpha_s - \beta_j)}{\beta_j \prod_{s=1,(j)}^{q}(\beta_s - \beta_j)}, \tag{4}$$

$$c_n = \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n}, \tag{5}$$

$$S_n = \frac{(\alpha_1 + n)(\alpha_2 + n) \cdots (\alpha_p + n)}{(\beta_1 + n)(\beta_2 + n) \cdots (\beta_q + n)}, \tag{6}$$

$$\tau_{n,k} = \frac{S_n}{\alpha_k + n}, \tag{7}$$

$$A = \sum_{s=1}^{p} \alpha_s, \tag{8}$$

$$B = \sum_{s=1}^{q} \beta_s. \tag{9}$$

An examination of (5) and (6) shows that

$$c_{n+1} = S_n c_n. \tag{10}$$

The relation $\alpha(\alpha+1)_n = (\alpha+n)(\alpha)_n$, together with the definitions of the contiguous functions, yields the formulas:

$$F = \sum_{n=0}^{\infty} \frac{c_n x^n}{n!},$$

$$F(\alpha_k+) = \sum_{n=0}^{\infty} \frac{\alpha_k + n}{\alpha_k} \frac{c_n x^n}{n!}, \quad F(\alpha_k-) = \sum_{n=0}^{\infty} \frac{\alpha_k - 1}{\alpha_k + n - 1} \frac{c_n x^n}{n!}, \tag{11}$$

$$F(\beta_k+) = \sum_{n=0}^{\infty} \frac{\beta_k}{\beta_k + n} \frac{c_n x^n}{n!}, \quad F(\beta_k-) = \sum_{n=0}^{\infty} \frac{\beta_k + n - 1}{\beta_k - 1} \frac{c_n x^n}{n!}.$$

4. **The $(p+q-1)$ simple relations.** Using the operator $\theta = x(d/dx)$, we see that

$$(\theta + \alpha_k)F = \sum_{n=0}^{\infty} (\alpha_k + n) \frac{c_n x^n}{n!}.$$

Hence, with the aid of (11),

$$(\theta + \alpha_k)F = \alpha_k F(\alpha_k +); \qquad k = 1, 2, \cdots, p. \tag{12}$$

Similarly, it follows that

$$(\theta + \beta_k - 1)F = (\beta_k - 1)F(\beta_k -); \qquad k = 1, 2, \cdots, q. \tag{13}$$

The $(p+q)$ equations (12) and (13) lead at once by elimination of θF to $(p+q-1)$ linear algebraic relations between F and pairs of its contiguous functions. Let us use $F(\alpha_1 +)$ as an element in each equation. The result is the set of simple relations,

$$(\alpha_1 - \alpha_k)F = \alpha_1 F(\alpha_1 +) - \alpha_k F(\alpha_k +); \qquad k = 2, 3, \cdots, p, \tag{14}$$

and

$$(\alpha_1 - \beta_k + 1)F = \alpha_1 F(\alpha_1 +) - (\beta_k - 1)F(\beta_k -); \qquad k = 1, 2, \cdots, q. \tag{15}$$

5. A relation involving $(q+1)$ contiguous functions. From

$$F = \sum_{n=0}^{\infty} \frac{c_n x^n}{n!},$$

it follows that

$$\theta F = \sum_{n=1}^{\infty} \frac{n c_n x^n}{n!} = x \sum_{n=0}^{\infty} \frac{c_{n+1} x^n}{n!}.$$

Thus, because of (10) and (6),

$$\theta F = x \sum_{n=0}^{\infty} S_n \frac{c_n x^n}{n!},$$

where

$$S_n = \frac{(\alpha_1 + n)(\alpha_2 + n) \cdots (\alpha_p + n)}{(\beta_1 + n)(\beta_2 + n) \cdots (\beta_q + n)}.$$

Now, if $p < q$, then the degree of the numerator of S_n is lower than the degree of the denominator and the elementary theory of rational fraction expansions yields

$$S_n = \sum_{j=1}^{q} \frac{\beta_j U_j}{\beta_j + n}, \qquad p < q,$$

in which the U_j is as defined in (3).

Therefore,

$$\theta F = x \sum_{n=0}^{\infty} \sum_{j=1}^{q} \frac{\beta_j U_j}{\beta_j + n} \frac{c_n x^n}{n!},$$

which in view of (11) becomes

$$\theta F = x\sum_{j=1}^{q} U_j F(\beta_j +). \tag{16}$$

The elimination of θF using (16) and the case $k=1$ of (12) leads to

$$\alpha_1 F = \alpha_1 F(\alpha_1 +) - x\sum_{j=1}^{q} U_j F(\beta_j +), \qquad p < q. \tag{17}$$

If $p=q$, the degree of the numerator of S_n equals that of the denominator. However, when $p=q$,

$$S_n = 1 + \frac{\prod_{s=1}^{p}(\alpha_s + n) - \prod_{s=1}^{q}(\beta_s + n)}{\prod_{s=1}^{q}(\beta_s + n)},$$

in which the fraction on the right has the desired property that its numerator is of lower degree than its denominator. Thus

$$S_n = 1 + \sum_{j=1}^{q} \frac{\beta_j U_j}{\beta_j + n}, \qquad p = q,$$

and it is easy to see that in this case

$$\theta F = xF + x\sum_{j=1}^{q} U_j F(\beta_j +),$$

so that (17) is replaced by the relation

$$(\alpha_1 + x)F = \alpha_1 F(\alpha_1 +) - x\sum_{j=1}^{q} U_j F(\beta_j +), \qquad p = q. \tag{18}$$

If $p=q+1$, then with the notation of (8) and (9) we may write

$$S_n = n + A - B + \frac{\prod_{s=1}^{p}(\alpha_s + n) - (n + A - B)\prod_{s=1}^{q}(\beta_s + n)}{\prod_{s=1}^{q}(\beta_s + n)},$$

in which the fraction on the right has the desired rational fraction expansion, so that

$$S_n = n + A - B + \sum_{j=1}^{q} \frac{\beta_j U_j}{\beta_j + n}.$$

Thus we conclude that, when $p=q+1$,

$$\theta F = x\theta F + (A - B)xF + x\sum_{j=1}^{q} U_jF(\beta_j +),$$

so that (17) is this time to be replaced by the relation

$$\begin{aligned}[(1 - x)\alpha_1 + (A - B)x]F = (1 - x)\alpha_1F(\alpha_1 +)\\ - x\sum_{j=1}^{q} U_jF(\beta_j +); \quad p = q + 1.\end{aligned} \tag{19}$$

6. **The remaining p relations.** Since

$$(\alpha_k - 1)_n = \frac{(\alpha_k - 1)(\alpha_k)_n}{\alpha_k + n - 1},$$

it is possible to write

$$\theta F(\alpha_k -) = \sum_{n=1}^{\infty} \frac{\alpha_k - 1}{\alpha_k + n - 1} \frac{nc_nx^n}{n!},$$

or

$$\theta F(\alpha_k -) = x\sum_{n=0}^{\infty} \frac{\alpha_k - 1}{\alpha_k + n} \frac{c_{n+1}x^n}{n!}.$$

Now, with the notation of (7),

$$\frac{c_{n+1}}{\alpha_k + n} = c_n\tau_{n,k},$$

where $\tau_{n,k}$ has, for $p\leqq q$, its numerator of lower degree than its denominator. Thus

$$\theta F(\alpha_k -) = (\alpha_k - 1)x\sum_{n=0}^{\infty} \tau_{n,k}\frac{c_nx^n}{n!},$$

where

$$\tau_{n,k} = \sum_{j=1}^{q} \frac{\beta_jW_{j,k}}{\beta_j + n}$$

in which the $W_{j,k}$ is as defined in (4).

We now have

$$\theta F(\alpha_k -) = (\alpha_k - 1)x\sum_{j=1}^{q} W_{j,k}F(\beta_j +); \quad p \leqq q;\ k = 1, 2, \cdots, p.$$

But, from (12),

$$\theta F(\alpha_k -) = (\alpha_k - 1)[F - F(\alpha_k -)].$$

The elimination of $\theta F(\alpha_k -)$ from the above two formulas yields the p relations:

$$(20)\qquad F = F(\alpha_k -) + x\sum_{j=1}^{q} W_{j,k}F(\beta_j +); \qquad p \leq q;\ k = 1, 2, \cdots, p.$$

When $p=q+1$ the fraction $\tau_{n,k}$ has its numerator and denominator of equal degree. Then we write

$$\tau_{n,k} = 1 + \frac{\prod_{s=1,(k)}^{p}(\alpha_s + n) - \prod_{s=1}^{q}(\beta_s + n)}{\prod_{s=1}^{q}(\beta_s + n)}$$

and conclude in the same manner as before that

$$\theta F(\alpha_k -) = (\alpha_k - 1)xF + (\alpha_k - 1)x\sum_{j=1}^{q} W_{j,k}F(\beta_j +).$$

Therefore, for $p=q+1$, (20) is to be replaced by

$$(21)\qquad (1 - x)F = F(\alpha_k -) + x\sum_{j=1}^{q} W_{j,k}F(\beta_j +); \qquad k = 1, 2, \cdots, p.$$

7. **Summary of contiguous function relations.** We have shown that for

$$F(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; x)$$

in which no two β's are equal and no β is a nonpositive integer, a canonical set of $(2p+q)$ contiguous function relations is as described below.

If $p<q$, (14), (15), (17), and (20) hold, with

$$U_j = \frac{\prod_{s=1}^{p}(\alpha_s - \beta_j)}{\beta_j \prod_{s=1,(j)}^{q}(\beta_s - \beta_j)},$$

and

$$W_{j,k} = \frac{U_j}{\alpha_k - \beta_j}.$$

If $p=q$, the relations are (14), (15), and (20) above together with (18) to replace (17).

If $p=q+1$, the relations are (14) and (15) above together with (19), in which

$$A = \sum_{s=1}^{p} \alpha_s, \qquad B = \sum_{s=1}^{q} \beta_s,$$

and (21).

8. **Application to Bateman's $J_n^{u,v}$.** Bateman[6] has discussed the polynomials $x^{-u}J_n^{u,v}$ defined as

$$\frac{\Gamma(v+u/2+n+1)}{n!\Gamma(u+1)\Gamma(v+u/2+1)} F(-n; u+1, v+u/2+1; x^2).$$

Since a ${}_1F_2$ is involved, we may write down four relations using (15), (17), and (20) above with $\alpha_1=-n$, $\beta_1=u+1$, $\beta_2=v+u/2+1$ and with x replaced by x^2. These four relations may be translated into recurrence formulas for the $J_n^{u,v}$. When a redundancy is eliminated, the contiguous function relations are seen to be equivalent to the set

$$J_n^{u,v} = J_{n-1}^{u,v} + J_n^{u,v-1}, \qquad uJ_n^{u,v-1/2} = xJ_n^{u-1,v} + xJ_{n-1}^{u+1,v},$$

and

$$(n+u)J_n^{u,v} = (n+v+u/2)J_{n-1}^{u,v} + xJ_n^{u-1,v+1/2}. \tag{22}$$

Bateman gives the first two of these relations, but (22) seems to be new.

The $J_n^{u,v}$ satisfy a pure recurrence relation,

$$\begin{aligned} n(n+u)J_n^{u,v} = {} & [(n+v+u/2)(n+u) \\ & + (n-1)(2n+v+3u/2-1) - x^2]J_{n-1}^{u,v} \\ & - (n+v+u/2-1)(3n+v+3u/2-3)J_{n-2}^{u,v} \\ & + (n+v+u/2-1)(n+v+u/2-2)J_{n-3}^{u,v}, \end{aligned} \tag{23}$$

which may be obtained from the three relations given above it or, more easily, directly from the series expression for $J_n^{u,v}$.

9. **Applications to Rice's $H_n(\zeta, p, v)$.** S. O. Rice[7] has studied the polynomials

$$H_n(\zeta, p, v) = F(-n, n+1, \zeta; 1, p; v).$$

Here a ${}_3F_2$ is involved so that eight contiguous function relations appear. Since in the ${}_3F_2$ one parameter is fixed and two others are related, some of the relations are not pertinent; that is, they relate H_n to other functions. All that comes out of the eight relations is the set of four:

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[6] Loc. cit. p. 575.

[7] Loc. cit. p. 108.

$$\begin{aligned}(24)\quad &(\zeta - p + 1)H_n(\zeta, p, v)\\ &\quad = \zeta H_n(\zeta + 1, p, v) - (p - 1)H_n(\zeta, p - 1, v),\end{aligned}$$

$$\begin{aligned}(25)\quad &(\zeta + n)H_n(\zeta, p, v) + (\zeta - n)H_{n-1}(\zeta, p, v)\\ &\quad = \zeta H_n(\zeta + 1, p, v) + \zeta H_{n-1}(\zeta + 1, p, v),\end{aligned}$$

$$\begin{aligned}(26)\quad &p[2\zeta - 1 - (\zeta + p - 1)v]H_n(\zeta, p, v)\\ &\quad = p\zeta(1 - v)H_n(\zeta + 1, p, v) + p(\zeta - 1)H_n(\zeta - 1, p, v)\\ &\qquad + (n + p)(n - p + 1)vH_n(\zeta, p + 1, v),\end{aligned}$$

$$\begin{aligned}(26\text{a})\quad &pH_n(\zeta, p, v) + pH_{n-1}(\zeta, p, v)\\ &\quad = (p + n)H_n(\zeta, p + 1, v) + (p - n)H_{n-1}(\zeta, p + 1, v).\end{aligned}$$

The polynomials H_n satisfy the pure recurrence relation

$$\begin{aligned}(27)\quad &n(2n - 3)(p + n - 1)H_n\\ &\quad = (2n - 1)[(n - 2)(p - n + 1) + 2(n - 1)(2n - 3)\\ &\qquad - 2(2n - 3)(\zeta + n - 1)v]H_{n-1}\\ &\qquad - (2n - 3)[2(n - 1)^2 - n(p - n + 1)\\ &\qquad + 2(2n - 1)(\zeta - n + 1)v]H_{n-2}\\ &\qquad - (n - 2)(2n - 1)(p - n + 1)H_{n-3},\end{aligned}$$

as can be verified directly.

10. A set of relations alternative to those in §6. There are times, as in treating Rice's H_n, when it is desirable to have contiguous function relations which omit some one of the $F(\beta_j+)$. Such a set is obtained by making the following replacements, in which

$$V_{j,k} = (\beta_q - \beta_j)W_{j,k}.$$

If $p<q$, replace (20) by

$$(28)\quad (2\alpha_k - \beta_q)F = \alpha_k F(\alpha_k +) + (\alpha_k - \beta_q)F(\alpha_k -) - x\sum_{j=1}^{q-1} V_{j,k}F(\beta_j +); \qquad k = 1, 2, \cdots, p.$$

If $p=q$, replace (20) by

$$\begin{aligned}(29)\quad &(2\alpha_k - \beta_q + x)F = \alpha_k F(\alpha_k +) + (\alpha_k - \beta_q)F(\alpha_k -)\\ &\qquad - x\sum_{j=1}^{q-1} V_{j,k}F(\beta_j +); \qquad k = 1, 2, \cdots, p.\end{aligned}$$

If $p=q+1$, replace (21) by

$$\begin{aligned}&[(2\alpha_k-\beta_q)(1-x)+(A-B)x]F\\ &\qquad=\alpha_k(1-x)F(\alpha_k+)+(\alpha_k-\beta_q)F(\alpha_k-)\\ &\qquad\quad-x\sum_{j=1}^{q-1}V_{j,k}F(\beta_j+);\qquad k=1,2,\cdots,p.\end{aligned}\tag{30}$$

University of Michigan

ON THE GROWTH OF THE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

A. ROSENBLATT

In a recent paper,[1] N. Levinson gave four theorems concerning the behaviour of the solutions of the differential equation of elastic vibrations

$$d^2x/dt^2+\phi(t)x=0\tag{1}$$

as $t\to+\infty$. It is the purpose of this note to give generalizations of the Theorems I and III of Levinson by making use of certain inequalities concerning homogeneous equations of the first order

$$\frac{dx_i}{dt}+\sum_{k=1}^{n}a_{ik}x_k=0,\qquad i=1,\cdots,n.\tag{2}$$

Theorems I and III of Levinson run as follows:

Theorem I. *If* $\alpha(t)$ *denotes the integral*

$$\alpha(t)=\int_0^t\left|\phi(t)-c^2\right|dt,\tag{3}$$

then

$$x(t)=O\{\exp(\alpha(t)/2c)\}.\tag{4}$$

Theorem III. *If* $\alpha(t)$ *is* $O(t)$ *then*

$$\limsup_{t\to\infty}\left|x(t)\exp(\alpha(t)/2c)\right|>0.\tag{5}$$

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[1] *The growth of the solutions of a differential equation*, Duke Math. J. vol. 8 (1941) pp. 1–11.

The Polish mathematician Z. Butlewski in a paper written in Polish[2] sets

$$r = \left(\sum_1^n x_i^2\right)^{1/2}, \tag{6}$$

$$\phi(t) = \sum_1^n a_{ii}\left(\frac{x_i}{r}\right)^2 + \sum_{i,k=1}^n (a_{ik} + a_{ki}) \frac{x_i x_k}{r^2}, \tag{7}$$

where $i \neq k$, $i < k$, and obtains immediately from (2)

$$r = C \exp\left(-\int_{t_0}^t \phi(\tau) d\tau\right). \tag{8}$$

Setting

$$\alpha_{ii} = \int_{t_0}^t | a_{ii} | d\tau, \qquad \beta_{ij} = \int_{t_0}^t | a_{ij} + a_{ji} | d\tau, \tag{9}$$

we have

$$r \leqq C \exp\left(\sum_1^n \alpha_{ii} + \frac{1}{2} \sum_{i,j=1;\, i<j}^n \beta_{ij}\right) \tag{10}$$

and from this we have the following theorem.

THEOREM XVIII OF BUTLEWSKI. *If* $\alpha_{ii} < +\infty$ *and* $\beta_{ij} < +\infty$, *all the systems* x_i, $i = 1, \cdots, n$, *of solutions of* (2) *are bounded.*

With the designation

$$M(t) = \max \int_{t_0}^t | a_{ij} | d\tau, \tag{11}$$

we have

$$r \leqq C \exp(nM(t)). \tag{12}$$

We can complete Butlewski's theorem by remarking that

$$r \geqq C \exp\left(-\sum_1^n \alpha_{ii} - \frac{1}{2} \sum_{i,j=1;\, i<j}^n \beta_{ij}\right), \tag{13}$$

and

$$r \geqq C \exp(-nM(t)). \tag{14}$$

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[2] O całkach rzeczywistych równań różniczkowych zwyczajnych, Wiadomości Matematyczne vol. 44 (1937) pp. 17–81.

In the particular case $n=2$ Butlewski introduces polar coordinates

$$x_1 = \rho \cos \phi, \qquad x_2 = \rho \sin \phi \tag{15}$$

and obtains

$$\rho = C \exp\left(-\int_{t_0}^{t} \{a_{11} \cos^2 \phi + (a_{12} + a_{21}) \sin \phi \cos \phi + a_{22} \sin^2 \phi)\} d\tau\right). \tag{16}$$

The maximum of $F(\phi) = -\{a_{11} \cos^2 \phi + (a_{12}+a_{21}) \sin \phi \cos \phi + a_{22} \sin^2 \phi\}$ is

$$2^{-1}\{-a_{11} - a_{22} + ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2}\},$$

so that

$$\rho \leqq C \exp\left(\frac{1}{2}\int_{t_0}^{t} \{-a_{11} - a_{22} + ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2}\} d\tau\right). \tag{17}$$

Thus we obtain the following theorem.

THEOREM XIX OF BUTLEWSKI. *x_1, x_2 are limited if*

$$\frac{1}{2}\int_{t_0}^{t} \{-a_{11} - a_{22} + ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2}\} d\tau) \tag{18}$$

is limited.

In the same manner we obtain

$$\rho \geqq C \exp\left(\frac{1}{2}\int_{t_0}^{t} \{-a_{11} - a_{22} - ((a_{11} - a_{22})^2 + (a_{12} + a_{21})^2)^{1/2}\} d\tau\right), \tag{19}$$

completing Butlewski's results.

The linear differential equation

$$x'' + \psi(t)x' + \phi(t)x = 0 \tag{20}$$

with continuous ϕ and ψ can be transformed into a system (2) by setting

$$x' = \lambda x_2, \qquad x = x_1, \tag{21}$$

$\lambda \neq 0$, continuous in t. We obtain the system

$$\frac{dx_1}{dt} - \lambda x_2 = 0, \qquad \frac{dx_2}{dt} + \frac{\phi}{\lambda} x_1 + \left(\frac{\lambda'}{\lambda} + \psi\right) x_2 = 0, \tag{22}$$

$a_{11}=0$, $a_{12}=-\lambda$, $a_{21}=\phi/\lambda$, $a_{22}=\lambda'/\lambda+\psi$.

The inequalities (17), (19) now take the form

$$\rho \leqq C \exp\left(\frac{1}{2}\int_{t_0}^{t}\left\{-\frac{\lambda'}{\lambda} - \psi + \left(\left(\frac{\lambda'}{\lambda}+\psi\right)^2 + \left(\frac{\phi}{\lambda}-\lambda\right)\right)^{2\,1/2}\right\}d\tau\right), \tag{23}$$

$$\rho \geqq C \exp\left(\frac{1}{2}\int_{t_0}^{t}\left\{-\frac{\lambda'}{\lambda} - \psi - \left(\left(\frac{\lambda'}{\lambda}+\psi\right)^2 + \left(\frac{\phi}{\lambda}-\lambda\right)\right)^{2\,1/2}\right\}d\tau\right). \tag{24}$$

Taking $\lambda = c$, Butlewski obtains

$$\rho \leqq C \exp\left(\frac{1}{2}\int_{t_0}^{t}\left\{-\psi + \left(\psi^2 + \left(\frac{\phi}{c}-c\right)\right)^{2\,1/2}\right\}d\tau\right) \tag{25}$$

and we can add the inequality

$$\rho \geqq C \exp\left(\frac{1}{2}\int_{t_0}^{t}\left\{-\psi - \left(\psi^2 + \left(\frac{\phi}{c}-c\right)\right)^{2\,1/2}\right\}d\tau\right). \tag{26}$$

We shall now generalize Levinson's Theorem III for the equation (20).

We can suppose $x>0$, $x'<0$, otherwise we should have an infinite number of values $t=t_i$, $i=1, 2, \cdots$; $t_i \to \infty$, with

$$|x(t_i)| \geqq C \exp\left(\frac{1}{2}\int_{t_0}^{t}\left\{-\psi - \left(\psi^2 + \left(\frac{\phi}{c}-c\right)\right)^{2\,1/2}\right\}d\tau\right). \tag{27}$$

Consider the intervals $n \leqq t \leqq n+1$. We have

$$x(n+1) - x(n) = \int_n^{n+1} x'(t)\,dt,$$

and denoting by x_n' the maximum of $x'(t)$ in $\langle n, n+1\rangle$,

$$x(n) \geqq -x_n', \qquad x_n' = x'(t_n), \qquad n \leqq t_n \leqq n+1.$$

We have

$$x'(n) - x_n' = \int_{t_n}^{n} x''(t)dt = -\int_{t_n}^{n} (\phi x + \psi x')dt,$$

$$\begin{aligned} | x'(n) | &\leqq x(n) + x(n) \int_{n}^{n+1} | \phi | \, dt + | \psi |_{\max} x(n) \\ &= x(n) \left\{ 1 + \int_{n}^{n+1} | \phi | \, dt + | \psi |_{\max} \right\}. \end{aligned}$$

$|\psi|_{\max}$ is the maximum of $|\psi|$ in $n \leqq t \leqq n+1$.

We obtain

$$x(n) \geqq \frac{C \exp \left(\frac{1}{2} \int_{t_0}^{t} \left\{ -\psi - \left(\psi^2 + \left(\frac{\phi}{c} - c \right)^2 \right)^{1/2} \right\} d\tau \right)}{1 + \frac{1}{c^2} \left(1 + | \psi |_{\max} + \int_{n}^{n+1} | \phi | \, dt \right)}. \tag{28}$$

The result is the following theorem.

THEOREM. *There exist infinite values of* $t = t_i$, $t_i \to \infty$, *for which we have* (27) *if the following conditions are satisfied*:

1. $\alpha(t)$ *is of order* $O(t)$.
2. $|\psi|$ *is bounded.*

UNIVERSITY OF SAN MARCOS

SOME REMARKS ON THE MEASURABILITY OF CERTAIN SETS

PAUL ERDÖS

The present note contains some elementary remarks on sets defined by simple geometric properties. Our main tool will be the Lebesgue density theorem.

First we introduce a few notations: $d(a, b)$ denotes the distance from a to b and $S(x, r)$ the open sphere of center x and radius r. A point x of a set A is said to be of metric density 1 if to every ϵ there exists a δ such that $A \cap S(x, r)$, $r < \delta$, has measure greater than $(1-\epsilon)$ times the volume of $S(x, r)$. $\bar{A}$ denotes the closure of A.

(1) Let E be any closed set in n-dimensional euclidean space. Denote by E_r the set of points whose distance from E is r $(r>0)$. We shall prove that E_r has measure 0.

The set E_r is clearly closed and therefore measurable. If it had positive measure it would contain a point of metric density 1. Let x be any point of E_r and $y \in E$ be one of the points in E at distance r from x. Then $S(y, r)$ cannot contain any point of E_r. Thus x cannot be a point of metric density 1, which completes the proof. This proof is due to T. Radó.

(2) Let A be any set of measure 0 on the positive real axis. Denote by E_A the set of points whose distance from E is in A. We shall show that E_A has measure 0. As is well known A is contained in a G_δ, say G of measure 0. Thus it suffices to show that E_G has measure 0. E_G is clearly a G_δ and thus measurable, so that again it will suffice to show that E_G has no point of metric density 1. Let x be any point of E_G and y any one of the points of E closest to it. Denote by $C_x(\eta_1, \eta_2)$ the half cone defined as follows: $z \in C_x(\eta_1, \eta_2)$ if $d(z, x) < \eta_1$ and the angle zxy is less than η_2. Let R be any ray in C_x from x. Denote by z a variable point of R. We assert that if η_1 and η_2 are sufficiently small, $d(z, E)$ is a decreasing function of $d(z, x)$ for which the upper limit of the difference quotient with respect to $d(z, x)$ is less than $-\delta$, with some $\delta > 0$. Let $y_1 \in E$ be one of the points closest to z in E. We assert that $d(y, y_1)$ is small if η_2 is small. Clearly by definition y_1 is contained in $\{S(z, d(z, y))\}$ but not in $S(x, d(x, y))$. Since $d(x, z) < \eta_1$ the difference of these two spheres has small diameter if η_2 is small, which shows that $d(y, y_1)$ is small. Now it is geometrically clear that for sufficiently small η_1, η_2 there exists a $\delta > 0$ such that the upper limit of the difference quotient of $d(z, y_1)$ with respect to $d(z, x)$ is less

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than $-\delta$. A fortiori the upper limit of the difference quotient of $d(z, E)$ with respect to $d(z, x)$ is less than $-\delta$. Thus it follows that the set of points on R for which $d(z, E)$ is in A is of measure 0. Thus by a trivial modification of Fubini's theorem we obtain that $C_x \cap E_g$ has measure 0. Thus x could not have been a point of metric density 1, which completes our proof.

Let S be any measurable set on the positive real axis; it is easy to see that E_S is also measurable. For S can be written as $F+A$ where F is an F_σ and A is of measure 0. Now clearly $E_S=E_F+E_A$. E_F is measurable since it is also an F_σ and E_A is of measure 0. Therefore, E_S is measurable.

(3) Denote by M the set of those points for which there is more than one closest point in E. It is known that the necessary and sufficient condition for E to be convex is that M be empty. We shall prove that M has measure 0.

For $x \in M$, denote by $\phi(x)$ the set of points closest to x. Clearly the set of points M_c for which the diameter of $\phi(x)$ is not less than c is closed, thus M is an F_σ and thus measurable. It suffices to show that M_c has measure 0, or that it can have no point of metric density 1. Let $y \in \phi(x)$ be arbitrary ($\phi(x)$ is of course closed). Define $C_x(\eta_1, \eta_2)$ as in (2). We shall prove that no point of $C_x(\eta_1, \eta_2)$ (except x) belongs to M_c and this will show that x cannot have metric density 1. If $z \in M_c \cap C_x(\eta_1, \eta_2)$ there exists a sphere $S(z, r)$, $r \leqq d(z, y)$, such that $S(z, r)$ contains no points of E in its interior and $\{S(z, r)\}$ contains two points u and v of E with $d(u, v) \geqq c$. But u and v cannot be in the interior of $S(x, d(x, y))$. Hence they must be in $[\text{Comp}\,(S(x, d(x, y)))] \cap \{S(z, r)\}$ (Comp A denotes the complement of A), but for $\eta_2=\eta_2(c)$ small enough the diameter of this set is less than c, which is a contradiction. This completes the proof.

The problems in (1) and (3) were suggested to me by Deane Montgomery.

(4) Let x be any point in the complement of E. As before we denote by $\phi(x)$ the set of points in E closest to x. We shall prove that $\sum_{x \notin E} \phi(x)$ has measure 0.

It will be sufficient to prove that no point $z \in \sum_{x \notin E} \phi(x)$ has upper metric density 1.[1] If $z \in \phi(x)$ then $S(x, d(x, z))$ contains no point of E in its interior (and $\sum_{x \notin E} \phi(x) \subset E$), which proves our theorem.

(5) Denote by M_k the set of points for which $\phi(x)$ contains k points not all in a $(k-2)$-dimensional euclidean subspace. In (3) we proved that M_2 has n-dimensional measure 0. I conjecture that M_k

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[1] Let E be any set. Then the upper metric density is 1 at almost all points of E. (See, for example, Hildebrandt, Bull. Amer. Math. Soc. vol. 32 (1926) p. 451.)

has Hausdorff dimension $n+1-k$. At present I can prove this only for $k=n+1$. In fact we shall prove that M_{n+1} is denumerable. For the sake of simplicity we shall restrict ourselves to $n=2$. The proof for the general case is not an easy generalization of the case $n=2$, but we omit details.

Suppose then that M_3 is nondenumerable. Then it must contain a point of condensation, x say. Put $r=d(x, E)$. There exist nondenumerably many points z such that $r-\epsilon<d(z, E)<r+\epsilon$, and $S(z, d(z, E))$ contains at least three points of E on its boundary. $S(z, d(z, E))\cap E$ is closed. Denote by t_z the maximum of the smallest side of all possible triangles formed from points of $S(z, d(z, E))\cap E$. By a well known argument there exists a constant $c>0$ such that for every $\delta>0$ there are uncountably many points z satisfying

$$d(z, x) < \epsilon, \qquad c \leqq t_z < c + \delta. \tag{1}$$

Choose δ small, and consider $\bigcup_z S(z, d(z, E))$ with z satisfying (1). Denote the boundary of this domain by B. Let p be any point of B and denote by $C_p(\eta)$ the half cone whose vertex is at p and whose center line is the extension of the line from x to p. It is easy to see that for sufficiently small ϵ there exists an $\eta>0$ such that for any point p on B, $C_p(\eta)$ does not contain any point of B other than p. From this it can be shown by straightforward methods that B is a rectifiable curve[2] and hence can contain only countably many arcs of circles. This we shall show to be false. Let z_1 be any point satisfying (1). Denote by (a, b) the arc on $S(z_1, d(z_1, E))$ determined by the side of length t_{z_1}. Since we can choose z_1 in uncountably many ways, we can assume that z_1 has been chosen so that the arc (a, b) does not lie on B. But since $a\in B$ and $b\in B$ there must exist a point z_2 satisfying (1) such that $S(z_2, (d(z_2, E))$ intersects $S(z_1, d(z_1, E))$ in two points u and v on the arc (a, b). Therefore if δ is a sufficiently small fraction of c,

$$t_{z_2} < c$$

which shows that z_2 does not satisfy (1), an evident contradiction. This completes the proof.

(6) In (1) we proved that E_r has n-dimensional measure 0. Let us now assume that E is bounded, then we shall sketch a proof of the fact that E_r has finite $(n-1)$-dimensional measure.

Let D be the diameter of E. Assume first that r is large. Let x be a fixed point of E and p any point of E_r. Then it is easy to see that $C_p(\eta)$

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[2] Pauc, J. Reine Angew. Math. vol. 185 (1943) pp. 127–128. Pauc proves a more general theorem.

does not contain any point of E_r other than p. $C_p(\eta)$ is defined as in (5). From this it can be shown that E_r has finite $(n-1)$-dimensional measure. Let us not assume now that r is large. We then write $E=\bigcup_{k=1}^{m}E^{(k)}$ where the E^k's are closed and their diameter is less than ϵ. Then, by what has been shown before, if ϵ is small enough $E_r^{(k)}$ has finite $(n-1)$-dimensional measure. Clearly $E_r\subset\bigcup_{k=1}^{m}E_r^{(k)}$. But E_r is closed, therefore its $(n-1)$-dimensional measure exists, and it clearly can not be 0, since it separates the space. Thus E_r must have finite $(n-1)$-dimensional measure.

Added in proof. The author has recently discovered that the following two theorems have been stated by C. Pauc, Revue Scientifique vol. 77 (1939) no. 8: Let the set E be in the plane then M_2 is contained in the sum of countably many Jordan curves and M_3 is countable.

UNIVERSITY OF MICHIGAN

REMARK ON TAYLOR'S FORMULA

PHILIP HARTMAN

Taylor's formula

$$f(a) = \sum_{k=0}^{n-1} f^{(k)}(0)a^k/k! + f^{(n)}(\xi)a^n/n!, \qquad 0 < \xi < a, \tag{1}$$

is usually proved under the assumptions that

(I) $f(x)$ is continuous on the closed interval $[0, a]$;

(II) $f(x)$ possesses $n-1$ derivatives on the half closed interval $[0, a)$;

(III) $f^{(n-1)}(x)$ is continuous at $x=0$; and

(IV) $f(x)$ has an nth derivative on the open interval $(0, a)$.

In the case $n=1$, the assumption (III) that $f^{(0)}(x)\equiv f(x)$ be continuous at $x=0$ is essential but is contained in condition (I). In the case $n>1$, it will be shown below that the assumption (III) is entirely superfluous, so that (1) *is valid whenever* (I), (II) *and* (IV) *hold.*

The proof of (1) is usually reduced to an application of the mean value theorem to the $(n-1)$th derivative of $f(x)$ on an interval $[0, c]$, $0<c<a$. Thus, to prove the italicized statement, it is sufficient to show that if $f(x)$, defined on the interval $[0, a]$, is the derivative of a function and $f(x)$ itself possesses a derivative on the open interval $(0, a)$, then there exists a number ξ such that

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$$f(a) - f(0) = af'(\xi) \quad \text{and} \quad 0 < \xi < a.$$

Without loss of generality, it may be supposed that $f(a) = f(0) = 0$. It must then be proved that there exists a number ξ, $0 < \xi < a$, such that $f'(\xi) = 0$. Now if $f(x)$ is identically zero, nothing remains to be proved. Suppose, therefore, that for some η, $0 < \eta < a$, $f(\eta) \neq 0$. Since the function $f(x)$ is a derivative and, consequently, assumes every value between $0 = f(0)$ and $f(\eta) \neq 0$ at least once on the interval $(0, \eta)$, there exists a number x_1, such that

$$f(x_1) = f(\eta)/2 \quad \text{and} \quad 0 < x_1 < \eta.$$

Similarly, there exists a number x_2 such that

$$f(x_2) = f(\eta)/2 \quad \text{and} \quad \eta < x_2 < a.$$

Since $f(x)$ possesses a derivative on $(0, a)$, it is continuous on the interval $[x_1, x_2]$, and so by the mean value theorem, there exists a number ξ such that

$$0 = f(x_1) - f(x_2) = (x_2 - x_1)f'(\xi) \quad \text{and} \quad x_1 < \xi < x_2.$$

This completes the proof.

QUEENS COLLEGE

THE COEFFICIENTS OF UNIVALENT FUNCTIONS

M. S. ROBERTSON

1. Introduction. Let the function

$$f(z) = c_0 + c_1z + c_2z^2 + \cdots + c_nz^n + \cdots, \qquad c_n \text{ real}, \tag{1.1}$$

be regular and convex in the direction of the imaginary axis for $|z| < 1$. Thus each circle $|z| = r$, $0 < r < 1$, is mapped by $f(z)$ into a contour C_r which has the property that straight lines parallel to the imaginary axis cut C_r in at most two points. Since the coefficients are all real, C_r is symmetric about the real axis. For

$$f(re^{i\theta}) = U(r, \theta) + iV(r, \theta)$$

we have $\partial U(r, \theta)/\partial\theta \leqq 0$ for $0 < \theta < \pi$. In other words, $zf'(z)$ is typically real for $|z| < 1$. It is well known [1, 2][1] that the coefficients c_n are bounded, $|c_n| \leqq |c_1|$, $n = 1, 2, \cdots$, and [3] have the representation

$$c_n = \frac{c_1}{n\pi} \int_0^\pi \frac{\sin n\theta}{\sin \theta}\, d\alpha(\theta) \tag{1.2}$$

where $\alpha(\theta)$ is a nondecreasing function of θ in $(0, \pi)$ normalized so that

$$\frac{1}{\pi}\int_0^\pi d\alpha(\theta) = 1.$$

A sufficient condition that $f(z)$, given by the series (1.1), be regular and convex in the direction of the imaginary axis for $|z| < 1$ is that the sequence $\{c_n\}$ be monotonic of order 4, a theorem due to L. Fejér [4]. A sequence $\{c_n\}$ is said to be monotonic of order p if each of the differences

$$\Delta^{(k)}c_n = c_n - C_{k,1}c_{n+1} + C_{k,2}c_{n+2} - \cdots + (-1)^k C_{k,k}c_{n+k} \tag{1.3}$$

are non-negative for $k = 0, 1, 2, \cdots, p$; $n = 0, 1, 2, \cdots$. This sufficiency test implies, among other inequalities, that $0 \leqq c_n - c_{n+1}$. This suggests the problem of finding an upper bound for $c_n - c_{n+1}$ for functions $f(z)$ given by (1.1) which are convex in the direction of the imaginary axis for $|z| < 1$. The example $c_1z(1+z)^{-1}$ shows that the upper bound $2|c_1|$ is sharp. However, if we consider the differences $c_{n-1} - c_{n+1}$ we obtain an inequality which is not so immediately obvious. This inequality is stated in the following theorem.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

THEOREM A. *Let the function*

$$f(z) = z + c_2z^2 + \cdots + c_nz^n + \cdots$$

be regular and convex in the direction of the imaginary axis for $|z| < 1$ *and be real on the real axis. Then the coefficients* c_n *satisfy the inequalities*

$$c_{n-1} - c_{n+1} \leqq 4n(n^2 - 1)^{-1}(1 - |c_n|), \qquad n = 2, 4, 6, \cdots; c_1 = 1;$$

$$|c_{n-1} - c_{n+1}| \leqq 4n(n^2 - 1)^{-1}(1 - c_n), \qquad n = 3, 5, 7, \cdots.$$

The factor $4n(n^2-1)^{-1}$ *cannot be replaced by a smaller one and the equality signs are attained for the function* $z(1-z)^{-1}$.

2. **Some trigonometric inequalities.** For the proof of Theorem A we need the trigonometric inequality to follow.

LEMMA 1. *If n is a positive integer then for all real values of* θ

$$(2.1) \qquad (n + 1) \frac{\sin (n - 1)\theta}{\sin \theta} - (n - 1) \frac{\sin (n + 1)\theta}{\sin \theta} \leqq 4\left(n - \frac{\sin n\theta}{\sin \theta}\right).$$

PROOF. We consider first the case where n is an odd positive integer.

$$\begin{aligned} 8 \sum_{k=1}^{(n-1)/2} & \left[\cos\frac{n\theta}{2} - \cos (2k - 1) \frac{\theta}{2}\right]^2 \\ &= (4n - 4) \cos^2\frac{n\theta}{2} - 16 \cos\frac{n\theta}{2} \cdot \sum_{1}^{(n-1)/2} \cos (2k - 1) \frac{\theta}{2} \\ &\quad + 8 \sum_{1}^{(n-1)/2} \cos^2 (2k - 1) \frac{\theta}{2} \\ &= (4n - 4) \cos^2\frac{n\theta}{2} - 8 \cos\frac{n\theta}{2} \cdot \frac{\sin ((n - 1)\theta/2)}{\sin (\theta/2)} \\ &\quad + 8\left[\frac{n - 1}{4} + \frac{\sin (n - 1)\theta}{4 \sin \theta}\right] \\ &= (2n - 2)(1 + \cos n\theta) - 4\left[\frac{\sin ((2n - 1)\theta/2)}{\sin (\theta/2)} - 1\right] \\ &\quad + 2n - 2 + 2 \frac{\sin (n - 1)\theta}{\sin \theta} \end{aligned}$$

$$= 4n + (2n - 2)\cos n\theta + 4\left[\frac{\sin(n-1)\theta}{\sin\theta} - \frac{\sin((2n-1)\theta/2)}{\sin(\theta/2)}\right]$$

$$- 2\frac{\sin(n-1)\theta}{\sin\theta}$$

$$= 4n + (2n - 2)\cos n\theta - 4\frac{\sin n\theta}{\sin\theta} - 2\frac{\sin(n-1)\theta}{\sin\theta}$$

$$= 4\left(n - \frac{\sin n\theta}{\sin\theta}\right) - (n+1)\frac{\sin(n-1)\theta}{\sin\theta} + (n-1)\frac{\sin(n+1)\theta}{\sin\theta}.$$

Again, if n is an even integer we have

$$4\left(1 - \cos\frac{n\theta}{2}\right)^2 + 8\sum_{k=1}^{(n-2)/2}\left[\cos\frac{n\theta}{2} - \cos k\theta\right]^2$$

$$= \left(4 - 8\cos\frac{n\theta}{2} + 4\cos^2\frac{n\theta}{2}\right) + (4n - 8)\cos^2\frac{n\theta}{2}$$

$$- 16\cos\frac{n\theta}{2}\sum_{1}^{(n-2)/2}\cos k\theta + 8\sum_{1}^{(n-2)/2}\cos^2 k\theta$$

$$= 4 - 8\cos\frac{n\theta}{2} + 2(1 + \cos n\theta) + (2n - 4)(1 + \cos n\theta)$$

$$- 16\cos\frac{n\theta}{2}\left[-\frac{1}{2} + \frac{\sin((n-1)\theta/2)}{2\sin(\theta/2)}\right]$$

$$+ 8\left[\frac{n-3}{4} + \frac{\sin(n-1)\theta}{4\sin\theta}\right]$$

$$= 4n - 4 + (2n - 2)\cos n\theta - 8\cos\frac{n\theta}{2}\cdot\frac{\sin((n-1)\theta/2)}{\sin(\theta/2)}$$

$$+ 2\frac{\sin(n-1)\theta}{\sin\theta}$$

$$= 4\left(n - \frac{\sin n\theta}{\sin\theta}\right) - (n+1)\frac{\sin(n-1)\theta}{\sin\theta} + (n-1)\frac{\sin(n+1)\theta}{\sin\theta}.$$

Hence, in either case we have, as required,

$$4\left(n - \frac{\sin n\theta}{\sin\theta}\right) + (n-1)\frac{\sin(n+1)\theta}{\sin\theta} \tag{2.2}$$

$$- (n+1)\frac{\sin(n-1)\theta}{\sin\theta} \geqq 0.$$

LEMMA 2. *If n is a positive integer then for all real values of θ*

$$4\left[1+\frac{\sin 2\theta}{\sin\theta}+\frac{\sin 3\theta}{\sin\theta}+\cdots+\frac{\sin(n-1)\theta}{\sin\theta}\right] - (n-2)\frac{\sin n\theta}{\sin\theta} \leqq n^2. \tag{2.3}$$

Lemma 2 follows easily from the identity

$$\frac{\sin n\theta}{\sin\theta}+2\sum_{k=1}^{n-1}\frac{\sin k\theta}{\sin\theta}=\left(\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right)^2 \tag{2.4}$$

and the inequality $A_n(0)\geqq 0$ where

$$A_n(\theta)\equiv n^2+n\frac{\sin n\theta}{\sin\theta}-2\left(\frac{\sin(n\theta/2)}{\sin(\theta/2)}\right)^2. \tag{2.5}$$

To prove the inequality $A_n(\theta)\geqq 0$ we observe first that $A_1(\theta)=A_2(\theta)\equiv 0$. We prove next that $A_{n+1}(\theta)\geqq A_{n-1}(\theta)$ whence $A_n(\theta)\geqq 0$ by induction.

$$\begin{aligned} A_{n+1}(\theta)-A_{n-1}(\theta) &\\ &= 4n+(n+1)\frac{\sin(n+1)\theta}{\sin\theta}-(n-1)\frac{\sin(n-1)\theta}{\sin\theta}\\ &\quad -2\left[\left(\frac{\sin((n+1)\theta/2)}{\sin(\theta/2)}\right)^2-\left(\frac{\sin((n-1)\theta/2)}{\sin(\theta/2)}\right)^2\right]\\ &= 4n+(n-1)\left[\frac{\sin(n+1)\theta-\sin(n-1)\theta}{\sin\theta}\right]\\ &\quad +2\frac{\sin(n+1)\theta}{\sin\theta}-4\frac{\sin n\theta}{\sin\theta}(1+\cos\theta)\\ &= 4\left(n-\frac{\sin n\theta}{\sin\theta}\right)+(n-1)\frac{\sin(n+1)\theta}{\sin\theta}\\ &\quad -(n+1)\frac{\sin(n-1)\theta}{\sin\theta}\geqq 0 \end{aligned} \tag{2.6}$$

by Lemma 1.

3. **Proof of Theorem A.** From (2.1) we have

$$\frac{1}{(n-1)\pi}\int_0^{\pi}\frac{\sin(n-1)\theta}{\sin\theta}\,d\alpha(\theta) \tag{3.1}$$

$$
\tag{3.1}
\begin{aligned}
&-\frac{1}{(n+1)\pi}\int_0^\pi \frac{\sin (n+1)\theta}{\sin\theta}\,d\alpha(\theta)\\
&\leqq \frac{4n}{n^2-1}\left[\frac{1}{\pi}\int_0^\pi d\alpha(\theta) - \frac{1}{n\pi}\int_0^\pi \frac{\sin n\theta}{\sin\theta}\,d\alpha(\theta)\right].
\end{aligned}
$$

From (1.2) and (3.1) we have immediately

$$
\tag{3.2}
c_{n-1} - c_{n+1} \leqq \frac{4n}{n^2-1}(1-c_n), \qquad n = 2, 3, \cdots .
$$

Since

$$
\tag{3.3}
-f(-z) = z - c_2z^2 + c_3z^3 - c_4z^4 \cdots
$$

is also regular and convex in the direction of the imaginary axis for $|z| < 1$ it follows that (3.2) may be replaced by the inequalities given in Theorem A.

The function

$$
\tag{3.4}
f(z) = \sum_1^\infty \frac{1}{n}\frac{\sin n\theta}{\sin\theta} z^n
$$

is regular and convex in the direction of the imaginary axis for $|z| < 1$ since $zf'(z) = z(1-2z\cos\theta + z^2)^{-1}$ is typically real, indeed univalent and star-like for $|z| < 1$. For this function

$$
\begin{aligned}
&\lim_{\theta\to 0} \frac{c_{n-1} - c_{n+1}}{1 - c_n}\\
&\quad = \lim_{\theta\to 0}\left(\frac{n}{n^2-1}\right)\frac{(n+1)\sin(n-1)\theta - (n-1)\sin(n+1)\theta}{n\sin\theta - \sin n\theta}\\
&\quad = 4n\cdot(n^2-1)^{-1}.
\end{aligned}
$$

As a consequence of Theorem A we easily obtain the following theorem.

THEOREM B. *Let the function*

$$
F(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots
$$

have real coefficients and be regular and univalent for $|z| < 1$. *Then*

$$
\tag{3.5}
\begin{aligned}
&(n+1)a_{n-1} - (n-1)a_{n+1} \leqq 4(n - |a_n|), \quad n = 2, 4, 6, \cdots ,\\
&|(n+1)a_{n-1} - (n-1)a_{n+1}| \leqq 4(n - a_n), \quad n = 3, 5, 7, \cdots .
\end{aligned}
$$

The equality signs are attained by the function $z\cdot(1-z)^{-2}$ *which is univalent in the unit circle.*

4. Some additional inequalities for the coefficients.

From the inequality (2.3) by an argument similar to that used in the proof of Theorems A and B we obtain the following theorem.

THEOREM C. *With the same hypothesis as in Theorem* B, *the following inequalities for the coefficients of $F(z)$ hold*:

$$4(1 + a_2 + a_3 + \cdots + a_{n-1}) - (n - 2)a_n \leqq n^2. \tag{4.1}$$

In a somewhat similar way by using the trigonometric inequality

$$\begin{aligned}&(n^2 - 1)(n + 2)\left(2 - \frac{\sin 2\theta}{\sin \theta}\right) - 2(n - 1)(n + 2)\frac{\sin (n + 1)\theta}{\sin \theta}\\ &\qquad + (n + 3)(n + 2)\frac{\sin n\theta}{\sin \theta} + (n^2 - 3n - 2)\frac{\sin (n + 2)\theta}{\sin \theta} \geqq 0\end{aligned} \tag{4.2}$$

we may obtain another set of inequalities for the coefficients of $f(z)$ of Theorem A:

$$2(n^2 - 1)(1 - c_2 - c_{n+1}) + (n^2 + 3n)c_n + (n^2 - 3n - 2)c_{n+2} \geqq 0. \tag{4.3}$$

We omit the details of the proof.

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3. ———, *On the coefficients of a typically-real function*, Bull. Amer. Math. Soc. vol. 41 (1935) pp. 565–572.

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RUTGERS UNIVERSITY

NOTE ON APPELL POLYNOMIALS

I. M. SHEFFER

An interesting characterization of Appell polynomials by means of a Stieltjes integral has recently been given by Thorne.[1] We propose to give a second such representation, and to extend the result to the case of sets of polynomials of *type zero*, of which Appell sets form a subclass.

Appell sets may be defined by either of the following equivalent conditions: $\{P_n(x)\}$, $n=0, 1, \cdots$, is an Appell set (P_n being of degree exactly n) if either

(i) $P_n'(x)=P_{n-1}(x)$, $n=1, 2, \cdots$,

or

(ii) there exists a formal power series $A(t)=\sum_0^\infty a_nt^n$ $(a_0\neq 0)$ such that (again formally)

$$A(t)e^{tx} = \sum_0^\infty P_n(x)t^n.$$

The function $A(t)$ may be called the *determining function* for the set $\{P_n(x)\}$. The essence of Thorne's result is the following:

THEOREM OF THORNE. *A polynomial set $\{P_n(x)\}$ is an Appell set if and only if there exists a function $\alpha(x)$ of bounded variation on $(0, \infty)$ with the following properties:*

(i) *The moment integrals*

$$\mu_n = \int_0^\infty x^n d\alpha(x)$$

all exist.

(ii) $\mu_0\neq 0$.

(iii) $\int_0^\infty P_n^{(r)}(x)d\alpha(x) = \delta_{nr}$, $\delta_{nr}=1$ *for* $n=r$, $\delta_{nr}=0$ *for* $n\neq r$.

And for the set $\{P_n(x)\}$ the determining function $A(t)$ is given by

$$A(t) = \left[\sum_0^\infty \mu_n \frac{t^n}{n!}\right]^{-1} = \left[\int_0^\infty e^{tx}d\alpha(x)\right]^{-1}.$$

The Stieltjes integral characterization that we now give will be seen to be essentially different from that in (iii) above.

THEOREM 1. *A polynomial set $\{P_n(x)\}$ is an Appell set if and only*

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[1] C. J. Thorne, *A property of Appell sets*, Amer. Math. Monthly vol. 52 (1945) pp. 191–193.

if there is a function $\beta(x)$ *of bounded variation on* $(0, \infty)$ *with the following properties:*

(i) *The moment constants*

$$b_n = \int_0^\infty x^n d\beta(x) \tag{1}$$

all exist.

(ii) $b_0 \neq 0$.

(iii) *For* $n = 0, 1, \cdots,$

$$P_n(x) = \int_0^\infty \frac{(x+t)^n}{n!} d\beta(t). \tag{2}$$

If (i) and (ii) hold, then $P_n(x)$ as given by (2) exists for each n, and is a polynomial of degree exactly n. Moreover, differentiation under the integral sign is permissible, so that $P_n'(x) = P_{n-1}(x)$; that is, $\{P_n(x)\}$ is an Appell set.

Now suppose $\{P_n(x)\}$ is an Appell set, and let $A(t)$ be its determining function: $A(t) = \sum_0^\infty a_n t^n$. Define the sequence $\{b_n\}$ by

$$b_n = n!a_n.$$

By a theorem of Boas[2] there is a function $\beta(x)$ of bounded variation on $(0, \infty)$ whose moment constants are $\{b_n\}$; and since $a_0 \neq 0$, therefore $b_0 \neq 0$. If we denote the right side of (2), which exists, by $Q_n(x)$, then $\{Q_n(x)\}$ is an Appell set. Since (formally)

$$\sum Q_n(x)u^n = \int_0^\infty e^{u(x+t)} d\beta(t) = e^{ux} \int_0^\infty e^{ut} d\beta(t),$$

the determining function for $\{Q_n(x)\}$ is

$$\begin{aligned} A^*(u) = \int_0^\infty e^{ut} d\beta(t) &= \sum_0^\infty \left\{ \int_0^\infty t^n d\beta(t) \right\} \frac{u^n}{n!} = \sum_0^\infty b_n \frac{u^n}{n!} \\ &= \sum a_n u^n = A(u). \end{aligned}$$

It follows that $\{Q_n(x)\} \equiv \{P_n(x)\}$, so that (2) holds.

COROLLARY. *The determining function for* $\{P_n(x)\}$ *is*

$$A(u) = \int_0^\infty e^{ut} d\beta(t) = \sum_0^\infty \frac{b_n}{n!} u^n. \tag{3}$$

---

[2] Widder, *The Laplace transform*, p. 139. This result of Boas is stated for *real* sequences, but it extends immediately to complex sequences.

Let $\{I_n(x)\}$ be the Appell set

$$I_n(x) = x^n/n!; \tag{4}$$

then (2) can be written

$$P_n(x) = \int_0^\infty I_n(x+t)d\beta(t). \tag{5}$$

Note that $I_0(0)=1$, $I_n(0)=0$ $(n>0)$; it is this property that makes the determining function $A(t)$ so simply expressible in terms of the moment constants $\{b_n\}$. It is easily shown that in relation (5) the polynomials $I_n(x)$ can be replaced by *any other* Appell set, by a suitable change in the function $\beta(t)$. However, the determining function $A(t)$ is now not so easily expressed through the moment constants $\{b_n\}$. In fact, if $\{Q_n(x)\}$ is the Appell set defined by

$$Q_n(x) = \int_0^\infty I_n(x+t)d\gamma(t), \tag{6}$$

then

$$P_n(x) = \int_0^\infty Q_n(x+t)d\beta(t), \tag{7}$$

where the function $A(z)$ for $\{P_n(x)\}$ is given by

$$A(z) = \int_0^\infty e^{zt}\int_0^\infty e^{zu}d\gamma(u)d\beta(t) = \sum_0^\infty a_nz^n, \tag{8}$$

with

$$a_n = (1/n!)\{c_nb_0 + C_{n,1}c_{n-1}b_1 + \cdots + C_{n,n}c_0b_n\}. \tag{9}$$

Here $\{b_n\}$, $\{c_n\}$ are given by

$$b_n = \int_0^\infty x^nd\beta(x), \qquad c_n = \int_0^\infty x^nd\gamma(x), \qquad n = 0, 1, \cdots. \tag{10}$$

We turn now to sets of type zero. The polynomial set $\{P_n(x)\}$ is of type zero[3] if either of the following equivalent conditions holds:

(i) Formal series

$$A(t) = \sum_0^\infty a_nt^n \quad (a_0 \neq 0), \qquad H(t) = \sum_1^\infty h_nt^n \qquad (h_1 \neq 0) \tag{11}$$

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[3] I. M. Sheffer, *Some properties of polynomial sets of type zero*, Duke Math. J. vol. 5 (1939) pp. 590–622.

exist for which (formally)

$$A(t)e^{xH(t)} = \sum_0^\infty P_n(x)t^n. \tag{12}$$

(ii) An operator $L[y(x)]$ of form

$$L[y(x)] \equiv \sum_1^\infty l_n y^{(n)}(x) \qquad (l_1 \neq 0) \tag{13}$$

exists such that

$$L[P_n(x)] = P_{n-1}(x), \qquad n = 1, 2, \cdots. \tag{14}$$

If $L(t)$ is the series

$$L(t) = \sum_1^\infty l_n t^n, \tag{15}$$

then $H(t)$ and $L(t)$ are (formally) inverse functions:

$$L(H(t)) = H(L(t)) = t. \tag{16}$$

All polynomial sets satisfying (14) will be said to be associated with the operator L. Sets associated with a given L are distinguished one from another by their *determining function* $A(t)$ appearing in (12). Associated with a given L there is a unique set $B_L = \{B_n(x)\}$ for which $B_0(0) = 1$, $B_n(0) = 0$ $(n > 0)$. This set we call the *basic set* for L; its determining function is $A(t) \equiv 1$.

THEOREM 2. *In order that the polynomial set* $\{P_n(x)\}$ *be a set associated with the operator* L *it is necessary and sufficient that there exist a function* $\beta(x)$ *of bounded variation on* $(0, \infty)$, *with the following properties:*

(i) *The moment constants* $\{b_n\}$ *for* $\beta(x)$ *all exist.*

(ii) $b_0 \neq 0$.

(iii) *For* $n = 0, 1, \cdots$,

$$P_n(x) = \int_0^\infty B_n(x+t)d\beta(t), \tag{17}$$

where $\{B_n(x)\}$ *is the basic set for* L.

COROLLARY. *The determining function* $A(t)$ *for* $\{P_n(x)\}$ *has the expression*

$$A(t) = \int_0^\infty e^{uH(t)}d\beta(u), \tag{18}$$

which is equivalent to

$$A(L(z)) = \int_0^\infty e^{uz}d\beta(u) = \sum_0^\infty \frac{b_n}{n!}z^n. \tag{19}$$

If conditions (i), (ii) hold, it is clear that $\{P_n(x)\}$ as given by (17) is a set associated with the operator L. Now let $\{P_n(x)\}$ be a set relative to an operator L; to show that (i), (ii), (iii) hold. Let $A(t)$ be the determining function for $\{P_n(x)\}$, so that we have relation (12). Since A and L are given, the function $A(L(z))$ is known; let its (formal) power series be

$$A(L(z)) = \sum_0^\infty a_n^* z^n. \tag{20}$$

Now define $\{b_n\}$ so that $b_n = n!a_n^*$, and let $\beta(t)$ be the function of bounded variation on $(0, \infty)$, guaranteed by the Boas theorem, whose moments are $\{b_n\}$. With this $\beta(t)$, the right side of (17) defines a polynomial set $\{Q_n(x)\}$, associated with L, whose determining function is seen to be the function $A(t)$ given by (18). As (18) is carried into (19) by the transformation $z = H(t)$ (and therefore $t = L(z)$), it follows that $\{P_n\}$, $\{Q_n\}$ have the same determining function $A(t)$, and are therefore the same set. Hence (17) holds.

In Theorem 2 the representation (17) can be replaced by

$$P_n(x) = \int_0^\infty Q_n(x+t)d\beta(t), \tag{21}$$

where $\{Q_n(x)\}$ is *any* polynomial set associated with L; but in this case the determining function $A(t)$ for $\{P_n(x)\}$ has a more complicated representation than (18) or (19). In fact, if we write

$$Q_n(x) = \int_0^\infty B_n(x+t)d\gamma(t), \tag{22}$$

it is readily shown that

$$A(t) = \int_0^\infty \int_0^\infty e^{(u+w)H(t)}d\gamma(u)d\beta(w), \tag{23}$$

or, what is equivalent to this,

$$A(L(z)) = \int_0^\infty \int_0^\infty e^{(u+w)z}d\gamma(u)d\beta(w) = \sum_0^\infty a_n^* z^n, \tag{24}$$

where

$$a_n^* = (1/n!)\{c_nb_0 + C_{n,1}c_{n-1}b_1 + \cdots + C_{n,n}c_0b_n\}. \tag{25}$$

Here $\{b_n\}$, $\{c_n\}$ are the moment constants for $\beta(t)$, $\gamma(t)$ respectively.

We close with two observations.

REMARK 1. The Thorne theorem carries over to polynomial sets of type zero; we have only to replace the condition (iii) of that theorem by

$$\int_0^\infty L^{(r)}[P_n(x)]d\alpha(x) = \delta_{nr}, \tag{26}$$

where $L^{(1)}=L$, $L^{(2)}$, $\cdots$ are the iterates of L. Also, the determining function $A(t)$ for $\{P_n(x)\}$ is given by

$$A(t) = \left[\int_0^\infty e^{xH(t)}d\alpha(x)\right]^{-1}, \tag{27}$$

or, by the equivalent expression,

$$A(L(z)) = \left[\int_0^\infty e^{xz}d\alpha(x)\right]^{-1} = \left[\sum_0^\infty \frac{\mu_n}{n!}z^n\right]^{-1}. \tag{28}$$

REMARK 2. Throughout this note we have not hesitated to use *formal* power series. This does not invalidate the results obtained; it has permitted the results to be gotten faster. As to how we could have reasoned without the use of these formal series, we refer to the reference in footnote 3, where there is to be found on page 596 a note of justification.

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PERMUTATIONS WITHOUT 3-SEQUENCES

JOHN RIORDAN

1. Introduction. The enumeration of permutations of n distinct elements without rising 2-sequences 12, 23, $\cdots$, $n-1\ n$, is given by Whitworth [1],[1] who gives also the enumeration when $n1$ is added to this set of sequences. More recently, Kaplansky [2] and Wolfowitz [4] have enumerated permutations without rising or falling 2-sequences, that is, without 21, 32, $\cdots$, $n\ n-1$ as well as 12, $\cdots$, $n-1\ n$. An addition to these results, the enumeration of permutations without 3-sequences, 123, $\cdots$, $n-2\ n-1\ n$ is given here. This case, aside from its general interest as a natural extension of its predecessors, has a particular interest because it is a relatively simple example of failure of what Kaplansky has called quasi-symmetry. In the method of inclusion and exclusion, or its symbolic equivalent, a case is said to be quasi-symmetric when the number of permutations having k of the given properties is either zero or a function of k alone.

2. The enumeration setting. Employing the symbolic method, with q_{ijk} the probability that elements i, j and k are consecutive, the probability of finding a permutation without any of the 3-sequences in question is

$$P_0 = (1 - q_{123})(1 - q_{234}) \cdots (1 - q_{n-2\ n-1\ n}). \tag{1}$$

The meaning of this is that on expansion a product of q's represents the probability of permutations having a particular set of 3-sequences denoted by subscripts of these q's.

It is evident that the sequences chosen do not conflict; that is, it is possible to have any k of the $n-2$ simultaneously.

For a single q the number of permutations is $(n-3+1)!$, for 3 elements are required for the corresponding 3-sequence which may be permuted as a single entity; the corresponding probability is $(n-2)!/n!$.

For a product of two q's however, the case is otherwise. If the two are immediately adjacent, like 123, 234, the number of permutations is $(n-4+1)!$ or $(n-3)!$; if not, like 123, 345 or 123, 456, the number is either $(n-5+1)!$ or $(n-6+2)!$, in either case $(n-4)!$. If sequences are denoted by their initial numbers, the number of permutations is

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

$(n-3)!$ or $(n-4)!$ according as the two initial numbers in rising order are a 2-sequence or not.

In the same way, for three q's, the number of permutations with the corresponding three 3-sequences is $(n-4)!$, $(n-5)!$ or $(n-6)!$ according as the set of three initial numbers in rising order has 2, 1 or 0 2-sequences, with a 3-sequence counting as two 2-sequences.

In general, for a product of k q's, the number of permutations with k corresponding 3-sequences is $(n-2k+i)!$ when the set of k initial numbers in rising order has i 2-sequences or for euphony successions, a j-sequence counting as $j-1$ successions.

The enumeration is then completed by determining the number of combinations k at a time of the given $n-2$ 3-sequences which correspond to i successions in the set of their k initial numbers in rising order, as given by the following theorem:

THEOREM. *The number of combinations of n things k at a time such that each combination in rising order has i successions, a j-sequence counting as $j-1$ successions, is*

$$f_i(n, k) = C_{k-1,i}C_{n-k+1,k-i}.$$

3. **Proof of the theorem.** The proof depends essentially on the recurrence in the following lemma:[2]

LEMMA. $f_i(n, k)=f_i(n-1, k)+f_i(n-2, k-1)+f_{i-1}(n-1, k-1)$ $-f_{i-1}(n-2, k-1)$.

The combinations may be subdivided into those which do not contain n and those which do, and the latter in turn into those which do not contain $n-1$ and those which do. The number without n is $f_i(n-1, k)$. When n is present and $n-1$ absent, n is not in any succession, and the corresponding number is $f_i(n-2, k-1)$. When both n and $n-1$ are present, there must be $i-1$ successions among the combinations of $n-1$ elements $k-1$ at a time which contain $n-1$. The combinations of $n-1$ elements $k-1$ at a time with $i-1$ successions is $f_{i-1}(n-1, k-1)$, the number of these which do not contain $n-1$ is $f_{i-1}(n-2, k-1)$, hence the number which do is $f_{i-1}(n-1, k-1)$ $-f_{i-1}(n-2, k-1)$.

Then the formula in the theorem may be derived, step by step, for successive values of i. The case $i=0$ has been given by Kaplansky

---

[2] This lemma replaces a succession of lemmas, for successive values of i, in an earlier version, the change being a response to suggestions from Irving Kaplansky, S. O. Rice, and C. E. Shannon. To Dr. Kaplansky I owe also a simplification of the procedure for verification of the formula in the theorem as well as the correction of an error in the variance formula (6) below.

[3]; the recurrence

$$f_0(n, k) = f_0(n-1, k) + f_0(n-2, k-1)$$

which follows from the lemma with $f_{-1}=0$, and boundary conditions $f_0(n, 1)=n$, $f_0(k, k)=0$, $k>1$, lead to the formula in the theorem. In the general case the formula of the theorem must be checked for the lowest values of the other two variables of the induction, n and k. The lowest value of n is k and the corresponding boundary relation is $f_i(k, k)=1$, $i=k-1$, and zero otherwise. The lowest value of k is $i+1$ and $f_i(n, i+1)=n-i$.

4. **Completed enumeration.** These results may now be used to put (1) in an explicit form, as follows:

$$P_0 = \sum_{k=0}^{n-2} (-)^k \sum_{i=0}^{k-1} C_{k-1,i} C_{n-k-1,k-i} \frac{(n-2k+i)!}{n!}. \tag{2}$$

Note that in the present application of the theorem n is replaced by $n-2$, the number of 3-sequences in question.

Another writing of this result is

$$P_0 = \sum_{k=0}^{n-2} (-)^k a_{n,k} \tag{2.1}$$

with

$$\begin{aligned} a_{n,k} &= \sum_{i=0}^{k-1} C_{k-1,i} C_{n-k-1,k-i} \frac{(n-2k+i)!}{n!} \\ &= \sum_{i=0}^{k-1} C_{k-1,i} C_{n-k-1,i+1} \frac{(n-i-k-1)!}{n!}. \end{aligned}$$

This is a convenient form for writing the more general result for the probability that a permutation has x 3-sequences of the specified kind, which is

$$P_x = \sum_{k=0}^{n-2} (-)^{k+x} C_{k,x} a_{n,k} \tag{3}$$

with $C_{k,x}=0$, $k<x$, by a known process (Kaplansky [2]). Also the coefficients $a_{n,k}$ are related to kth factorial moments $(m)_k$ of the distribution P_x by

$$k!a_{n,k} = (m)_k. \tag{4}$$

Hence the expected number of sequences $(m)_1$ is

$$(m)_1 = m_1 = a_{n,1} = \frac{(n-2)(n-2)!}{n!} = \frac{n-2}{n(n-1)} \tag{5}$$

and the variance is

$$\begin{aligned} V &= (m)_2 + m_1 - m_1^2 = 2a_{n,2} + a_{n,1} - a_{n,1}^2 \\ &= \frac{1}{(n)_2(n)_3}\,[n^4 - 3n^3 + n^2 - 6n + 8], \qquad n > 3, \end{aligned} \tag{6}$$

where $(n)_i$ is the factorial $n(n-1)\cdots(n-i+1)$.

A short table of numbers $n!\,P_x$ follows.

PERMUTATIONS WITH x 3-SEQUENCES
NUMBERS $n!\,P_x$

n/x	0	1	2	3	4	5	6	7	8
3	5	1							
4	21	2	1						
5	106	11	2	1					
6	643	62	12	2	1				
7	4547	406	71	13	2	1			
8	36696	3046	481	80	14	2	1		
9	332769	25737	3708	559	89	15	2	1	
10	3349507	242094	32028	4414	640	98	16	2	1

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BELL TELEPHONE LABORATORIES

A NOTE ON SYSTEMS OF HOMOGENEOUS ALGEBRAIC EQUATIONS

RICHARD BRAUER

1. Introduction. Consider a system of algebraic equations

$$\begin{aligned} f_1(x_1, x_2, \cdots, x_n) &= 0, \\ f_2(x_1, x_2, \cdots, x_n) &= 0, \\ \cdots\cdots&\cdots\cdots \\ f_h(x_1, x_2, \cdots, x_n) &= 0, \end{aligned} \tag{1}$$

where f_i is a homogeneous polynomial of degree r_i with coefficients belonging to a given field K. We interpret $x_1, x_2, \cdots, x_n$ as homogeneous coordinates in an $(n-1)$-dimensional projective space. When $n>h$, the system (1) has non-trivial solutions $(x_1, x_2, \cdots, x_n)$ in an algebraically closed extension field of K, but there may not exist any such solutions in K itself. It is, in general, extremely difficult to decide whether adjunction of irrationalities of a certain type to K is sufficient to guarantee the existence of non-trivial solutions of (1) in the extended field. However, the situation is much simpler, when n is very large, in the sense that n lies above a certain expression depending on the number of equations h and the degrees $r_1, r_2, \cdots, r_h$.

We shall show:

THEOREM A. *For any system of h positive degrees $r_1, r_2, \cdots, r_h$ there exists an integer $\Phi(r_1, r_2, \cdots, r_h)$ such that for $n \geqq \Phi(r_1, r_2, \cdots, r_h)$ the system* (1) *has a non-trivial solution in a soluble extension field K_1 of K. The field K_1 may be chosen such that its degree N_1 over K lies below a value depending on $r_1, r_2, \cdots, r_h$ alone and that any prime factor of N_1 is at most equal to* $\max(r_1, r_2, \cdots, r_h)$.

This Theorem A is evidently contained in the following theorem.

THEOREM B. *For any system of positive integers $r_1, r_2, \cdots, r_h$ and any integer $m \geqq 0$, there exists an integer $\Phi(r_1, r_2, \cdots, r_h; m)$ with the following property: For $n \geqq \Phi(r_1, \cdots, r_h; m)$, there exists a soluble extension field K_2 of K such that all points $(x_1, x_2, \cdots, x_n)$ of an m-dimensional linear manifold L, defined in K_2, satisfy the equations* (1). *Here K_2 may be chosen so that its degree N_2 over K lies below a bound depending on $r_1, r_2, \cdots, r_h$ and m and that no prime factor of N_2 exceeds* $\max(r_1, r_2, \cdots, r_h)$.

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At the same time, we shall prove the theorem:

THEOREM C. *Assume that the field K has the following property:*

(*) *For every integer $r>0$, there exists an integer $\Psi(r)$ such that for $n \geqq \Psi(r)$ every equation*

$$a_1x_1^r + a_2x_2^r + \cdots + a_nx_n^r = 0 \tag{2}$$

with coefficients a_i in K has a non-trivial solution in K.

Then, for every system of positive degrees $r_1, r_2, \cdots, r_h$ and every integer $m \geqq 0$, there exists an expression $\Omega(r_1, r_2, \cdots, r_h; m)$ with the following property: For $n \geqq \Omega(r_1, r_2, \cdots, r_h; m)$, there exists an m-dimensional linear manifold M, defined in K, whose points satisfy the equations (1).

We shall prove Theorem C in §2. The changes necessary in order to obtain Theorem B are obvious. In §3, some applications are given. One of them is concerned with Hilbert's resolvent problem. We prove here a recent conjecture of B. Segre.[1]

2. **Proof of Theorem C.** 1. Assume that Theorem C is not true. We choose a system $r_1, r_2, \cdots, r_h; m$ for which no $\Omega(r_1, \cdots, r_h; m)$ exists. We select this system such that $\max(r_1, \cdots, r_h) = s$ has the smallest possible value, and that for fixed s the number h has the smallest possible value. If $r_1', r_2', \cdots, r_{h'}'$ is any system of positive integers and m' a non-negative integer, then $\Omega(r_1', r_2', \cdots, r_{h'}'; m')$ exists, if either

$$\max(r_1', r_2', \cdots, r_{h'}') < s \tag{3a}$$

or if

$$\max(r_1', r_2', \cdots, r_{h'}') = s, \qquad h' < h. \tag{3b}$$

Assume first that $h>1$. We may assume that $r_h = s$. It follows that $\Omega(r_1, r_2, \cdots, r_{h-1}; m)$ exists (cf. the conditions (3a) and (3b)) and also that $\Omega(s; m'-1)$ exists for any integer $m'>0$. We set $m' = \Omega(r_1, \cdots, r_{h-1}; m)$. If $n \geqq \Omega(s; m'-1)$, the equation $f_h = 0$ is satisfied by all points of an $(m'-1)$-dimensional linear manifold M_1. If we restrict ourselves to points of M_1, we may express $x_1, \cdots, x_n$ linearly and homogeneously by m' parameters $y_1, \cdots, y_{m'}$ with coefficients in K. Then $f_i(x_1, \cdots, x_n)$ becomes a homogeneous polynomial g_i of $y_1, \cdots, y_{m'}$. The degree of g_i is r_i; the coefficients of g_i belong to K. In particular, g_h vanishes identically. In order to solve

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[1] B. Segre, Ann. of Math. vol. 46 (1945) p. 287. *Added September* 10: In the meantime, I learned from Mr. Segre that he also found Theorem A from which the proof of the conjecture can be derived.

(1), we have to solve

$$g_1 = 0,\ g_2 = 0,\ \cdots,\ g_{h-1} = 0. \tag{4}$$

Since $m' = \Omega(r_1, \cdots, r_{h-1}; m)$, the equations (4) will be satisfied by all points of an m-dimensional manifold M_2 of the $(y_1, \cdots, y_{m'})$-space. This then gives an m-dimensional linear manifold of the $(x_1, \cdots, x_n)$-space for which the equations (1) hold. But this shows that the expression $\Omega(r_1, \cdots, r_h; m)$ exists; we may take

$$\Omega(r_1, \cdots, r_h; m) = \Omega(\max(r_1, \cdots, r_h); \Omega(r_1, \cdots, r_{h-1}; m) - 1).$$

Hence the case $h > 1$ is impossible.

2. We now consider the case $h = 1$. The system (1) consists of only one equation

$$f(x_1, x_2, \cdots, x_n) = 0$$

of degree $r_1 = s$.

From the way the number s was chosen it follows that $\Omega(s; m)$ does not exist while for every system $r_1', r_2', \cdots, r_{h'}'$ with $r_1' < s$, $r_2' < s, \cdots, r_{h'}' < s$ and all m' the existence of $\Omega(r_1', r_2', \cdots, r_{h'}'; m')$ may be assumed.

We first discuss the case $m = 0$. Denoting the point $(x_1, x_2, \cdots, x_n)$ by $\mathfrak{x}$, we write $f(x_1, x_2, \cdots, x_n) = f(\mathfrak{x})$.

If $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n$ are n points whose coordinates are independent indeterminates and if $u_1, u_2, \cdots, u_n$ are n further independent indeterminates, we may set

$$f(u_1\mathfrak{x}_1 + u_2\mathfrak{x}_2 + \cdots + u_n\mathfrak{x}_n) = \sum u_1^{\mu} u_2^{\nu} \cdots u_n^{\tau} f_{\mu\nu\cdots\tau}(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n), \tag{5}$$

where the sum on the right side extends over all systems of n nonnegative integers $(\mu, \nu, \cdots, \tau)$ with

$$\mu + \nu + \cdots + \tau = s. \tag{5a}$$

The expressions $f_{\mu,\nu,\cdots,\tau}(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n)$ (the polar forms of f) are homogeneous polynomials in the coordinates of each $\mathfrak{x}_i$. As is easily seen, $f_{\mu,\nu,\cdots,\tau}(\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n)$ is of degree μ in the coordinates of $\mathfrak{x}_1$, of degree ν in the coordinates of $\mathfrak{x}_2, \cdots$, of degree τ in the coordinates of $\mathfrak{x}_n$.

Let $\mathfrak{a}_1 \neq \mathfrak{o}$ be a fixed point.[2] Choose $n-1$ points $\mathfrak{e}_1, \mathfrak{e}_2, \cdots, \mathfrak{e}_{n-1}$ which together with $\mathfrak{a}_1$ form a full linearly independent system, and set $\mathfrak{y} = y_1\mathfrak{e}_1 + y_2\mathfrak{e}_2 + \cdots + y_{n-1}\mathfrak{e}_{n-1}$ where the coefficients $y_1, y_2, \cdots, y_{n-1}$ are indeterminates.

Consider the system of equations

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[2] We denote by $\mathfrak{o}$ the row $(0, 0, \cdots, 0)$ consisting of n numbers 0.

$$\begin{aligned} f_{s-1,1,0,\cdots,0}(\mathfrak{a}_1, \mathfrak{y}, \mathfrak{o}, \cdots, \mathfrak{o}) &= 0,\\ f_{s-2,2,0,\cdots,0}(\mathfrak{a}_1, \mathfrak{y}, \mathfrak{o}, \cdots, \mathfrak{o}) &= 0,\\ \cdots\cdots&\cdots\cdots\\ f_{1,s-1,0,\cdots,0}(\mathfrak{a}_1, \mathfrak{y}, \mathfrak{o}, \cdots, \mathfrak{o}) &= 0. \end{aligned} \tag{6}$$

These equations are homogeneous in $y_1, y_2, \cdots, y_{n-1}$; the degrees are $1, 2, \cdots, s-1$ respectively.

From the remarks above it follows that the expression $\Omega(1, 2, \cdots, s-1; 0)$ exists. Hence for sufficiently large[3] n the equations (6) will have a non-trivial solution. Let $\mathfrak{y}=\mathfrak{a}_2$ be the corresponding point $\mathfrak{y}$. Then $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are linearly independent.

Let $\mathfrak{e}_1', \mathfrak{e}_2', \cdots, \mathfrak{e}_{n-2}'$ be a system of points which together with $\mathfrak{a}_1$ and $\mathfrak{a}_2$ form a full linearly independent system and set

$$\mathfrak{z} = z_1\mathfrak{e}_1' + \cdots + z_{n-2}\mathfrak{e}_{n-2}'$$

with indeterminate coefficients $z_1, z_2, \cdots, z_{n-2}$. Consider next the equations

$$f_{\mu,\nu,\rho,0,\cdots,0}(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{z}, \mathfrak{o}, \cdots, \mathfrak{o}) = 0, \tag{7}$$

where μ, ν, ρ range over all systems of non-negative integers with

$$\mu + \nu + \rho = s, \qquad 0 < \rho < s.$$

Again, $\Omega(r_1', \cdots, r_{h'}; 0)$ exists for the degrees $r_1', \cdots, r_{h'}$ of these equations in $z_1, z_2, \cdots, z_{n-2}$. It follows for sufficiently large n that the system (7) has a non-trivial solution $(z_1, \cdots, z_{n-2})$. Let $\mathfrak{z}=\mathfrak{a}_3$ be the corresponding point. Then $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3$ are linearly independent.

Set $t=\Psi(s)$.[4] Assuming that n is sufficiently large we continue with our procedure until we obtain t linearly independent points $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_t$ such that[5]

$$f_{\mu,\nu,\cdots,\tau}(\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_t, \mathfrak{o}, \cdots, \mathfrak{o}) = 0$$

for every system of n non-negative indices $(\mu, \nu, \cdots, \tau)$ with $\mu+\nu+\cdots+\tau=s$ in which the first t of our indices are all less than s.

For $\mathfrak{x}_1=\mathfrak{a}_1, \mathfrak{x}_2=\mathfrak{a}_2, \cdots, \mathfrak{x}_t=\mathfrak{a}_t, \mathfrak{x}_{t+1}=\mathfrak{o}, \cdots, \mathfrak{x}_n=\mathfrak{o}$, the identity

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[3] In part 2 of the proof we mean by "sufficiently large n" all values of n which lie above a suitable lower bound $\Lambda(s)$ depending only on s.

[4] In the case of Theorem B, we take $t=2$. The equation (8) will have a solution if we extend the field K by the adjunction of an sth root.

[5] If one of the last $n-t$ indices in $(\mu, \nu, \cdots, \tau)$ does not vanish, this equation is trivial, since the left side then contains an $\mathfrak{x}_\iota=\mathfrak{o}$ to a positive degree.

(5) gives a relation

$$f(u_1\mathfrak{a}_1 + u_2\mathfrak{a}_2 + \cdots + u_t\mathfrak{a}_t) = \sum_{i=1}^{t} a_i u_i^{s},$$

where a_i is a certain number of K. Actually, $a_i = f(\mathfrak{a}_i)$.

Since $t = \Psi(s)$, the equation

$$\sum_{i=1}^{t} u_i^{s} a_i = 0 \tag{8}$$

has a non-trivial solution $(u_1, u_2, \cdots, u_t)$ in K. The corresponding point $\mathfrak{x} = \sum u_i\mathfrak{a}_i$ then yields a non-trivial solution of the equation $(\mathfrak{x}) = 0$ in K.

This argument shows the existence of $\Omega(s; 0)$.

3. We assume that the existence of $m' = \Omega(s; m-1)$ has already been shown. If n is sufficiently large,[6] the result of 2 shows that we may find a point $\mathfrak{a}_1 \neq 0$ such that

$$f(\mathfrak{a}_1) = 0. \tag{9}$$

Consider again the equations (6) where $\mathfrak{y}$ has the old significance. Again, $\Omega(1, 2, \cdots, s-1; m'-1)$ exists. If $n \geqq \Omega(1, 2, \cdots, s-1; m'-1)$, it follows that there exists an $(m'-1)$-dimensional linear space M_0 such that the equations (6) hold for all points $\mathfrak{y}$ of M_0, and that M_0 does not contain $\mathfrak{a}_1$.

The identity (5) for $\mathfrak{x}_1 = \mathfrak{a}_1$, $\mathfrak{x}_2 = \mathfrak{y}$, $\mathfrak{x}_3 = \mathfrak{o}, \cdots, \mathfrak{x}_n = \mathfrak{o}$ yields

$$f(u_1\mathfrak{a}_1 + u_2\mathfrak{y}) = u_2^{s} f(\mathfrak{y}), \tag{10}$$

on account of (6) and (9). Restricting the point $\mathfrak{y}$ to the linear manifold M_0, we may consider the coordinates of $\mathfrak{y}$ as linear homogeneous functions of m' parameters $z_1, z_2, \cdots, z_{m'}$. Since $m' = \Omega(s; m-1)$, it follows that there exists an $(m-1)$-dimensional linear subspace M_1 of M_0 such that $f(\mathfrak{y}) = 0$ for all points $\mathfrak{y}$ of M_1. But (10) shows that $\mathfrak{a}_1$ and M_1 together span an m-dimensional linear space M which consists entirely of solutions of $f(\mathfrak{x}) = 0$. This proves the existence of $\Omega(s; m)$ which contradicts the assumptions made above.

This finishes the proof of Theorem C. The same method yields the proof of Theorem B, and hence the Theorem A.

3. **Applications.**[7] Consider the general algebraic equation of degree

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[6] In part 3 of the proof we shall say that n is sufficiently large if it lies above a suitable lower bound $M(s, m)$, depending on s and m only.

[7] For Hilbert's resolvent problem, see the paper by Segre quoted in footnote 1 and the literature mentioned in this paper, also A. Wiman, Nova Acta Uppsala (1927).

n in one unknown

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n = 0.$$

If the roots are $\omega_1, \omega_2, \cdots, \omega_n$ and if we set

$$\theta_i = u_0 + u_1\omega_i + \cdots + u_{n-1}\omega_i^{n-1}$$

then the θ_i are the roots of an equation

$$g(x) = x^n + b_1x^{n-1} + \cdots + b_n = 0$$

and it is well known that the coefficient b_i of this Tschirnhaus transformation is a homogeneous polynomial $B_i(u_0, u_1, \cdots, u_{n-1})$ of degree i in the $u_0, u_1, \cdots, u_{n-1}$. For a fixed k, we determine the quantities $u_0, u_1, \cdots, u_{n-1}$ as a non-trivial solution of the equations

$$B_1(u_0, u_1, \cdots, u_{n-1}) = 0,$$
$$B_2(u_0, u_1, \cdots, u_{n-1}) = 0, \cdots, B_k(u_1, u_2, \cdots, u_{n-1}) = 0.$$

It follows from Theorem A that for sufficiently large n it is possible to take $u_0, u_1, \cdots, u_{n-1}$ in a field obtained from the field of the rational functions of $a_1, a_2, \cdots, a_n$ by adjunction of a finite number of radicals. The equation $g(x)$ then has the form

$$x^n + b_{k+1}x^{n-k-1} + \cdots + b_n = 0.$$

Its roots then may be considered as algebraic functions of $n-k$ quantities $b_{k+1}, b_{k+2}, \cdots, b_n$. Since ω_i can be expressed in terms of θ_i, it follows that the solution of the general equation of nth degree can be expressed in terms of the coefficients if we use radicals and one algebraic function of $n-k$ arguments.[8] Here k was a fixed number and n was to be taken sufficiently large.

Hilbert's resolvent problem deals with the question of finding the smallest number l_n for given n such that the roots of the general equation of degree n may be expressed in terms of the coefficients by means of algebraic functions of at most l_n parameters. Our above remark shows that $l_n \leqq n-k$ for fixed k and sufficiently large n. In other words, we have shown that[9]

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[8] Since we can make $b_n = 1$ through a simple transformation, we could replace the last function by one depending on $n-k-1$ arguments.

[9] This result shows that in Segre's notation an infinite series of theorems H_i exists. The same is true for the theorems B_i, if in the statement beside the adjunction of square roots and cube roots the adjunction of a finite number of other radicals is admitted. On the other hand, icosahedral irrationalities are superfluous. The existence of these infinite series of theorems H_i and B_i had been stated as a conjecture in Segre's paper.

$$\lim_{n\to\infty} (n - l_n) = \infty.$$

Hilbert's observation that $l_n \leqq n-5$, at least for $n \geqq 9$, and Segre's observation that $l_n \leqq n-6$, at least for $n \geqq 157$,[10] can be supplemented by an infinite number of analogous observations. The method of §2 would allow us to find explicit values n_k such that $l_n \leqq n-k$ for $n \geqq n_k$. However, the values obtained would probably be far too large.

As an example of a field which satisfies the assumption (*) of Theorem C, we may take any field K which is closed with regard to forming radicals $a^{1/m}$, a in K, $m=2, 3, 4, \cdots$. We have here $\Psi(r)=2$ for all r. In particular, any homogeneous equation $f(x_1, x_2, \cdots, x_n)=0$ of degree r has a non-trivial solution, provided that n lies above a certain number depending on r only.

An example of a somewhat less trivial nature is obtained by considering a $\mathfrak{p}$-adic field K. As is well known the multiplicative group of all α^r ($\alpha \neq 0$, α in K) is of finite index in the group of all α ($\alpha \neq 0$, α in K). From this it follows at once that the assumption (*) of Theorem C is satisfied, and the statement of Theorem C holds for K. In particular, *a homogeneous equation* $f(x_1, \cdots, x_n)=0$ *of degree* r *in a* $\mathfrak{p}$*-adic field has a non-trivial solution* $(x_1, x_2, \cdots, x_n)$, *if* n *is sufficiently large, say* $n \geqq N(r)$.[11]

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[10] The somewhat rough method of our proof does not allow us to derive this result. The bound obtained for n would be much larger.

[11] E. Artin has remarked that it follows at once from the existence of normal division algebras of rank r^2 over K that $N(r) > r^2$.

ON SIMPLE GROUPS OF FINITE ORDER. I

RICHARD BRAUER AND HSIO-FU TUAN

1. **Introduction.** Using the theory of representations of groups we have obtained a number of results for simple groups of certain types of orders. In the present paper, we shall prove the following result: If $\mathfrak{G}$ is a (non-cyclic) simple group of order $g=pq^bg^*$, where p and q are two primes and where b and g^* are positive integers with $g^*<p-1$, then either $\mathfrak{G}\cong LF(2, p)$[1] with $p=2^m\pm 1$, $p>3$, or $\mathfrak{G}\cong LF(2, 2^m)$ with $p=2^m+1$, $p>3$; conversely, these groups satisfy the assumptions. As an application, we determine all simple groups of order prq^b, where p, r, q are primes and where b is a positive integer. The only simple groups of this type are the well known groups of orders 60 and 168.

2. **Some known results concerning representations of groups.** 1. In this section, some known theorems are given without proof. Most of these results, which are needed in the following, have been obtained in the theory of modular representations of groups. However, all the statements are concerned with the *ordinary* group characters.[2]

2. If $\mathfrak{G}$ is a group of order g containing k classes $K_1, \cdots, K_\mu, \cdots, K_k$ of conjugate elements, then there exist exactly k distinct irreducible characters $\zeta_1(G), \cdots, \zeta_\mu(G), \cdots, \zeta_k(G)$, where G denotes a variable element of $\mathfrak{G}$. If we restrict G to a subgroup $\mathfrak{M}$ of order m of $\mathfrak{G}$, then each $\zeta_\mu(G)$ may be considered as a (reducible or irreducible) character of $\mathfrak{M}$. From the orthogonality relations for the characters of $\mathfrak{M}$, it follows that

$$\sum{}'\zeta_\mu(G) \equiv 0 \pmod{m}, \tag{2.1}$$

where the sum extends over all elements G of $\mathfrak{M}$. More generally, the same congruence holds, if ζ is a linear combination of the ζ_μ's with coefficients which are algebraic integers.

3. Let p be a prime number and let $\mathfrak{p}$ be a prime ideal divisor of p in the algebraic number field generated by all $\zeta_\mu(G)$. Denote by $h(G)$ the number of elements in the class K_μ containing G. If ζ_μ has degree z_μ, the number $h(G)\zeta_\mu(G)/z_\mu$ is an algebraic integer. Two characters ζ_μ and ζ_ν belong to the same p-*block*, if

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[1] We use the notation of L. E. Dickson, *Linear groups*, Leipzig, 1901.

[2] The fundamental properties of group characters are given in a large number of books. Here we mention only: W. Burnside, *The theory of groups of finite order*, 2d ed., Cambridge, 1911.

$$(2.2)\qquad h(G)\zeta_\mu(G)/z \equiv h(G)\zeta_\nu(G)/z \pmod{\mathfrak{p}},$$

for all G in $\mathfrak{G}$. In this manner, the k characters are distributed into a certain number of p-blocks $B_1(p)$, $B_2(p)$, $\cdots$. The *first p-block* $B_1(p)$ will always be taken as the block containing the 1-character $\zeta_1(G)=1$ (for all G). If for all characters ζ_μ of $B_\sigma(p)$ the degree z_μ of ζ_μ is divisible by a power p^α while at least one of the degrees z_μ is not divisible by $p^{\alpha+1}$, then $B_\sigma(p)$ is a block of *type* α. In particular, $B_\sigma(p)$ is of the *lowest type* if $\alpha=0$.

An element G is *p-regular*, if its order is prime to p; and G is p-singular in the other case. For every p-block $B_\sigma(p)$ we have[3]

$$(2.3)\qquad \sum_\mu \zeta_\mu(P)\zeta_\mu(Q) = 0,$$

where ζ_μ ranges over all characters of $B_\sigma(p)$ and where P is any p-singular element of $\mathfrak{G}$ and Q any p-regular element.

If we let ζ_μ range over all the k characters of $\mathfrak{G}$, then the orthogonality relations for group characters show that the sum in (2.3) vanishes for any two elements P and Q for which P and Q^{-1} are not conjugate. If P and Q^{-1} are conjugate, the sum does not vanish.

4. If we assume that the prime p divides g to the first power,[4] we can make more definite statements. It will be sufficient to restrict our attention to the first p-block $B_1(p)$. There exists a divisor t of $p-1$ such that $B_1(p)$ consists of $w=(p-1)/t$ "non-exceptional" characters $\zeta_1(G), \cdots, \zeta_w(G)$ and t "exceptional" characters $\zeta_{w+1}(G), \cdots, \zeta_{w+t}(G)$. The latter have all the same degree z_{w+1}. To each of these characters $\zeta_i(G)$, there belongs a certain sign $\delta_i=\pm 1$ such that the following relations hold:

$$(2.4)\qquad z_i \equiv \delta_i \pmod{p} \qquad \text{for } i = 1, 2, \cdots, w;$$

$$(2.5)\qquad tz_{w+1} \equiv \delta_{w+1} \pmod{p};$$

$$(2.6)\qquad \sum_{\mu=1}^{\omega+1} \delta_\mu z_\mu = 0 \qquad (\delta_1 = z_1 = 1).$$

Moreover, for p-singular elements P of G, we have

$$(2.7)\qquad \zeta_i(P) = \delta_i \qquad (i = 1, 2, \cdots, w).$$

There exist elements of order $w=(p-1)/t$ in $\mathfrak{G}$, hence

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[3] R. Brauer and C. Nesbitt, University of Toronto Studies, Mathematical Series, No. 4, 1937, Theorem VIII, p. 21.

[4] For the results quoted in this part, cf. R. Brauer, Amer. J. Math. vol. 64 (1942) pp. 401–420, especially p. 420, §8, and p. 417, Formula (47a).

(2.8) $$g \equiv 0 \pmod{w}.$$

5. If $\mathfrak{G}$ coincides with its commutator subgroup $\mathfrak{G}'$, in particular if $\mathfrak{G}$ is simple and non-cyclic, then $\zeta_1 = 1$ is the only character of degree 1. It follows that the number w above must be larger than 1. Indeed, if we had $w = 1$, the equation (2.6) would read $\delta_1 z_1 + \delta_2 z_2 = 0$, and as $\delta_1 = 1$, $\delta_2 = \pm 1$, $z_1 = 1$, it would follow that the positive number z_2 must be 1 which is impossible. Thus, in particular, $p \neq 2$.

6. Finally we quote the following results obtained in previous papers which yield characterizations of certain groups $LF(2, m)$.

Theorem A.[5] *If a non-cyclic simple group* $\mathfrak{G}$ *has an order which contains the prime* p *to the first power and if the exceptional degree* z_{w+1} *in the first* p*-block* $B_1(p)$ *satisfies the condition* $z_{w+1} \leqq (p+1)/2$, *then* $\mathfrak{G} \cong LF(2, p)$ $(p \neq 2, 3)$.

Theorem B.[6] *If a non-cyclic simple group* $\mathfrak{G}$ *has an order* g *of the form*

$$g = (p - 1)p(1 + mp)/\tau,$$

where p *is a prime and where* τ *and* m *are non-negative integers such that* τ *divides* $p-1$ *and* $m < (p+3)/2$, *then either* $\mathfrak{G} \cong LF(2, p-1)$ *and* p *is a prime of the form* $p = 2^b + 1 > 3$, *or* $\mathfrak{G} \cong LF(2, p)$ *and* p *is any prime larger than* 3.

3. Proof of the main result.

We now begin to prove the following theorem.

Theorem 1. *If* $\mathfrak{G}$ *is a simple group of order*

(3.1) $$g = pq^b g^*,$$

where p *and* q *are two primes and where* b *and* g^* *are positive integers with*

(3.2) $$g^* < p - 1,$$

then either $\mathfrak{G} \cong LF(2, p)$ *with* $p = 2^m \pm 1$, $p > 3$, *or* $\mathfrak{G} \cong LF(2, 2^m)$ *with* $p = 2^m + 1$, $p > 3$. *Conversely, these groups satisfy the assumptions.*

Remark. It may be added that for both alternatives we have $q = 2$ and m is the highest exponent of q dividing g.

Proof. 1. If g as given by (3.1) is the order of a non-cyclic simple group $\mathfrak{G}$, then by (3.2) certainly $p\neq 2$ and without restriction we may assume that

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[5] H. F. Tuan, Ann. of Math. vol. 45 (1944) pp. 110–140, especially p. 135, Theorem 4, and p. 139, Formulas (10.i) and (10.i′) for $i = 1, \cdots, 7$. Notice that Formulas (10.2′) and (10.3′) are printed there in the wrong order.

[6] R. Brauer, Ann. of Math. vol. 45 (1944) pp. 57–79, especially p. 76, Theorem 11.

$$(g^*, q) = 1. \tag{3.3}$$

The case $p=q$ is impossible on account of Sylow's theorem since the p-Sylow subgroup of a group $\mathfrak{G}$ of order $g=p^{b+1}g^*$ would be normal in $\mathfrak{G}$, if $g^*<p-1$. Hence $p\neq q$, and p divides g only to the first power. From (2.6) it follows that in the first p-block $B_1(p)$ we have a degree $n\neq 1$ which is prime to q. Then

$$n \mid g^*, \tag{3.4}$$

and hence $n<p-1$. This shows that n must be the exceptional degree $n=z_{w+1}$ (cf. (2.5)), since all the other degrees are congruent to ± 1 (mod p) according to (2.4).

2. If $n\leqq (p+1)/2$, Theorem A gives $\mathfrak{G}\cong LF(2, p)$, $p\neq 3$, and $g=p(p+1)(p-1)/2$. Hence

$$(p+1)(p-1) = 2q^b g^*.$$

As $p+1$ and $p-1$ have the greatest common divisor 2, it follows that one of the two numbers $p-1$ and $p+1$ is divisible by q^b. The other number then divides $2g^*$. But $p\pm 1>g^*$ by (3.2) and we have one of the two cases

$$p - 1 = q^b, \qquad p + 1 = 2g^*; \tag{I}$$

$$p + 1 = q^b, \qquad p - 1 = 2g^*. \tag{II}$$

In either case, $q=2$, and this leads to the first alternative of our theorem.

3. We may now assume that

$$p - 1 > n > (p + 1)/2. \tag{3.5}$$

By (2.5), we have $nt\equiv \pm 1$ (mod p) where t divides $p-1$. It follows that $n\equiv \mp (p-1)/t$ (mod p), and (3.5) gives

$$n = p - (p - 1)/t. \tag{3.6}$$

From (2.8), it follows that we may set

$$(p - 1)/t = q^\beta h, \tag{3.7}$$

$$h \mid g^*, \tag{3.8}$$

where $\beta\leqq b$ is a non-negative integer. Combining (3.6) and (3.7), we obtain

$$n = p - q^\beta h, \tag{3.9}$$

which implies $(n, h)=1$. Now (3.4) and (3.8) give

$$nh \mid g^*. \tag{3.10}$$

The relations (3.6) and (3.7) also yield

$$1+n = 1+p-(p-1)/t = 1+(1+tq^\beta h)-q^\beta h,$$
$$1+n = 2+q^\beta h(t-1). \tag{3.11}$$

4. If $\beta=0$, then (3.9) gives $p=n+h$. On the other hand, (3.10) and (3.2) show that $nh \leqq g^* < p-1$. Hence $nh<n+h$, and then at least one of the two positive integers n, h must be 1. But we have $n\neq 1$, and therefore h must be equal to 1. However, this would lead to $n=p-1$, which contradicts (3.5). Thus,

$$\beta > 0. \tag{3.12}$$

5. From (3.6), it follows that $nt\equiv 1 \pmod{p}$. Since $n=z_{w+1}$, we have $\delta_{w+1}=1$ in (2.5). The relation (2.6) then has the form

$$(1+n+\cdots)-(\cdots)=0, \tag{3.13}$$

where the missing terms are the non-exceptional degrees greater than 1 of the first p-block $B_1(p)$.

If $q\neq 2$, then (3.11) shows that at least one of these degrees must be prime to q, and hence a divisor of $g^*<p-1$. But the only degree m with $1<m<p-1$ is the exceptional degree (cf. (2.4)). This is a contradiction; we must have

$$q = 2. \tag{3.14}$$

6. Assume next that $\beta\geqq 2$. Then (3.11) shows that $1+n\equiv 2 \pmod{4}$, and at least one of the missing degrees in (3.13) is not divisible by 4. This degree then has the form $m=\mu$ or $m=2\mu$ with $\mu \mid g^*$. Hence $m\leqq 2g^*<2(p-1)$. On the other hand, we have $m\equiv \pm 1 \pmod{p}$ on account of (2.4). As $m\neq 1$, the only possibilities are $m=p\pm 1$. The number g^* is a multiple of $m/2$, and from (3.2) it now follows that $g^*=(p\pm 1)/2$. But then the divisor n of g^* is at most $(p\pm 1)/2$ which contradicts (3.5).

Hence the only possible case is the case $\beta=1$.

7. For $q=2$, $\beta=1$, the equation (3.9) reads

$$n = p-2h. \tag{3.15}$$

The number g^* now is odd and so are its divisors n and h. Combining (3.10), (3.2) and (3.15), we find

$$nh \leqq g^* < p-1 < n+2h. \tag{3.16}$$

If $h \neq 1$, then $h \geqq 3$ and (3.16) gives,

$$n > (n-2)h \geqq 3n - 6,$$

which is impossible as n is odd and $n \neq 1$. This proves that $h = 1$. Now (3.15) reads $n = p-2$. The multiple g^* of n then must be $p-2$; we have

$$g^* = n = p - 2, \tag{3.17}$$

$$g = p2^b(p - 2). \tag{3.18}$$

From (3.6), it follows that

$$w = (p-1)/t = 2.$$

In the equation (2.6) only three terms appear. The first two terms are 1 and $n = p-2$. The missing term therefore is $-(p-1)$ and (2.6) reads

$$1 + (p-2) - (p-1) = 0. \tag{3.19}$$

As the degree of an irreducible character, the number $p-1$ is a divisor of g. Then (3.18) shows that $p-1$ is a power of 2, say

$$p - 1 = 2^c, \qquad c \leqq b. \tag{3.20}$$

8. In order to finish the proof, we need three lemmas which we state here in a more general form than actually needed for our present purpose. The proof of these lemmas will be given in the next section.

Lemma 1. *Let $\mathfrak{G}$ be a group which is identical with its commutator subgroup $\mathfrak{G}'$, and assume that the first p-block $B_1(p)$ contains an irreducible* 1-1 *representation $\mathfrak{Z}$ of degree $z < 2p$. Then the order of the centralizer $\mathfrak{C}(\mathfrak{P})$ of a p-Sylow subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is a power of p.*

Lemma 2. *If a group $\mathfrak{G}$ with center* 1 *has an irreducible* 1-1 *representation $\mathfrak{Z}$ of degree $z = p^r$ (p a prime) and if the center $\mathfrak{C}$ of the p-Sylow subgroup $\mathfrak{P}$ of $\mathfrak{G}$ has the order p^s, then $\mathfrak{Z}$ belongs to a p-block of type at least s. In particular, $r \geqq s$. Also, $s \geqq 1$ except when $\mathfrak{G} = 1$.*

Lemma 3. *Let $\mathfrak{G}$ be a group of order $g = p^a q^b g^*$ where p and q are different primes and a, b and g^* are positive integers, $(g^*, pq) = 1$. Assume that $\mathfrak{G}$ does not contain elements of order pq. Then for every p-singular element P of $\mathfrak{G}$, we have*

$$\sum{}' z_\mu \zeta_\mu(P) \equiv 0 \pmod{q^b},$$

where the sum extends over all characters ζ_μ which belong simultaneously to a fixed p-block $B_\sigma(p)$ and to a fixed q-block $B_\tau(q)$. Here, z_μ denotes the degree of ζ_μ.

9. Assuming these lemmas, we conclude the proof of Theorem 1 as follows:

Lemma 1 shows that under the assumptions of the theorem, $\mathfrak{G}$ does not contain any element of order $2p$. From Lemma 2, it follows that the degree $p-1=2^c$ in (3.19) belongs to a 2-block $B_\tau(2)$ which is not of the lowest kind. Now apply Lemma 3 to the first p-block $B_1(p)$ and the 2-block $B_\tau(2)$. The only character in common to these two blocks is the character ζ of degree $p-1$ in (3.19), since the other degrees occurring in $B_1(p)$ are the odd numbers 1 and $p-2$ which therefore cannot occur in $B_\tau(2)$. Now the statement of Lemma 3 gives

$$z\zeta(P) = (p-1)\zeta(P) \equiv 0 \pmod{2^b}$$

for any p-singular element P of $\mathfrak{G}$. But (2.7) and (2.4) give $\zeta(P)=-1$, and hence $p-1\equiv 0 \pmod{2^b}$. Combining this with (3.20) and (3.18) we find

$$p - 1 = 2^b, \tag{3.21}$$

$$g = p(p-1)(p-2) = 2p(1 + p(p-3)/2). \tag{3.22}$$

Now Theorem B can be applied with $r=(p-1)/2$ and $m=(p-3)/2$. We have either $\mathfrak{G}\cong LF(2, p-1)$ with $p=2^b+1>3$ or $\mathfrak{G}\cong LF(2, p)$. In the second case, $g=p(p-1)(p+1)/2$. Comparison with (3.22) then gives $p=5$. In any case, $\mathfrak{G}$ is of the form stated in Theorem 1. As the converse is trivial, this finishes the proof.

4. **Proof of the lemmas.**

To complete the proof of Theorem 1, it now remains to prove the three lemmas used and formulated in the preceding section.

PROOF OF LEMMA 1. If ζ is the (irreducible) character of a 1-1 representation $\mathfrak{Z}$ of the first p-block $B_1(p)$, we have (cf. (2.2))

$$h(G)\zeta(G)/z \equiv h(G) \pmod{\mathfrak{p}}. \tag{4.1}$$

For all elements G of the centralizer $\mathfrak{C}(\mathfrak{P})$ of a p-Sylow subgroup $\mathfrak{P}$, the number $h(G)$ is prime to p and can therefore be cancelled in (4.1) and we have then

$$\zeta(G) \equiv z \pmod{\mathfrak{p}}. \tag{4.2}$$

Assume that the order of $\mathfrak{C}(\mathfrak{P})$ contains a prime factor $v\neq p$. Then $\mathfrak{C}(\mathfrak{P})$ contains a cyclic subgroup $\mathfrak{V}$ of order v. Now $\zeta(G)$ for G in $\mathfrak{V}$ may be considered as a (reducible) character of $\mathfrak{V}$ and the same is true for $\zeta^*(G)=z=1+1+\cdots+1$ (z terms). We cannot have $\zeta(G)=z$ for all G in $\mathfrak{V}$, as $\mathfrak{Z}$ then would represent every G in $\mathfrak{V}$ by the unit matrix while $\mathfrak{Z}$ is assumed to be a 1-1 representation. The congruence (4.2) implies[7]

$$\zeta(G) = z_0 + p\theta(G), \tag{4.3}$$

where z_0 is the least non-negative residue of z_0 (mod p) and where $\theta(G)$ is a reducible or irreducible character of $\mathfrak{V}$.

If the degree z of $\mathfrak{Z}$ is less than $2p$, the degree of $\theta(G)$ is 1. It follows that for every G in $\mathfrak{V}$, the matrix $\mathfrak{Z}(G)$ has z_0 characteristic roots 1, and p roots $\theta(G)$. If $\mathfrak{G}$ coincides with its commutator subgroup $\mathfrak{G}'$, every representation of $\mathfrak{G}$ represents elements of $\mathfrak{G}$ with matrices of determinant 1. Hence

$$1^{z_0}\cdot\theta(G)^p = 1.$$

But $\theta(G)$ is a vth root of unity with $(v, p) = 1$. It follows that $\theta(G) = 1$ and all characteristic roots of $\mathfrak{Z}(G)$ are 1. But this means that $\mathfrak{Z}(G) = I$ which gives a contradiction for $G \neq 1$. Hence, under the assumptions of Lemma 1, the order of $\mathfrak{C}(\mathfrak{P})$ cannot contain a prime factor $v \neq p$, and this proves Lemma 1.

Proof of Lemma 2. Let $\mathfrak{Z}$ with the character ζ be an irreducible 1-1 representation of degree $z = p^r$ of $\mathfrak{G}$. If ζ belongs to a p-block $B = B(p)$ of type α, we may find a character ζ_0 of degree z_0 in B such that z_0 is divisible by p^α but not by $p^{\alpha+1}$. According to (2.2), we have for any element G of $\mathfrak{G}$,

$$h(G)\zeta(G)/z \equiv h(G)\zeta_0(G)/z_0 \pmod{\mathfrak{p}}. \tag{4.4}$$

Now, for any element G of the center $\mathfrak{C}$ of $\mathfrak{P}$, the number $h(G)$ is prime to p, and $z = p^r$ divides $\zeta(G)$ which is a sum of p^r roots of unity. A well known argument of Burnside[8] shows that either $\mathfrak{Z}(G)$ is a scalar multiple of I or $\zeta(G) = 0$. The first possibility cannot arise for $G \neq 1$, if the center of $\mathfrak{G}$ consists only of 1. In the case $\zeta(G) = 0$, (4.4) yields

$$\zeta_0(G) \equiv 0 \pmod{\mathfrak{p}p^\alpha} \qquad (\text{for } G \neq 1 \text{ in } \mathfrak{C}), \tag{4.5}$$

since $z_0 \not\equiv 0 \pmod{p^\alpha}$. On the other hand,

$$\zeta_0(1) = z_0. \tag{4.6}$$

Adding (4.6) and (4.5) for all elements $G \neq 1$ of $\mathfrak{C}$, we obtain

$$\sum \zeta_0(G) \equiv z_0 \pmod{\mathfrak{p}p^\alpha}, \tag{4.7}$$

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[7] It follows from the orthogonality relations for group characters and (4.2) that $\zeta(G)$ contains every irreducible character of $\mathfrak{V}$ with a multiplicity divisible by p with the exception of the 1-character only. In the case of the 1-character, the multiplicity is congruent to z (mod p).

[8] Burnside, p. 322, Theorem I.

where the sum extends over all elements G of $\mathfrak{C}$. The expression on the left side is divisible by the order p^s of $\mathfrak{C}$ (cf. (2.1)). Since $z_0 \not\equiv 0 \pmod{p^{\alpha+1}}$, the congruence (4.7) shows that $s \leqq \alpha$, and this proves the main assertion of Lemma 2. It is then clear that $r \geqq s$. If $s=0$, then g would be prime[9] to p and hence $r=0$, $z=1$. But then $\mathfrak{G}$ would be Abelian and would not have the center 1, except for $\mathfrak{G}=1$.

COROLLARY.[10] *If in Lemma 2 the p-Sylow subgroup is Abelian, then z must be the highest power of p which divides the order of $\mathfrak{G}$.*

PROOF OF LEMMA 3. Let P be a p-singular element of a group $\mathfrak{G}$, let $B_\sigma(p)$ be a fixed p-block of $\mathfrak{G}$ and set

$$(4.8) \qquad \xi_\mu = \zeta_\mu(P) \quad \text{for} \quad \zeta_\mu \text{ in } B_\sigma(p), \qquad \xi_\mu = 0 \quad \text{for} \quad \zeta_\mu \text{ not in } B_\sigma(p).$$

The equation (2.3) can be written in the form

$$(4.9) \qquad \sum_\mu \xi_\mu \zeta_\mu(Q) = 0.$$

Here Q is an arbitrary p-regular element of $\mathfrak{G}$, and we may let ζ_μ range over all characters of $\mathfrak{G}$. We can determine $a_1, a_2, \cdots, a_k$ from

$$(4.10) \qquad \xi_\mu = \sum_\kappa a_\kappa \zeta_\mu(G_\kappa),$$

where $G_1, G_2, \cdots, G_k$ represent the different classes of $\mathfrak{G}$ (this because the determinant $|\zeta_\mu(G_\kappa)|$ ($\mu, \kappa = 1, 2, \cdots, k$) does not vanish). Multiplication of (4.10) with $\zeta_\mu(Q)$ and addition over μ gives

$$\sum_\mu \xi_\mu \zeta_\mu(Q) = \sum_\kappa a_\kappa \sum_\mu \zeta_\mu(G_\kappa)\zeta_\mu(Q).$$

The expression on the left side vanishes on account of (4.9). The orthogonality relations for the group characters show that the inner sum on the right side is different from 0 only for that element G_κ which is conjugate to Q^{-1}. Hence $a_\kappa = 0$ when G_κ is conjugate to Q^{-1}. Now since Q^{-1} as well as Q may be any p-regular element, it follows that $a_\kappa = 0$ when G_κ is p-regular. It will consequently suffice to let G_κ in (4.10) range over all p-singular elements.

Take Q now as a q-singular element of $\mathfrak{G}$. Since $\mathfrak{G}$ does not contain elements of order pq, the element Q must be p-regular, and can therefore be used in (4.9). Applying (2.3) to the q-block $B_\tau(q)$ of $\mathfrak{G}$, we have for any q-regular element G_κ the equation

$$(4.11) \qquad \sum_\rho{}^{*} \zeta_\rho(G_\kappa)\zeta_\rho(Q) = 0,$$

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[9] Burnside, p. 119, Theorem I.

[10] R. Brauer, loc. cit. footnote 6, p. 78, Lemma 5.

where ζ_ρ ranges over all characters of $B_\tau(q)$. In particular, this will hold for p-singular elements G_κ, that is for all G_κ actually appearing in (4.10). Multiplication of (4.11) with a_κ and subsequent addition over all p-singular G_κ gives

$$\sum_\kappa \sum_\rho{}^* a_\kappa \zeta_\rho(G_\kappa)\zeta_\rho(Q) = 0.$$

On account of (4.10) this may be written in the form

$$\sum_\rho{}^* \xi_\rho \zeta_\rho(Q) = 0. \tag{4.12}$$

Set now

$$S(G) = \sum_\rho{}^* \xi_\rho \zeta_\rho(G) \tag{4.13}$$

for any G in $\mathfrak{G}$. Then $S(G)$ is a linear combination of the characters of $\mathfrak{G}$, the coefficients ξ_ρ are algebraic integers as follows from (4.8). On account of (4.12), $S(G)$ vanishes for all $G \neq 1$ belonging to a q-Sylow subgroup $\mathfrak{Q}$; we obtain

$$\sum S(G) = S(1) = \sum_\rho{}^* \xi_\rho z_\rho,$$

where G ranges over all elements of $\mathfrak{Q}$. The left side is a sum of the kind studied in (2.1) and hence it is divisible by the order q^b of $\mathfrak{Q}$. Consequently,

$$\sum{}^* \xi_\rho z_\rho \equiv 0 \pmod{q^b}.$$

Here, ρ ranges over those values for which ζ_ρ lies in $B_\tau(q)$. As defined in (4.8), ξ_ρ is 0 if ζ_ρ does not belong to $B_\sigma(p)$. We thus obtain

$$\sum{}' \zeta_\rho(P) z_\rho \equiv 0 \pmod{q^b},$$

here the sum extends over those values of ρ for which ζ_ρ belongs to both $B_\sigma(p)$ and $B_\tau(q)$. This proves Lemma 3.

COROLLARY. *If the order of a group $\mathfrak{G}$ is divisible by two different primes p and q, and if $\mathfrak{G}$ does not contain elements of order pq, then the first p-block $B_1(p)$ of $\mathfrak{G}$ and the first q-block $B_1(q)$ of $\mathfrak{G}$ have at least one character $\zeta_\mu \neq 1$ in common.*

5. **Simple groups of order prq^b.** We now prove the following theorem.

THEOREM 2.[11] *If a simple group $\mathfrak{G}$ has an order of the form $g = prq^b$*

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[11] This result was announced without proof in R. Brauer, Proc. Nat. Acad. Sci. U. S. A. vol. 25 (1939) p. 290.

where p, q and r are primes and where b is a positive integer, then

$$\mathfrak{G} \cong LF(2, 5),\ g = 60, \quad or \quad \mathfrak{G} \cong LF(2, 7),\ g = 168.$$

PROOF. It follows from a well known theorem of Burnside[12] that the primes p, q, r must be distinct. As then both p and r must be odd, we may assume without restriction that $r < p-1$. Now Theorem 1 can be applied. Two cases are possible:

$$\text{(I)} \qquad g = prq^b = p(p-1)(p+1)/2, \qquad p = 2^b \pm 1;$$

$$\text{(II)} \qquad g = prq^b = p(p-1)(p-2), \qquad p = 2^b + 1;$$

with $q=2$ and $p \neq 3$. In both cases, g is divisible by 3 and hence $r=3$. In the first case, this gives

$$3 \cdot 2^{b+1} = (p-1)(p+1).$$

Not both factors on the right are divisible by 4. Hence either $p+1$ or $p-1$ divides 6. Then $p=5$ or $p=7$ and this leads to $\mathfrak{G} \cong LF(2, 5)$ or $\mathfrak{G} \cong LF(2, 7)$. In the second case, we obtain

$$3 \cdot 2^b = (p-1)(p-2).$$

It follows that the odd number $p-2$ must be 3 and this gives $p=5$, $\mathfrak{G} \cong LF(2, 4) \cong LF(2, 5)$. Thus Theorem 2 is proved.

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[12] Burnside, p. 323, Theorem I, Corollary 3.

BOUNDS FOR THE CHARACTERISTIC ROOTS OF A MATRIX

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In a recent paper [1],[1] A. B. Farnell reiterated the possibility of the existence of a certain upper bound for the characteristic roots of a matrix. This bound was first conjectured by W. V. Parker (cf. [1]), and was, in fact, proved in the case of matrices of order 2 by Farnell. We shall here, in Theorem 1 below, establish that bound for matrices of *any* order n. Two previous results on bounds for characteristic roots, one due to Parker [2], and one to Farnell [1], will be stated for purposes of comparison with Theorem 1. But first, we make certain definitions, following, for the most part, those of Farnell, for the sake of uniformity.

We let $A = \|a_{rs}\|$, of order n, be any matrix of complex numbers, with a characteristic root $\lambda = \alpha + i\beta$ different from 0. Then there exists a non-null vector $\{x_r\}$, $r = 1, \cdots, n$, such that

$$(1) \qquad \lambda x_r = \sum_s a_{rs} x_s.$$

With A^* denoting the hermitian-conjugate of A, set

$$(A + A^*)/2 = B = \|b_{rs}\|, \qquad (A - A^*)/2i = C = \|c_{rs}\|,$$

both of which matrices are hermitian. Finally, define

$$R_r = \sum_s |a_{rs}|, \qquad T_s = \sum_r |a_{rs}|,$$

$$R = \max_{(r)} R_r, \qquad T = \max_{(s)} T_s,$$

$$S_r' = \sum_s |b_{rs}| = \sum_s |b_{sr}|, \qquad S_r'' = \sum_s |c_{rs}| = \sum_s |c_{sr}|.$$

Now the theorems of Parker and Farnell may be stated briefly as follows:

PARKER'S THEOREM. *If S, S', and S'' are the greatest of the $(R_r+T_r)/2$, S_r', and S_r'', respectively, then*

$$|\lambda| \leqq S, \qquad |\alpha| \leqq S', \qquad |\beta| \leqq S''.$$

FARNELL'S THEOREM.[2] *$|\lambda|^2$ is bounded above by RT.*

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] This is not the strongest bound that Farnell obtains.

It should be pointed out that both these theorems were proved for any λ in the field of values of the matrix A, and thereby in particular for the characteristic roots. On the other hand, the results that we shall establish below are not necessarily true for all numbers in the field of values. One observes that these theorems, and all that follow, are trivially true when $\lambda = 0$. We therefore assume throughout that $\lambda \neq 0$.

We now prove the following theorem.

THEOREM 1. $|\lambda|^2$ *is bounded above by* $\max_{(r)} (R_r T_r)$.

From the equalities (1) we deduce immediately the inequalities

$$(2) \qquad |\lambda||x_r| \leqq \sum_s |a_{rs}||x_s| = \sum_s |a_{rs}|^{1/2} \cdot |a_{rs}|^{1/2}|x_s|.$$

Applying Schwarz's inequality to the right-hand side in the manner indicated by the preparation, we obtain

$$|\lambda||x_r| \leqq \left\{\left(\sum_s |a_{rs}|\right)\left(\sum_s |a_{rs}||x_s|^2\right)\right\}^{1/2}$$
$$= R_r^{1/2}\left(\sum_s |a_{rs}||x_s|^2\right)^{1/2};$$

or, squaring this,

$$(3) \qquad |\lambda|^2|x_r|^2 \leqq R_r \sum_s |a_{rs}||x_s|^2.$$

Suppose first that no R_r vanishes. Then we divide the rth inequality by R_r, and sum over r, so obtaining

$$|\lambda|^2 \sum_r \frac{|x_r|^2}{R_r} \leqq \sum_{r,s} |a_{rs}||x_s|^2 = \sum_s T_s |x_s|^2.$$

This may be written

$$\sum_r \left(\frac{|\lambda|^2}{R_r} - T_r\right)|x_r|^2 \leqq 0.$$

It follows that, for some m,

$$\frac{|\lambda|^2}{R_m} - T_m \leqq 0$$

or

$$|\lambda^2| \leqq R_m T_m \leqq \max_{(r)} (R_r T_r),$$

and the theorem is proved if $R_r \neq 0$ for each r.

Now suppose that some R_r vanishes. By a permutation of the inequalities (3), and a like permutation of the terms in the right-hand member of each inequality (a similarity transformation of the matrix A!), we may take $R_n=0$. Then, since $\lambda\neq 0$, we must have $x_n=0$. The inequalities (3), which are n in number, and in each of which the summation extends over the range $s=1, 2, \cdots, n$, are thus reduced to the $n-1$ inequalities

$$|\lambda|^2|x_r|^2 \leqq R_r\sum_{s=1}^{n-1}|a_{rs}||x_s|^2, \qquad r = 1, 2, \cdots, n-1.$$

If, now, $R_r\neq 0$ for $r=1, 2, \cdots, n-1$, we apply to these inequalities the arguments above, and obtain

$$|\lambda|^2 \leqq \max_{(r)}\left(R_r\cdot\sum_{s=1}^{n-1}|a_{sr}|\right).$$

But $R_n=0$ implies that $|a_{nr}|=0$ for each $r=1, 2, \cdots, n$. Hence, for each r,

$$\sum_{s=1}^{n-1}|a_{sr}| = \sum_{s=1}^{n}|a_{sr}| = T_r$$

and so, again, as was to be proved,

$$|\lambda|^2 \leqq \max_{(r)}(R_rT_r).$$

In general, by the procedure just illustrated, one shows that

$$|\lambda|^2 \leqq \max_{(r)}\left(R_r\cdot\sum_{\tau}|a_{\tau r}|\right),$$

where τ ranges over those indices κ such that $R_\kappa\neq 0$. But for those indices s, such that $R_s=0$, one has $|a_{sr}|=0$ for each $r=1, 2, \cdots, n$, so that

$$\sum_{\tau}|a_{\tau r}| = \sum_{s=1}^{n}|a_{sr}| = T_r.$$

Thus, the theorem is proved in every case.

Parker's bound on $|\lambda|$ follows from Theorem 1, since if $|\lambda|^2\leqq R_mT_m$, then

$$|\lambda| \leqq (R_mT_m)^{1/2} \leqq (R_m+T_m)/2 \leqq S.$$

Furthermore, if $|\lambda|^2\leqq R_mT_m$, then $R_m\leqq R$ and $T_m\leqq T$, and we have $|\lambda|^2\leqq RT$, which is Farnell's result.

By the method employed above, one can establish the following, somewhat stronger theorem.

THEOREM 2. *Let* $0 \leqq k \leqq 2$, *and define* $R_r^{(k)} = \sum_s |a_{rs}|^k$, $T_s^{(k)} = \sum_r |a_{rs}|^k$. *Then, for each fixed* k *in the specified interval,* $|\lambda|^2$ *is bounded above by* $\max_{(r)} (R_r^{(k)} T_r^{(2-k)})$.

To prove this, one applies the very same train of arguments as in the proof of Theorem 1, starting with the inequalities (2) in the form

$$|\lambda| |x_r| \leqq \sum_s |a_{rs}|^{k/2} \cdot |a_{rs}|^{1-k/2} |x_s|.$$

By permitting k to vary with r and s in these last inequalities, one obtains a still stronger generalization of Theorem 1.

We state one more result, which obtains without the aid of Schwarz's inequality, and which again implies Farnell's theorem.

THEOREM 3. $|\lambda|$ *is bounded above by both* R *and* T.

Let $|x_m| = \max_{(r)} |x_r|$; then, dividing the inequality

$$|\lambda| |x_m| \leqq \sum_s |a_{ms}| |x_s|$$

through by $|x_m|$, we have

$$|\lambda| \leqq \sum_s |a_{ms}| \frac{|x_s|}{|x_m|} \leqq \sum_s |a_{ms}| = R_m \leqq R.$$

Now, sum all the following inequalities:

$$|\lambda| |x_r| \leqq \sum_s |a_{rs}| |x_s|;$$

we obtain

$$|\lambda| \sum_r |x_r| \leqq \sum_s T_s |x_s|$$

or

$$\sum_s (|\lambda| - T_s) |x_s| \leqq 0.$$

Hence, for some m,

$$|\lambda| \leqq T_m \leqq T.$$

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NOTE ON FACTORIZATION IN A QUADRATIC FIELD

GORDON PALL

1. **Introduction.** In this note we shall prove certain theorems relating to the "existence" and "uniqueness" of factorization in a quadratic field (cf. §§2 and 3); and shall maintain that the introduction of ideals should be regarded as restoring *existence* rather than *uniqueness* of factorization into primes.

To illustrate this, let us consider first the case of quaternions. Let $x=x_0+i_1x_1+i_2x_2+i_3x_3$ be a *primitive* quaternion, that is, let the coordinates $x_0, \cdots, x_3$ be relative-prime rational integers. Let the norm $Nx=\sum x_i^2$ be factored into a product of rational primes $p_1 \cdots p_s$. Then, by a theorem of Lipschitz,[1] there exist prime quaternions $t', t'', \cdots, t^{(s)}$, of respective norms $p_1, \cdots, p_s$ such that $x=t't'' \cdots t^{(s)}$. This factorization is unique, for any given ordering of the primes $p_1, \cdots, p_s$, except that we can insert unit factors in the trivial way illustrated by the example $t't''t'''=(t'i_1)(i_1t''i_3)(i_3t''')$ $=(-t')(t''i_2)(i_2t''')=\cdots$.

It is proved elsewhere that a similar uniqueness of factorization holds in every system of "generalized quaternions," but that the existence of such a factorization will fail if certain rational primes p_i are not norms.

As is well known there exists a very satisfactory arithmetic of ordinary quaternions, without the necessity of introducing ideals. Nevertheless, factorization of imprimitive quaternions is not unique. For example,

$$\begin{aligned}6 &= (1-i_1-i_2)(1-i_1)(1+i_1)(1+i_1+i_2)\\ &= (1-i_1-i_3)(1-i_1)(1+i_1)(1+i_1+i_3),\end{aligned}$$

where the primes $1-i_1-i_2$ and $1-i_1-i_3$ do not differ only by unit factors.

Similarly, in the quadratic field $R(\rho)$, where $\rho^2=-5$, we have

$$6 = (1+\rho)(1-\rho) = 2\cdot 3,$$

where the factors are essentially different prime integers of the field, and hence factorization is not unique. Yet a uniqueness theorem analogous to that for ordinary quaternions holds for the factoriza-

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[1] Lipschitz, Journal de Mathématiques (4) vol. 2 (1886) pp. 373–439; Hurwitz, *Vorlesungen uber die Zahlentheorie der Quaternionen*, 1919.

tion of *primitive* integers in every quadratic field.

Indeed, let Δ be a non-square integer, $\Delta\equiv\epsilon \pmod 4$, $\epsilon=0$ or 1, $\Delta/(2-\epsilon)^2$ be squarefree. Set $\rho=(-\epsilon+\Delta^{1/2})/2$. Then the integers of the field $R(\rho)$ have the form $x=x_0+x_1\rho$, where x_0 and x_1 are rational integers. The norm of x is $Nx=x_0^2-\epsilon x_0x_1+(\epsilon^2-\Delta)x_1^2/4$. Suppose that x is primitive, that is, $(x_0, x_1)=1$, and factor $Nx=p_1 \cdots p_s$ as a product of ordinary primes p_i. Then (as a corollary of Theorem 1) the factorizations $x=t't'' \cdots t^{(s)}$ with $N(t^{(i)})=p_i$ are unique apart from unit factors. However, the prime factors p_i of Nx need not be norms, and so such factorizations need not exist.

When ideals are introduced into the quadratic field, there are ideals of every prime norm which can divide the norm of a primitive integer. For, if $(\Delta|p)=-1$, then $p|Nx$ implies that $p|x$. Hence the principal service performed by the introduction of ideals is to restore existence rather than uniqueness of factorization into "primes." It is the fact that every prime is a sum of four squares that makes the arithmetic of quaternions satisfactory; and the fact that every prime p such that $(\Delta|p)\neq-1$ is a norm (in essentially only one way) that makes the fundamental theorem of arithmetic hold in those quadratic fields in which there is only one class of forms of discriminant Δ.

2. **Quadratic integers; unique factorization.** Let Δ be a non-square integer, $\Delta\equiv\epsilon \pmod 4$, where $\epsilon=0$ or 1. Set $\rho=(-\epsilon+\Delta^{1/2})/2$. We shall consider factorization in the ring of quadratic integers $x=x_0+x_1\rho$, where x_0 and x_1 are rational integers. The letters $t, \cdots, z$ (without subscripts) will be reserved for such integers. We do not restrict attention to the case where $\Delta/(2-\epsilon)^2$ is squarefree. This restriction is commonly made in books on quadratic fields, and does in fact make the arithmetical theory simpler than it would otherwise be, since it excludes from consideration certain complicated cases. But, arithmetically, we are as much interested in the complicated cases (such as, say $x_0+x_1(13)^{1/2}$) as in the others, and with very little effort can remove the restriction. It is to be noted that if $\Delta/(2-\epsilon)^2$ is squarefree, our set of integers is the same as that in the classical theory.

It will be observed that x (along with its conjugate $\bar{x}=x_0+x_1\bar{\rho}$, where $\bar{\rho}=(-\epsilon-\Delta^{1/2})/2$) satisfies the equation

$$x^2-(2x_0-\epsilon x_1)x+(x_0^2-\epsilon x_0x_1+(\epsilon^2-\Delta)x_1^2/4)=0.$$

The rational integers $x+\bar{x}=2x_0-\epsilon x_1$ and $x\bar{x}=x_0^2-\epsilon x_0x_1+(\epsilon^2-\Delta)x_1^2/4$ are called the trace and norm of x, respectively, the latter denoted by Nx.

If $x=yz$, evidently $\bar{x}=\bar{y}\bar{z}$, whence $Nx=Ny\cdot Nz$. Hence if t is a divisor of x, $Nt|Nx$. If θ denotes any unit, that is if $N\theta=1$, the associates θt are (with t) factors of x, all with the same norm.

THEOREM 1. *If x is primitive (that is, x_0, x_1 coprime), and*

$$x = ut = vt', \qquad Nt = Nt',$$

then t and t' are associates, except possibly when Nt is divisible by a prime p for which Δ/p^2 is an integer congruent to 0 *or* 1 mod 4.[2]

The proof is made up of three lemmas:

LEMMA 1. *If $x\equiv y$ (mod m), then x and y have the same factors of norm m.*

PROOF. If $x=ut$ and $Nt=m$, then $x+zm=(u+z\bar{t})t$.

LEMMA 2. *If $ut=vt'$, $Nt=Nt'=m$, and Nv is prime to m, then t and t' are associates.*

PROOF. Let $k\cdot Nv\equiv 1$ (mod m). Then $k\bar{v}ut=k\cdot Nv\cdot t'$. By Lemma 1, t' has t for a factor, $t'=wt$, $Nt'=Nw\cdot Nt$, $Nw=1$.

LEMMA 3. *If x is primitive, and $x=ut$ where $Nt=m$, then we can find an integral z such that $\{N(x+zm)\}/m$ is prime to m, except possibly when m is not semiprime to Δ.*

PROOF. Set $x\bar{x}=km$, that is, $(2x_0-\epsilon x_1)^2-\Delta x^2=4km$. We have

$$\begin{aligned} N(x + zm) &= (x + zm)(\bar{x} + \bar{z}m) = x\bar{x} + (x\bar{z} + z\bar{x})m + z\bar{z}m^2 \\ &= m\{k + (2x_0 - \epsilon x_1)z_0 \\ &\qquad + (-\epsilon x_0 + (\epsilon^2 - \Delta)x_1/2)z_1 + mNz\}. \end{aligned}$$

We can evidently choose z_0 and z_1 to make $\{N(x+zm)\}/m$ prime to m, except when for some prime p dividing m and k,

$$(1) \qquad 2x_0 - \epsilon x_1 \equiv 0 \equiv -\epsilon x_0 + (\epsilon^2 - \Delta)x_1/2 \pmod{p}.$$

Since $(x_0, x_1)=1$ this requires that $p^2|\Delta$ if $p>2$. If $p=2$ and $\epsilon=1$, (1) implies $2|x_0$ and $2|x_1$, a contradiction. If $p=2$ and $\epsilon=0$, then $4|x_0{}^2-\Delta x_1{}^2/4$, implying x_0 and x_1 even, if $\Delta/4\equiv 2$ or 3 (mod 4).

3. **Conditions for the existence of factors of given norm.** For a given x, and a given rational integer m which satisfies the obviously necessary conditions

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[2] Hence there is no exception if $\Delta/(2-\epsilon)^2$ is squarefree. We shall say that m is semiprime to Δ if m is divisible by no prime p such that $p^2|\Delta$ $(p>2)$, nor by the prime 2 if $\Delta/4\equiv 0$ or 1 (mod 4). All integers are semiprime to Δ if $\Delta/(2-\epsilon)^2$ is squarefree.

$$(2)\qquad m \text{ is a norm}, \qquad m \mid Nx,$$

we investigate the set of divisors t of x having norm m. Consider then

$$(3)\qquad x = ut, \qquad Nt = m.$$

These conditions on t are equivalent to the following:

$$(4)\qquad x\bar{t} \equiv 0 \ (\text{mod } m), \qquad Nt = m.$$

The condition $x\bar{t} \equiv 0$ expands into the following:

$$(5)\qquad x_1t_0 - x_0t_1 \equiv 0 \ (\text{mod } m),$$

$$(6)\qquad x_0t_0 - (\epsilon x_0 - (\epsilon^2 - \Delta)x_1/4)t_1 \equiv 0 \ (\text{mod } m).$$

Hereafter we assume, for simplicity, that:

$$(7)\qquad x \text{ is primitive, and } m \text{ is semiprime to } \Delta.$$

Since $m \mid x_0^2 - \epsilon x_0x_1 + (\epsilon^2 - \Delta)x_1^2/4$, $(m, x_1) = 1$. Hence (5) can be written $t_0 \equiv \lambda t_1$ (mod m), where $\lambda \equiv x_0/x_1$. If this is put into (6), (6) reduces to $(t_1/x_1)Nx \equiv 0$ (mod m), a consequence of (2_2).

Hence, under assumptions (2) and (7_1), (3) holds if and only if

$$(8)\qquad Nt = m, \qquad t_0 \equiv \lambda t_1 \ (\text{mod } m).$$

Putting $t_0 = my_0 + \lambda y_1$, $t_1 = y_1$, into (8_1), we get

$$(9)\qquad my_0^2 + (2\lambda - \epsilon)y_0y_1 + ky_1^2 = 1,$$

where k is the integer defined by

$$km = \lambda^2 - \epsilon\lambda + (\epsilon^2 - \Delta)/4.$$

The binary form $\phi = my_0^2 + (2\lambda - \epsilon)y_0y_1 + ky_1^2$ has discriminant Δ.

Now it is well known that only one class of binary quadratic forms of discriminant Δ can represent 1, namely the principal class, containing the form $\phi_0 = x_0^2 - \epsilon x_0x_1 + (\epsilon^2 - \Delta)x_1^2/4$.

There are certain cases in which we can be sure that ϕ belongs to the principal class. An integer m semiprime to Δ is represented by forms in at most one genus of discriminant Δ. Hence if ϕ_0 is in a genus of one class, ϕ, which represents m, must be equivalent to ϕ_0. In particular the form $x_0^2 + 5x_1^2$ is included in this case.

Secondly, a prime is represented in at most one class and its reciprocal. Hence $\phi \sim \phi_0$ if $\pm m$ is a prime.

Finally we shall state without proof necessary and sufficient conditions that an integer m which is semiprime to Δ be isolated, that is, be represented by at most one class and its reciprocal class under composition. First suppose that the primes $p_1, \cdots, p_r$ are represented in non-ambiguous classes $G_1, \cdots, G_r$, and that $a_1, \cdots, a_r$ are positive integers. Then $n_1 = p_1^{a_1} \cdots p_r^{a_r}$ is isolated if and only if either $r=0$, or $r=1=a_1$, or

$$a_1 + a_2 + \cdots + a_r \text{ is odd}, G_1^2 = \cdots = G_r^2, G_1^4 = \cdots = G_r^4 = 1.$$

Next, if n_1 contains only primes represented in non-ambiguous classes, and n_2 only primes represented only in ambiguous classes (these include primes dividing the discriminant but semiprime to it), then n_1n_2 is isolated if and only if n_1 is isolated. Finally, the only other primes which may divide an integer m which is semiprime to Δ are primes such that $(\Delta|p) = -1$; such primes must appear in m to an even power, if m is represented at all, and can be cancelled out of m and its representations.

It is only when there is but one class of forms of discriminant Δ that every prime p such that $(\Delta|p) \neq -1$, and p is semiprime to Δ, is a norm. Further, every such prime is a norm in only one way, so that its factors are unique. Hence in this case we can easily deduce from the preceding theory that every quadratic integer whose norm is semiprime to Δ has a unique expression into factors, which are either of prime norm or are themselves rational primes such that $(\Delta|p) = -1$.

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TRANSFINITE RATIONALS

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The standard method of extending a system with an operation to a system with the operation and its inverse depends heavily on the law of cancellation: $ab=ac$ implies $b=c$ (possibly with some limitation on a). The most familiar example is undoubtedly the extension of the integers to the rational numbers. If the law of cancellation fails, not only does the method fail, but the extension cannot be a group. However, one may ask whether some method of extension is available in certain cases where the cancellation law fails but is replaced by something else, for example an order relation. It is the purpose of this paper to present an extension of the system of cardinal numbers (the positive cardinals, excluding zero). This will be an extension of both the cardinal numbers and the positive rational numbers with respect to the operations of addition and multiplication and the relation of ordering. Furthermore, this extension is the smallest extension subject to certain conditions. We shall assume the axiom of choice in the form of the simple ordering of the cardinals.

The means for obtaining the extension is suggested by a treatment of ratios by Eudoxus. (See for instance E. T. Bell, *Development of mathematics*, New York, 1940, p. 61.)

We define first an equivalence relation between ordered pairs of cardinal numbers: $(a, b)\rho(c, d)$ *if and only if* (1) $a>b$ *and* $c>d$, *or* (2) $a=b$ *and* $c=d$, *or* (3) $a<b$ *and* $c<d$.

In terms of the relation ρ we define the fundamental equivalence relation between ordered pairs of cardinal numbers:

DEFINITION. $(a, b)\sim(c, d)$ *if and only if for every pair of cardinal numbers, m and n, $(ma, nb)\rho(mc, nd)$.*

The relation $\sim$ is readily shown to be an equivalence relation. Before discussing the equivalence classes defined by this relation we shall state a theorem, assuming for the first two parts that $(a_1, b_1)\sim(a_2, b_2)$.

THEOREM. I. *If a_1 and b_1 are finite so are a_2 and b_2.* II. *If $a_1<b_1$ and b_1 is infinite, then $b_1=b_2$. If $a_1>b_1$ and a_1 is infinite, then $a_1=a_2$.* III. *If $a_1<b$, $a_2<b$, and b is infinite, then $(a_1. b)\sim(a_2, b)$. If $a>b_1$, $a>b_2$, and a is infinite, then $(a, b_1)\sim(a, b_2)$.* IV. *If $(a, a)\sim(b, b)$ and a and b are infinite, then $a=b$.*

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As a sample proof we shall prove the first half of part II. Choose $m=b_1$, $n=1$. Since $b_1a_1=1\cdot b_1$, therefore $b_1a_2=1\cdot b_2$. But $a_2<b_2$, and therefore $b_1=b_2$.

Each equivalence class $\{(a, b)\}$ defined by the relation $\sim$ will be called a *rational*, and the entire set will be denoted by R. If a and b, for any equivalence class representative, are finite, the class will be called a *finite rational*; otherwise, a *transfinite rational*. We shall now define ordering and multiplication of rationals. In each case it is necessary to show that the definition is independent of the representatives chosen. The proofs for the two cases are similar, and only that for multiplication will be given.

DEFINITION OF ORDERING. $\{(a, b)\} \leqq \{(c, d)\}$ *if and only if for every pair of cardinal numbers*, m *and* n,

$$ma > nb \quad \textit{implies} \quad mc > nd, \qquad ma = nb \quad \textit{implies} \quad mc \geqq nd.$$

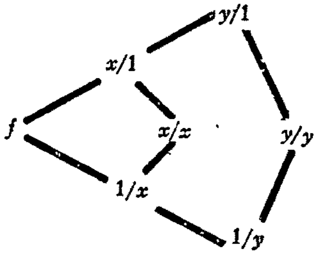
Remarks. 1. The definition of $\leqq$ is independent of the equivalence class representatives. 2. If $\{(a,b)\}=\{(c,d)\}$, then $\{(a,b)\}\leqq\{(c,d)\}$. 3. If $\{(a, b)\}\leqq\{(c, d)\}$ and $\{(a, b)\}\geqq(c, d)\}$, then $\{(a, b)\} = \{(c, d)\}$. 4. $\{(a, b)\}\leqq\{(c, d)\}$ if and only if $\{(b, a)\}\geqq\{(d, c)\}$. 5. The relation $<$ is defined in terms of $\leqq$ and $=$. 6. The rationals form a partially ordered system.

DEFINITION OF MULTIPLICATION. $\{(a, b)\}\cdot\{(c, d)\}=\{(ac, bd)\}$.

Remarks. 1. $(a_1, b_1)\sim(a_2, b_2)$ implies $(a_1c, b_1d)\sim(a_2c, b_2d)$. [In order to show that $(ma_1, nb_1)\rho(ma_2, nb_2)$ implies $(qa_1c, rb_1d)\rho(qa_2c, rb_2d)$, let $m=qc$ and $n=rd$.] 2. $(a_1, b_1)\sim(a_2, b_2)$ and $(c_1, d_1)\sim(c_2, d_2)$ imply $(a_1c_1, b_1d_1)\sim(a_2c_2, b_2d_2)$. $[(a_1c_1, b_1d_1)\sim(a_2c_1, b_2d_1)\sim(a_2c_2, b_2d_2)$.] 3. Multiplication is commutative and associative. 4. $\{(a, b)\}=\{(ac, bc)\}$ only if c is not too large.

In order to imbed the cardinals in R we define the correspondence $a\rightleftarrows\{(a, 1)\}$. It is readily shown that this correspondence is one-one, preserves ordering, and is isomorphic with respect to multiplication. Furthermore, the definitions of ordering and multiplication in the case of the finite rationals are equivalent to those normally given for rational numbers. The finite rational $\{(1, 1)\}$ is a multiplicative unit for the entire system R.

Adopting the notation a/b for $\{(a, b)\}$, letting f represent any finite rational, and using x and y to represent any two transfinite cardinals where $x<y$, we can indicate the order structure of R by means of an abbreviated Hasse diagram:



The complete diagram would contain three sequences of transfinite rationals radiating from f. From the above diagram it can be seen that R is a non-modular lattice. It might be noted that the product of any two rationals, not both finite, is represented on the diagram by the one which is farther to the right, unless the two are distinct and situated on a common vertical, in which case the product is the first element to the right of both. (One could obtain a new ordering by giving the complete Hasse diagram a quarter turn, the result being a distributive lattice where multiplication is identical with the lattice operation $\cup$. In this case it is to be understood that only one finite rational is included.)

DEFINITION OF DIVISION. $a/b \div c/d = ad/bc$.

Since $a/1 \div b/1 = a/b$, the notation A/B will be used for quotients of elements of R as well as for equivalence classes of pairs, without danger of confusion. Although many familiar laws involving division persist, let us observe that multiplication and division are not inverse operations in the usual sense. For example, a rational divided by itself is the unit element only if the rational is finite.

The definition of addition which first comes to mind is $a/b + c/d = (ad+bc)/bd$, particularly since it can be shown that this is independent of the representatives chosen. However, this possibility will be rejected for two reasons. In the first place, the expected distributive law fails, and secondly the corresponding formula for adding quotients of elements of R fails. A more satisfactory method is to impose certain requirements and prove that the resulting operation exists and is unique. To furnish a more concise arrangement of material, a third method has been chosen which is equivalent to the second. The definition will be given in a form which is apparently arbitrary, but it will be justified by properties given subsequently.

DEFINITION OF ADDITION. *$A+B=A\cup B$ unless A and B are both finite rationals, in which case the sum corresponds to the sum of the corresponding rational numbers.*

We now state and prove the principal theorem which enumerates properties of R and establishes its minimal character.

THEOREM. *The following conditions are satisfied by the system of rationals R. Furthermore, R can be imbedded isomorphically with respect to the operations of multiplication, addition, division, $\cup$, and $\cap$ in any lattice L with operations of multiplication, addition, and division defined, subject to the following conditions*: I. *L is an extension of the positive rational numbers and the cardinals with respect to addition, multiplication, division, and countable $\cup$ and $\cap$, wherever these are defined among the positive rational numbers and the cardinals.* II. *Addition and multiplication are commutative and associative. The following distributive laws hold: multiplication over addition, over $\cup$, and over $\cap$; addition over $\cup$.* III. *There exists a unit I such that $AI = A/I = A$.* IV. $(A/B)(C/D) = (A/B)/(D/C) = AC/BD$. V. *$A < B$ implies $I/A > I/B$.* VI. *$I < A < B$ implies $A/B < I$.* VII. $A \leqq A + B$.

Proof that R actually has the properties claimed will be omitted, since there is no difficulty present. Proof of the distributive laws consists of verification of all cases, but this is simplified by the partial equivalence between addition and $\cup$, and the duality between $\cup$ and $\cap$ where multiplication is involved.

Assume now that L is a lattice with the given properties. The distributive laws for addition and multiplication over $\cup$ will be used in the forms: $A < B$ implies $A + C \leqq B + C$ and $AC \leqq BC$. For the sake of compactness the following notational conventions will be made: the letter F will denote an element of L, corresponding to a positive rational number, P an element corresponding to a positive integer, and T, X, and Y elements corresponding to transfinite cardinals with the added assumption that $X < Y$. Proof that L is an extension of R is divided into four parts.

I. *Ordering*. 1. By property I, $P < T$. 2. $F < T$, since F is less than some P. 3. By property V, $I/Y < I/X < F < X < Y$. 4. By property V and the fact that T/T is not a multiplicative unit, there can be no order relation between I and T/T. 5. By property VI, $I < Y/X$. Since multiplication is distributive over $\cup$, $I < Y/X \leqq Y/I = Y$. Assume $I < Y/X < Y$. Then by property VI, $(Y/X)/Y = Y/Y < I$. Since this is impossible, $Y/X = Y$. 6. Similarly, $T/F = FT = T$ and $F/T = I/T$. 7. By property V, there is no order relation between X/X and Y/Y unless they are equal. Assume $Y/Y = X/X$. Then $(Y/Y)(Y/X) = (X/X)(Y/X)$ or $Y/Y = Y/X > I$. This is impossible. 8. Similarly, there can be no order relation between X and Y/Y, or between I/X and Y/Y. 9. $I/T < T/T < T$. In establishing these order relations between quotients of certain pairs of elements of L, we have incidentally proved the more fundamental fact that the obvious imbedding correspondence is one-one. For example, the distinct elements $x/1$, $1/y$, and y/y of R correspond to distinct elements X, I/Y, and Y/Y of L.

II. *Multiplication and division.* Proof is trivial by property IV and by what has been proved in part I.

III. *Addition.* A sample collection of pairs of quotients of elements of L will be chosen, and the sum of each pair determined. 1. $T \leqq F+T \leqq T+T$. By property I, $T+T=T$, and therefore $F+T=T$. 2. By a distributive law, $I/T+I/T=(I+I)/T=F/T=I/T$. More generally, any element of L corresponding to a transfinite rational is idempotent with respect to addition as well as multiplication, $\cup$, and $\cap$. 3. It follows from the inequalities $T/T \leqq I/T+T/T \leqq T/T+T/T=T/T$ that $I/T+T/T=T/T$. 4. Assume $I=I+T/T$. Multiplication by I/T gives $I/T=I/T+T/T=T/T$. This is false, and therefore, since $I \leqq I+T/T$, $I<I+T/T$. 5. Since $I<T$ and $T/T<T$, $I+T/T \leqq T+T=T$. Assume $I<I+T/T<T$. Then by property VI, $(I/T)(I+T/T)<I$, or $I/T+T/T=T/T<I$. This is impossible, and therefore $I+T/T=T$. 6. $F+T/T=F(I+T/T)=FT=T$. 7. $X/X+Y/Y=(X/X)(I+Y/Y)=(X/X)(Y)=Y$. 8. $F+I/T \leqq F+F'$, where F and F' correspond to arbitrary finite rationals. From property I, since $\bigcap$[all finite rationals$>f$]$=f$, it follows that $\bigcap_{F'}[F+F']$ exists and equals F. Therefore $F \leqq F+I/T \leqq F$, and $F+I/T=F$.

IV. $\cup$ *and* $\cap$. In R countable $\cup$ and $\cap$ are trivial except where finite rationals are concerned, and the isomorphism proof in this case is a consequence of property I. Finite $\cup$ and $\cap$ cause no difficulty.

UNIVERSITY OF MINNESOTA

HYDRODYNAMIC PROBLEMS ARISING FROM THE INVESTIGATION OF THE TRANSVERSE CIRCULATION IN THE ATMOSPHERE

H. J. STEWART

Introduction. The motion of the atmosphere can be considered as a mean flow which has a very large scale and is only slowly changed and, superimposed on this, the low level, smaller scale phenomena usually associated with the polar front. If the mean pressures over a period of about a week are plotted, it is seen that the latter disturbances are averaged out and only the large scale mean motion is shown. Such a plot of the northern hemisphere shows, in addition to the mean westerly flow of air, large scale closed isobaric systems spaced at comparatively regular intervals over the surface of the earth. These include the Aleutian and Icelandic low pressure areas to the north of the westerlies and the Pacific and Bermuda high pressure areas to the south of the westerlies.

It has been noticed that the position and strength of these systems control the paths of the low level storms. This fact has been used in the development of a long range forecasting technique which has proved to be very successful for periods as long as three months. A knowledge of the properties of these large scale systems is thus not only of academic interest but of considerable value in the development of long range forecasting techniques.

In the investigation of these large scale transverse motions in the atmosphere, two controlling factors have been suggested and discussed by the author [1] and by Rossby [2] and Haurwitz [3]. These are first, the dynamical instability of the shearing motion on either side of the belt of westerly winds and second, certain forced oscillations in the atmosphere. The first of these and several related problems are discussed here.

In making these calculations several approximations are made. The principal ones and the reasons for their adoption are as follows:

(1) As the systems are very deep, that is, the wind momentum vector for a vertical section is roughly constant, it was felt that the horizontal field of motion was the dominant factor; consequently, vertical velocities are neglected; and horizontal momentum is assumed not to vary with height.

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(2) With this dynamical setup, the effect of variation of density with altitude would probably be small; consequently the atmosphere is treated as a layer of fluid of constant density with the depth being determined by the hydrostatic law.

(3) All frictional effects are neglected.

(4) In most of these calculations it is assumed that the fluid can be considered as being on a rotating disc rather than on a rotating sphere. As on the earth, the gravitational field is assumed to be deformed by an amount large enough to cancel out the centrifugal acceleration acting on a particle which is stationary with respect to the rotating disc or sphere.

In the analysis the following notation is used:

x, y = rectangular coordinates on a rotating disc,
r, θ = polar coordinates on a rotating disc,
u = velocity in the x direction,
v = velocity in the y direction,
v_θ = velocity in the tangential direction,
h = depth of fluid layer,
ω = angular velocity of the disc,
g = acceleration due to gravity.

The fundamental relations. Using the assumptions discussed in the preceding section, the equations determining the fluid motion on a rotating disc (see [4, p. 317]) are, for a Cartesian coordinate system, the dynamical equations,

$$\frac{Du}{Dt} - 2\omega v = -g\frac{\partial h}{\partial x} \tag{1}$$

and

$$\frac{Dv}{Dt} + 2\omega u = -g\frac{\partial h}{\partial y}, \tag{2}$$

and the equation of continuity,

$$\frac{Dh}{Dt} + h\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\} = 0 \tag{3}$$

where $D(\ )/Dt$ is the derivative following the fluid particle. If h is eliminated from (1) and (2) by cross-differentiation, the result may be combined with (3) to obtain

$$\frac{D}{Dt}\left\{\frac{\zeta + 2\omega}{h}\right\} = 0 \tag{4}$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{5}$$

This is the law of conservation of vorticity and could just as well have been taken as one of the fundamental equations. If the motion of the fluid could have been started from rest with a uniform depth, h_0, the last equation can be integrated and the result obtained that

$$\frac{\zeta + 2\omega}{h} = \frac{2\omega}{h_0}. \tag{6}$$

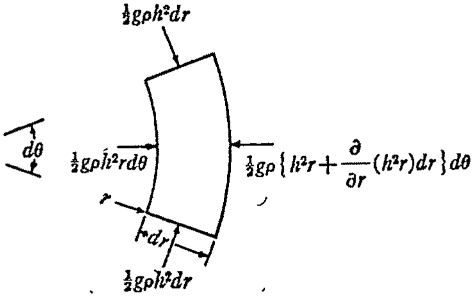
$\frac{1}{2}g\rho h^2 dr$
$d\theta$
$\frac{1}{2}g\rho h^2 r d\theta$
$\frac{1}{2}g\rho\{h^2 r + \frac{\partial}{\partial r}(h^2 r)dr\}d\theta$
r
dr
$\frac{1}{2}g\rho h^2 dr$

FIG. 1
Diagram of forces for flow in circles

As the dynamical equations are nonlinear, general solutions to these expressions are not readily found; however, the solution for the case of steady motion in circles can be obtained quite easily. The equilibrium of forces in the radial direction (see Figure 1) requires that

$$(1/2)gr\frac{d}{dr}(h^2) = hv_\theta^2 + 2\omega r h v_\theta. \tag{7}$$

From (6),

$$h = h_0\{1 + \zeta/2\omega\} = h_0\left\{1 + \frac{1}{2\omega r}\frac{d}{dr}(rv_\theta)\right\}. \tag{8}$$

The equation which determines the velocity is thus

$$\frac{d^2 v_\theta}{dr^2} + \frac{1}{r}\frac{dv_\theta}{dr} - \left\{\frac{4\omega^2}{gh_0} + \frac{1}{r^2}\right\}v_\theta = \frac{2\omega}{gh_0 r}v_\theta^2. \tag{9}$$

This can be put into a somewhat simpler form by using the dimensionless variables, $\rho = 2\omega r/(gh_0)^{1/2}$ and $V_\theta = v_\theta/(gh_0)^{1/2}$. With these variables, (9) becomes

$$\frac{d^2V_\theta}{d\rho^2} + \frac{1}{\rho}\frac{dV_\theta}{d\rho} - \left\{1 + \frac{1}{\rho^2}\right\}V_\theta = -\frac{1}{\rho}V_\theta^2. \tag{10}$$

The desired solutions of this equation are those which vanish at infinity and thus correspond to vortex motion. If V_θ is positive, this is the equation for cyclonic rotation; if V_θ is negative, this is the equation for anticyclonic rotation. These solutions were found numerically at the Massachusetts Institute of Technology by means of the Differential Analyzer. These results are plotted in Figure 2 together with $K_1(\rho)$ which they approach asymptotically. These results were calculated from this asymptotic solution and thus can be expected to diverge more and more from the true solutions as ρ decreases. From physical considerations it appears improbable that the anticyclonic solution should have the velocity approach negative infinity at the origin. There exists a solution which is finite at the origin and which has a series expansion which starts as follows:

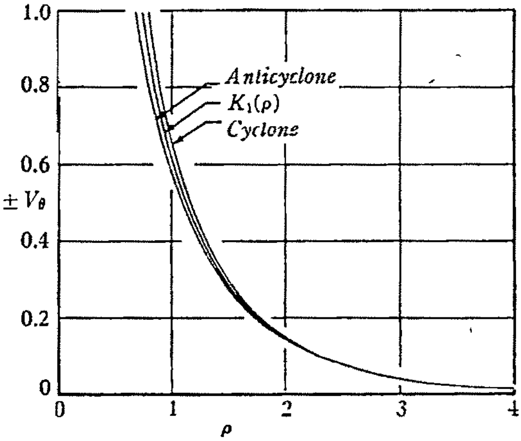

FIG. 2
Velocity profiles for vortices

$$V_\theta = a\left\{\rho + \frac{1+a}{8}\rho^2 + \frac{(1+a)(1+2a)}{192}\rho^3 + \cdots\right\}. \tag{11}$$

It seems probable that the anticyclonic solution must start out this way for some particular value of a, but this conjecture has not been verified.

An approximate solution to this same problem may be found by linearizing the quadratic terms of (1), (2), and (3), that is, by replacing $D(\ )/Dt$ by $\partial(\ )/\partial t$. If the linearized equations are solved for h, it is seen that

$$\frac{\partial^2 h}{\partial t^2} - gh_0\left\{\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right\} + 4\omega^2(h - h_0) = 0. \tag{12}$$

In terms of dimensionless variables $X = 2\omega x/(gh_0)^{1/2}$, $Y = 2\omega y/(gh_0)^{1/2}$ and $\eta = (h - h_0)/h_0$, this equation becomes

$$\frac{\partial^2 \eta}{\partial t^2} - \left\{\frac{\partial^2 \eta}{\partial X^2} + \frac{\partial^2 \eta}{\partial Y^2}\right\} + \eta = 0. \tag{13}$$

The only steady state solution to this equation which vanishes at infinity and which represents flow in circles about the origin is

$$\eta = AK_0(\rho) \tag{14}$$

where $\rho = (X^2 + Y^2)^{1/2}$ is the same as in (10), A is an arbitrary constant, and $K_0(\rho)$ is a modified Bessel function of the second type [5, p. 21]. From (1), it is seen that

$$V_\theta = \frac{v_\theta}{(gh_0)^{1/2}} = \frac{\partial \eta}{\partial \rho} = -AK_1(\rho). \tag{15}$$

This is plotted for a unit cyclone, $A = -1$, in Figure 2 together with the exact solutions which retained the quadratic terms. From (15) it can be seen that if $\rho \ll 1$, the velocity varies inversely as the radius, just as in an ordinary line vortex in an incompressible nonviscous fluid, and if distances between the vortices of (14) are small, results identical to those found with ordinary vortices will be obtained. However, the linearization of the equations of motion breaks down in this range; so that results obtained with ordinary vortices cannot be applied directly.

If h_0 is taken as being the depth of the homogeneous atmosphere, about 8 km, a distance of about 2,000 km corresponds to $\rho = 1$. Figure 2 or (10) shows that at distances as large as this, the error caused by neglecting the quadratic terms is not large. As the spacing of the pressure centers being investigated is of the order of 2,000 km, the quadratic terms will be neglected throughout, and the linearized form, (13), will be used. In addition to this, the motions will be considered to be either stationary or changing very slowly so that the local acceleration can be neglected as compared to the Coriolis term. This is equivalent to saying that the fluid motions to be considered can be built up through superposition of vortices of the type given by (14). With these approximations, (1) and (2) can be written as follows using dimensionless velocities $U=u/(gh_0)^{1/2}$ and $V=v/(gh_0)^{1/2}$:

$$V = \frac{\partial\eta}{\partial X}, \qquad U = -\frac{\partial\eta}{\partial Y}. \tag{16}$$

These are the well known geostrophic wind equations.

It is of interest to note that (13) is the same as the equation for the deflection of a membrane which is elastically supported. The character of its solutions may often be estimated by use of this analogy.

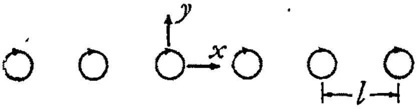

FIG. 3
Single row of vortices

Stability of rows of vortices. The most noticeable feature of the general circulation of the atmosphere is the belt of westerly winds. On both sides of the westerlies where the wind intensity diminishes rapidly there is a region of concentrated vorticity, cyclonic to the north and anticyclonic to the south of the westerlies. As a first approximation to this, the westerlies may be considered as a uniform jet with stationary air masses on either side. Separating the stationary and moving air masses are vortex sheets. That such vortex sheets are unstable has been shown by Pekeris and others. When an unstable vortex sheet breaks up into discrete eddies, the vorticity may either diffuse throughout the fluid mass or, if there is a stable vortex formation, the vorticity may collect in a definite pattern. As an example of this type of motion, the vortex sheets shed from a two-dimensional bluff body break down and then form the well known Kármán "vortex street."

A vortex sheet such as bounds the jet of westerlies can break up into only one system which is in equilibrium. This is a row of equal vortices equally spaced as in Figure 3. In terms of the dimensionless variables introduced in the preceding section, the surface deflection for such a system of vortices of unit strength is, by (14),

$$\eta = \sum_{n=-\infty}^{\infty} K_0[(x - nl)^2 + y^2]^{1/2}. \tag{17}$$

If the nth vortex is displaced by amounts Δx_n and Δy_n in the x and y directions, respectively, the surface deflection in the displaced position is

$$\eta = \sum_{n=-\infty}^{\infty} K_0[(x - nl - \Delta x_n)^2 + (y - \Delta y_n)^2]^{1/2}. \tag{18}$$

The velocity of a vortex is the sum of the velocities induced by all the other vortices. By (16), the velocity of the vortex at the origin is

$$\begin{aligned} \frac{d}{dt}(\Delta x_0) &= -\frac{\partial}{\partial y} \sum_{n=-\infty}^{\infty}{}' K_0[(x - nl - \Delta x_n)^2 + (y - \Delta y_n)^2]^{1/2}, \\ \frac{d}{dt}(\Delta y_0) &= \frac{\partial}{\partial x} \sum_{n=-\infty}^{\infty}{}' K_0[(x - nl - \Delta x_n)^2 + (y - \Delta y_n)^2]^{1/2} \end{aligned} \tag{19}$$

where x and y are to be replaced by Δx_0 and Δy_0, respectively, after the differentiation has been performed. The $(\ )'$ on the summation sign indicates that the term for $n = 0$ is to be omitted. If the displacements are small, these expressions can be written as

$$\begin{aligned} \frac{d}{dt}(\Delta x_0) &= \sum_{n=-\infty}^{\infty}{}' \frac{K_1|nl|}{|nl|}(\Delta y_n - \Delta y_0), \\ \frac{d}{dt}(\Delta y_0) &= \sum_{n=-\infty}^{\infty}{}' \left\{\frac{K_1|nl|}{|nl|} + K_0|nl|\right\}(\Delta x_n - \Delta x_0). \end{aligned} \tag{20}$$

Similar expressions for the velocities of the other vortices could be written by symmetry. These form a set of simultaneous differential equations which must be solved in order to determine the motion of the vortices.

As the equations for the infinitesimal displacements are linear, any arbitrary perturbation can be represented as a sum of terms of the type $\Delta x_n = \Delta x_0 e^{in\phi}$ and $\Delta y_n = \Delta y_0 e^{in\phi}$ where $0 \leqq \phi \leqq 2\pi$, and the individual harmonics can be investigated separately. From this,

$$\begin{aligned} \frac{d}{dt}(\Delta x_0) &= -2\Delta y_0 \sum_{n=1}^{\infty} \frac{K_1(nl)}{nl}(1 - \cos n\phi), \\ \frac{d}{dt}(\Delta y_0) &= -2\Delta x_0 \sum_{n=1}^{\infty} \left\{\frac{K_1(nl)}{nl} + K_0(nl)\right\}(1 - \cos n\phi). \end{aligned} \tag{21}$$

If these two equations are combined, it is seen that

$$\frac{d^2}{dt^2}(\Delta x_0) - \lambda^2 \Delta x_0 = 0, \tag{22}$$
$$\frac{d^2}{dt^2}(\Delta y_0) - \lambda^2 \Delta y_0 = 0$$

where

$$\lambda^2 = 4\left\{ \sum_{n=1}^{\infty} \frac{K_1(nl)}{nl}(1 - \cos n\phi) \right\} \cdot \left\{ \sum_{n=1}^{\infty} \left[\frac{K_1(nl)}{nl} + K_0(nl) \right] (1 - \cos n\phi) \right\}. \tag{23}$$

From (22) it is seen that any infinitesimal disturbance will grow in amplitude unless $\lambda^2 < 0$; however from (23) it is evident that λ^2 is never negative. A single row of vortices is thus unstable.

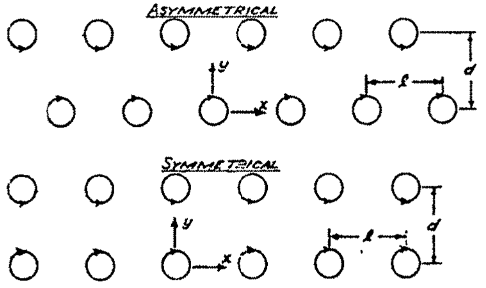

FIG. 4
Double rows of vortices

As there are two shear regions in the northern hemisphere, cyclonic shear to the north of the westerlies and anticyclonic shear to the south, it is possible that a double row of vortices arising from these might be stable. As shown in Figure 4, there are two possible arrangements of equal and opposite vortices which do not change shape with time.

The stability of the symmetrical system will be investigated first. If $\overline{\Delta x_n}$ and $\overline{\Delta y_n}$ are the displacements of the nth vortex in the upper row and Δx_n and Δy_n are the displacements of the nth vortex in the lower row, the surface deflection is given by

$$\eta = \sum_{n=-\infty}^{\infty} \left\{ K_0\left[(x - nl - \Delta x_n)^2 + (y - \Delta y_n)^2\right]^{1/2} - K_0\left[(x - nl - \overline{\Delta x_n})^2 + (y - d - \overline{\Delta y_n})^2\right]^{1/2} \right\}. \tag{24}$$

From this expression the velocities of the vortices can be obtained in the same manner as in the preceding calculation. By writing $\Delta x_n=\Delta x_0e^{in\phi}$, $\Delta y_n=\Delta y_0e^{in\phi}$ $\overline{\Delta x}_n=\overline{\Delta x}_0e^{in\phi}$ and $\overline{\Delta y}_n=\overline{\Delta y}_0e^{in\phi}$ as before, the velocity of the zero vortex in the lower row is seen to be

$$\begin{aligned} \frac{d}{dt}(\Delta x_0) &= A\Delta y_0 + B\overline{\Delta x}_0 + C\overline{\Delta y}_0, \\ \frac{d}{dt}(\Delta y_0) &= D\Delta x_0 - B\overline{\Delta y}_0 + E\overline{\Delta x}_0 \end{aligned} \tag{25}$$

where

$$\begin{aligned} A &= -K_0(d) - \frac{K_1(d)}{d} - 2\sum_{n=1}^{\infty}\left\{\frac{K_1(nl)}{nl}(1-\cos n\phi)\right. \\ &\qquad \left. + d^2\frac{K_0[n^2l^2+d^2]^{1/2}}{n^2l^2+d^2} - (n^2l^2-d^2)\frac{K_1[n^2l^2+d^2]^{1/2}}{[n^2l^2+d^2]^{3/2}}\right\}, \\ B &= 2i\sum_{n=1}^{\infty}\frac{nld\sin n\phi}{n^2l^2+d^2}\left\{2\frac{K_1[n^2l^2+d^2]^{1/2}}{[n^2l^2+d^2]^{1/2}} + K_0[n^2l^2+d^2]^{1/2}\right\}, \\ C &= K_0(d) + \frac{K_1(d)}{d} + 2\sum_{n=1}^{\infty}\left\{d^2\frac{K_0[n^2l^2+d^2]^{1/2}}{n^2l^2+d^2}\right. \\ &\qquad \left. - (n^2l^2-d^2)\frac{K_1[n^2l^2+d^2]^{1/2}}{[n^2l^2+d^2]^{3/2}}\right\}\cos n\phi, \\ D &= A - 2\sum_{n=1}^{\infty}K_0(nl)(1-\cos n\phi) + \sum_{n=-\infty}^{\infty}K_0[n^2l^2+d^2]^{1/2}, \\ E &= C - K_0(d) - 2\sum_{n=1}^{\infty}K_0[n^2l^2+d^2]^{1/2}\cos n\phi. \end{aligned} \tag{26}$$

From symmetry the equations for the motion of the zero vortex in the upper row can be written as

$$\begin{aligned} \frac{d}{dt}(\overline{\Delta x}_0) &= -A\overline{\Delta y}_0 + B\Delta x_0 - C\Delta y_0, \\ \frac{d}{dt}(\overline{\Delta y}_0) &= -D\overline{\Delta x}_0 - B\Delta y_0 - E\Delta x_0. \end{aligned} \tag{27}$$

It may be noticed that the second pair of equations is identical to the first in either of two cases, first if $\Delta x_0=\overline{\Delta x}_0$ and $\Delta y_0=-\overline{\Delta y}_0$ and second if $\Delta x_0=-\overline{\Delta x}_0$ and $\Delta y_0=\overline{\Delta y}_0$. The first case corresponds to symmetrical perturbations with respect to the center line and the second case corresponds to antisymmetrical perturbations. In order for the system to be stable, both types of motion must be stable.

$$(28)\quad \left(\frac{d}{dt} - B\right)\Delta x_0 = (A - C)\Delta y_0, \qquad \left(\frac{d}{dt} - B\right)\Delta y_0 = (D + E)\Delta x_0$$

for the symmetrical case. As B is an imaginary quantity, the condition for stability in this case is that

$$(29)\qquad (A - C)(D + E) \leqq 0.$$

For the antisymmetrical perturbations the corresponding stability condition is that

$$(30)\qquad (A + C)(D - E) \leqq 0.$$

For the special case where $\phi = \pi$ it is easily seen that the first of these is violated, for from (26) both $(A - C)$ and $(D+E)$ are negative. From this, it follows that the symmetrical vortex system of Figure 4 is unstable.

For the asymmetrical system of Figure 4 it is found by a similar procedure that (25) and (27) and thus (29) and (30) hold if

$$(31)\quad \begin{aligned}
A &= -2\sum_{n=1}^{\infty} \frac{K_1(nl)}{nl}(1 - \cos n\phi) \\
&\quad + 2\sum_{n=0}^{\infty}\left\{\frac{(n + 1/2)^2 l^2 - d^2}{R_n^3} K_1(R_n) - \frac{d^2}{R_n^2} K_0(R_n)\right\}, \\
B &= 2i\sum_{n=0}^{\infty} \frac{(n + 1/2)ld}{R_n^2}\left\{2\frac{K_1(R_n)}{R_n} + K_0(R_n)\right\} \sin (n + 1/2)\phi, \\
C &= 2\sum_{n=0}^{\infty}\left\{\frac{d^2}{R_n^2} K_0(R_n) - \frac{(n + 1/2)^2 l^2 - d^2}{R_n^3} K_1(R_n)\right\} \\
&\quad \cdot \cos (n + 1/2)\phi, \\
D &= A - 2\sum_{n=1}^{\infty} K_0(nl)(1 - \cos n\phi) + 2\sum_{n=0}^{\infty} K_0(R_n), \\
E &= C - 2\sum_{n=0}^{\infty} K_0(R_n) \cos (n + 1/2)\phi,
\end{aligned}$$

where $R_n = [(n+1/2)^2 l^2 + d^2]^{1/2}$. For $\phi = \pi$, the critical case for stability, $C = E = 0$. The stability criterion thus becomes $A(\pi)D(\pi) = 0$. From (31) it is apparent that this condition will be satisfied for a given value of l for a range of values of d between the limits which correspond respectively to $A(\pi)=0$ and $D(\pi)=0$. These limits are given in Table 1 and are shown graphically in Figure 5. It should be noted that for small values of l, the result that $d/l=0.281$ is exactly that obtained by Kármán in his work with vortex streets behind two-dimensional bluff bodies (see [4, p. 228]).

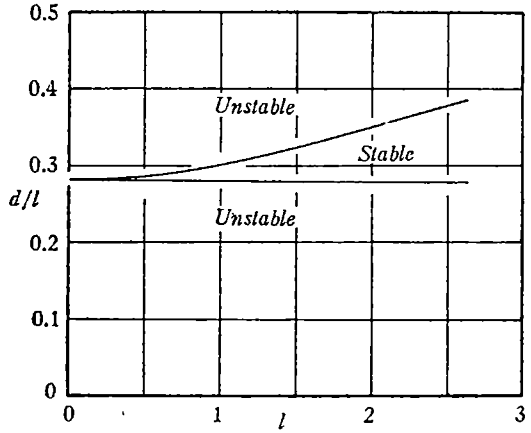

FIG. 5
Stability diagram for double vortex rows

TABLE 1

Stable range of values for d/l for asymmetric vortex street

l	Upper limit	Lower limit
0	0.281	0.281
1	0.300	0.281
2	0.350	0.280

This calculation shows that stable configurations do exist. These results would be directly applicable to motions in the earth's atmosphere if the width of the vortex street were small compared to the radius of the earth. Unfortunately, this is not so, and the curvature of the shear fields on the edges of the westerlies must be considered. These effects will be considered in the next section.

Stability of rings of vortices. The analysis of the stability of a double ring system of vortices, the system one would use to represent the vortex pattern accompanying the belt of westerly winds, is unfortunately so complex that a successful solution has not yet been carried out. The effects of the curvature on the stability of such systems can, however, be estimated from the investigation of the stability of a single ring of vortices. This corresponds to a condition where the shear field to the north of the westerlies is of negligible strength. As the shear field to the north of the westerlies is much weaker than that to the south, this single ring of anticyclonic vortices furnishes a much better picture of the atmospheric motions than the double row of equal vortices investigated in the preceding section.

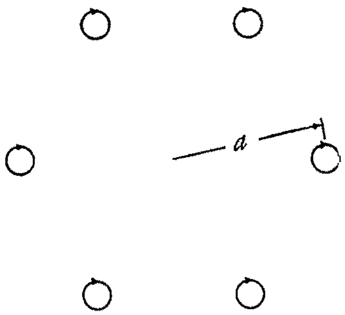

FIG. 6
Ring system of vortices

If the shear field north of the westerlies is neglected, a ring of equally strong anticyclones evenly spaced around a latitude circle is the only formation of vortices which will not change its shape with time. The stability of such a system can be investigated approximately by considering the stability of a ring of vortices of the type given by (14). If there are N such vortices in a ring of radius a, spaced at equal angles $\tau = 2\pi/N$, as shown in Figure 6, the surface deflection for such a system in its equilibrium state is

$$\eta = \sum_{n=1}^{N} K_0\left[a^2 + r^2 - 2ar\cos(\theta - n\tau)\right]^{1/2}. \tag{32}$$

For the calculation of the velocities of the system it is useful to know the form taken by (16) in polar coordinates. If v_r and v_θ are the dimensionless velocities in the radial and tangential directions, respectively, (16) can be written as

$$v_r = -\frac{1}{r}\frac{\partial \eta}{\partial \theta}, \qquad v_\theta = \frac{\partial \eta}{\partial r}. \tag{33}$$

From the second of these, the system shown in Figure 6 is seen to have an angular velocity Ω given by

$$\Omega = -(1/a)\sum_{n=1}^{N-1} K_1[2a \sin(n\tau/2)] \sin(n\tau/2). \tag{34}$$

If the nth vortex is displaced by Δr_n in the radial direction and by an angle of $\Delta\theta_n$ in the tangential direction, the surface deflection in the displaced condition is

$$\eta = \sum_{n=1}^{N} K_0[(a + \Delta r_n)^2 + r^2 - 2(a + \Delta r_n) r \cos(\theta - n\tau - \Delta\theta_n)]^{1/2}. \tag{35}$$

If the displacements are small, the change in velocity from the equilibrium value given by (34) can be readily calculated. For the Nth vortex it is found that

$$\begin{aligned} \Delta v_r &= \frac{d}{dt}(\Delta r_N) = (1/2)\sum_{n=1}^{N-1} \Delta r_n K_0(R_n) \sin n\tau \\ &\quad + \sum_{n=1}^{N-1} a(\Delta\theta_n - \Delta\theta_N)\left\{\frac{K_1(R_n)}{R_n} + K_0(R_n)\cos^2(n\tau/2)\right\}, \\ \Delta v_\theta &= a\,\frac{d}{dt}(\Delta\theta_N) + \Omega\Delta r_N \\ &= \Delta r_N \sum_{n=1}^{N-1}\left\{K_0(R_n)\sin^2(n\tau/2) - \frac{K_1(R_n)}{R_n}\cos n\tau\right\} \\ &\quad + \sum_{n=1}^{N-1}\left[\Delta r_n\left\{\frac{K_1(R_n)}{R_n} + K_0(R_n)\sin^2(n\tau/2)\right\}\right. \\ &\quad \left. + (1/2)a\Delta\theta_n K_0(R_n)\sin n\tau\right] \end{aligned} \tag{36}$$

where $R_n = 2a \sin(n\tau/2)$. Similar expressions for the velocities of the other vortices could be written from symmetry. These would form a set of simultaneous differential equations for the displacements.

If the N equations in each of the two sets indicated in (36) are added, it is seen that

$$\begin{aligned} &\frac{d}{dt}\left\{\sum_{n=1}^{N}\Delta r_n\right\} = 0, \\ &\frac{d}{dt}\left\{\sum_{n=1}^{N}\Delta\theta_n\right\} = \left\{\sum_{n=1}^{N}\Delta r_n\right\} \\ &\qquad\cdot\left\{\sum_{n=1}^{N-1}[2\sin^2(n\tau/2)K_0(R_n) + (1/a)\sin(n\tau/2)K_1(R_n)]\right\}. \end{aligned} \tag{37}$$

If the system is initially in equilibrium so that initially $\Delta r_n = \Delta\theta_n = 0$, then (37) shows that

$$\sum_{n=1}^{N} \Delta r_n = \sum_{n=1}^{N} \Delta\theta_n = 0. \tag{38}$$

These results correspond to the similar equations for two-dimensional line vortices which state that the impulse of a system having no external forces remains constant (see [4, p. 220]).

The method of harmonic analysis used in the preceding section can also be used to advantage in the solution of (36). If $\Delta r_n = \Delta r_N e^{in\phi}$ and $\Delta\theta_n = \Delta\theta_N e^{in\phi}$ where ϕ is a member of the series $2\pi/N, 4\pi/N, \cdots, (N-1)2\pi/N, 2\pi$, then (36) can be written as

$$\begin{aligned} \frac{d}{dt}(\Delta r_N) &= A\Delta r_N - Ba\Delta\theta_N, \\ a\frac{d}{dt}(\Delta\theta_N) &= C\Delta r_N + Aa\Delta\theta_N \end{aligned} \tag{39}$$

where

$$\begin{aligned} A &= (1/2)\sum_{n=1}^{N-1} e^{in\phi} \sin n\tau K_0(R_n), \\ B &= \sum_{n=1}^{N-1} (1 - e^{in\phi})\left\{\frac{K_1(R_n)}{R_n} + \cos^2(n\tau/2)K_0(R_n)\right\}, \\ C &= \sum_{n=1}^{N-1}\left\{(1 + e^{in\phi})\sin^2(n\tau/2)K_0(R_n)\right. \\ &\qquad\qquad \left. + (1 - 2\cos n\tau + e^{in\phi})\frac{K_1(R_n)}{R_n}\right\}. \end{aligned} \tag{40}$$

Since A is a purely imaginary quantity for any of the specified values of ϕ, the condition that the system be stable imposed by (39) is that

$$BC \geqq 0. \tag{41}$$

From (40) it may be seen that B is never negative and that C is always positive for $N=2, 3, 4, 5$, or 6. For $N=7$, C is positive if $a>71$. For $N>7$, C is always negative for one or more of the values of ϕ. A value of $a>71$ corresponds either to disturbances of such great wave length or to motions of such a shallow layer of air in the earth's atmosphere that it probably is of no significance. The results of this calculation may then be summarized by the statement that six or less equal vortices placed at the corners of a regular polygon form a stable vortex formation. This result is in agreement with the work of J. J. Thomson who investigated the stability of similar formations of two-dimensional line vortices in a frictionless incompressible fluid (see [6, p. 94]). The effect of the curvature of the row of vortices is thus seen to be stabilizing providing the number of the vortices in the ring is small enough.

As a single ring is stable, the effect of the shear field to the north of the westerlies can roughly be taken into account by placing a fixed vortex at the pole as in Figure 7. The effect of such a vortex is to change the rate of rotation of the system and to add additional turns to (40). If P is the ratio of polar (cyclonic) vorticity to the anticyclonic vorticity, then (39) still holds if

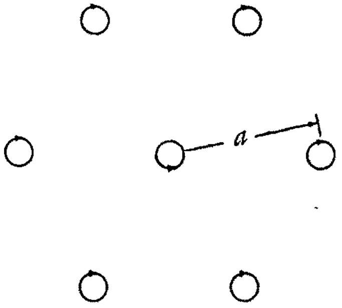

FIG. 7
Ring system of vortices with polar cyclone

$$C = C(P = 0) - NP\left\{K_0(a) + 2\,\frac{K_1(a)}{a}\right\}. \tag{42}$$

The fixed polar vortex is thus seen to cause a decrease in stability and if P is large enough the motion becomes unstable. As P is increased from zero it is found that the maximum number of vortices which form a stable formation diminishes from six to five, to four, to three and then to zero.

Vortex motion on a rotating sphere. All the calculations discussed in the preceding sections have neglected the effect of the variation of the Coriolis acceleration with latitude, that is, they correspond to conditions on a rotating disc rather than on a rotating sphere. In this section vortex motion in an atmosphere on a rotating sphere will be discussed.

In terms of polar coordinates on a sphere rotating with an angular velocity ω with θ being the angle from the north pole and ϕ being the longitudinal angle, the equation representing the conservation of angular momentum and corresponding to (4) is

$$\frac{D}{Dt}\left\{\frac{\zeta + 2\omega\cos\theta}{h}\right\} = 0 \tag{43}$$

where ζ is the radial component of the curl of the velocity vector. If u is the velocity to the east and v is the velocity to the north, the expression for ζ is

$$\zeta = \frac{1}{a\sin\theta}\left\{\frac{\partial}{\partial\theta}(u\sin\theta) + \frac{\partial v}{\partial\phi}\right\}. \tag{44}$$

This principle will be used to investigate the velocity distribution which would accompany a vortex with its core at the north pole. For this case it is permissible to assume symmetry about the polar axis with v being zero in the steady state and u and h being functions only of θ.

If the motion started from rest with the atmosphere having a uniform depth, h_0, (43) can be integrated to give

$$\frac{\zeta + 2\omega\cos\theta}{h} = \frac{2\omega}{h_0}\cos\theta_0 \tag{45}$$

where θ_0 is the colatitude at which the particle originated. As the mass of the fluid must be conserved,

$$\int_\theta^\pi h_0 \sin\theta\, d\theta = \int_\theta^\pi h\sin\theta\, d\theta. \tag{46}$$

From this,

$$\cos\theta_0 = -1 + \int_\theta^\pi \frac{h}{h_0}\sin\theta\, d\theta, \tag{47}$$

and the vorticity at any latitude is thus

$$\zeta = 2\omega\left\{-\frac{h}{h_0} - \cos\theta + \frac{h}{h_0}\int_\theta^\pi \frac{h}{h_0}\sin\partial\, d\theta\right\}. \tag{48}$$

The equation of motion is

$$\frac{1}{a}\cot\theta u^2 + 2\omega\cos\theta u = \frac{g}{a}\frac{dh}{d\theta}, \tag{49}$$

and from (44)

$$\zeta = \frac{1}{a \sin \theta} \frac{d}{d\theta} (u \sin \theta). \tag{50}$$

Equations (48), (49), and (50) must be solved simultaneously for u, h, and ζ. An exact solution is extremely difficult, but an approximate solution can be obtained if the centrifugal acceleration is neglected in (49) and if the depth of the atmosphere is only slightly different from the equilibrium value h_0.

If the centrifugal acceleration is neglected, these equations may be combined to give

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left\{ \tan \theta \frac{d\eta}{d\theta} \right\} = \alpha^2 \left\{ \eta \cos \theta + (1 + \eta) \int_\theta^\pi \eta \sin \theta \, d\theta \right\} \tag{51}$$

where

$$\alpha^2 = 4\omega^2 a^2/(gh_0) \quad \text{and} \quad \eta = (h - h_0)/h_0.$$

Since η is small compared to unity, (51) may be linearized by replacing the factor $1+\eta$ by 1. If a new independent variable, $\mu = \cos \theta$, is introduced, (51) can be then written in the form

$$\frac{d^2}{d\mu^2} \left\{ \frac{1 - \mu^2}{\mu} \frac{d\eta}{d\mu} \right\} - \alpha^2 \left\{ \mu \frac{d\eta}{d\mu} + 2\eta \right\} = 0. \tag{52}$$

This equation can be integrated once, and it then becomes

$$\mu^2 \frac{d}{d\mu} \left\{ \frac{1 - \mu^2}{\mu^2} \frac{d\eta}{d\mu} \right\} - \alpha^2 \mu^2 \eta = C \tag{53}$$

where C is an arbitrary constant. This equation (with $C=0$) is identical with an equation used in the development of the dynamical theory of the long period fortnightly and semiannual tides (see [4, p. 334]). The desired solution of (53) is finite except for $\mu = 1$ and satisfies the continuity principle; that is,

$$\int_{-1}^{1} \eta d\mu = 0.$$

For standard atmospheric conditions with the sea level pressure and density being 1013.3 mb and 1.276×10^3 gm/cc, respectively, the constant α^2 has the value 10.9. The velocity distribution obtained from the solution of (53) for this value of α^2 which satisfies the given boundary conditions is plotted in Figure 8 together with the corresponding solution for the velocity distribution for a vortex on a rotating disc as given by (15).

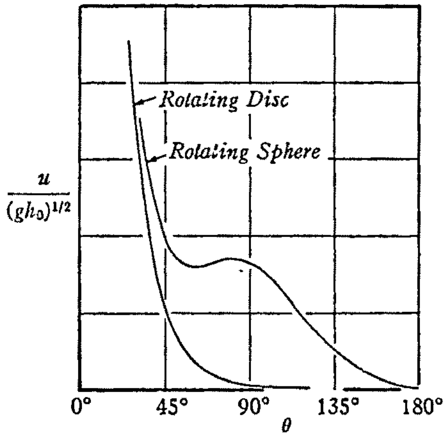

FIG. 8
Velocity distribution for a polar vortex on a rotating sphere

It would be desirable to extend these calculations to obtain the velocity distribution for a vortex arbitrarily placed on a sphere and to use this result to discuss the stability of vortex formations on a sphere. It may be seen from this calculation of the velocity field of a polar vortex that such an investigation would be very difficult to carry out satisfactorily.

Concluding remarks. It has been shown that a ring of equal anticyclones formed by the "rolling-up" of the shear field south of the westerlies is stable provided that the number of anticyclones is not more than six and that the shear field north of the westerlies is not strong. The most unsatisfactory feature of this dynamic picture is that if it were strictly true, the anticyclones would have a small velocity to the west; however it is apparent that the effect of the vorticity north of the westerlies would be to decrease this velocity. The "vortex street," for which the cyclonic and anticyclonic vorticities are equal, has a velocity to the east; so that an analysis which properly took into account the relatively small amount of cyclonic vorticity to the north of the westerlies might be expected to show stable systems which are practically stationary. This is in rough agreement with the idea that the position of such systems ought to be determined primarily by thermodynamic considerations, so that the stable vortex formations found in the atmosphere might be expected to be those that are stationary.

Since the cyclonic vorticity does not normally show concentrations like the Pacific and the Azores anticyclones—the Aleutian and Icelandic lows apparently being surface phenomena caused by the passage of large numbers of low level-storms—the representation of the cyclonic vorticity by a single polar vortex is probably a fairly good approximation. As this vortex was seen to cause a decrease in the number of anticyclones that could form stable formations, the maximum number which can exist in the atmosphere is probably less than six with the most stable formation of three anticyclones being the most probable. There is about 120° of longitude separating the mean positions of the Pacific and the Azores highs, and there is some evidence of the existence of a similar system equally distant from these two over Asia, but this evidence is far from conclusive.

These stability calculations were made with vortices on a rotating disc rather than on a rotating sphere, and the possibility of coupling between northern and southern hemispheric systems was neglected. It is not possible to state "a priori" whether or not these modifications are important; however it is believed that their effect is not large.

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THE FUNDAMENTAL LIMIT THEOREMS IN PROBABILITY

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1. Introduction. The main purpose of this address is to explain the mathematical content and meaning of the two most important limit theorems in the modern theory of probability: the central limit theorem[1] and the recently discovered precise form of what was generally known as "Kolmogoroff's celebrated law of the iterated logarithm." The former traces its origin to the very beginnings of the theory of probability and is often called after Laplace and Ljapunov. For a long time it was clouded in mystery, and Poincaré once remarked that mathematicians regard it as a physical law, whereas physicists hold mathematicians responsible for it. A great many mathematicians have contributed to the gradual recognition of the mathematical content of the theorem and to the establishment of the precise conditions of its validity. The complete solution came finally in 1935 and was possible only by an elimination of all classical restrictions and a reconsideration of the problem in a new generality.

The central limit theorem (like its little brother, the weak law of large numbers) is a statement on distribution functions, and can be formulated, either as such or in terms of Fourier analysis, without any appeal to probability or measure. This is not true of the infinitely more delicate law of the iterated logarithm and its generalizations (or of the strong law of large numbers): these are essentially measure-theoretic. The starting point of the long series of papers which lead to the present form of the iterated logarithm was not a problem in probability but, surprisingly enough, a problem in Diophantine approximations treated by Hardy and Littlewood [1914].[2] Their original estimate has gradually been improved for their particular number-theoretical case and, as a matter of fact, even the precise form of the iterated logarithm has first been checked for this particular case. It is therefore instructive to realize that, from the point of view of the general theory, the Hardy-Littlewood problem constitutes an exceedingly special case comparable only to the role of the linear function within the domain of all real functions. Such special

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[1] The name "central limit theorem" is due to Pólya [1920].

[2] Author's names, and years, appearing in brackets refer to the references cited at the end of the paper.

cases are in many respects misleading, and usually do not lend themselves for generalizations. Thus in its number-theoretical application our problem reduces to an evaluation of certain sequences of binomial coefficients, and such special techniques are not applicable even to the most trivial generalization. The history of probability shows that our problems must be treated in their greatest generality: only in this way can we hope to discover the most natural tools and to open channels for new progress. This remark leads naturally to that characteristic of our theory which makes it attractive beyond its importance for various applications: a combination of an amazing generality with algebraic precision.

The analytical formulation of our limit theorems seems unfortunately to obscure the fact that a great many individual problems can be treated as special cases. This fact seems little appreciated and often an unnecessary effort is spent on treating such problems. A few illustrative mathematical applications will be found in §4. Better examples are furnished by physical applications, but it would be too time consuming to explain them. The applicability of the central limit theorem to problems in number theory has been amply demonstrated in papers by Erdös, Hartman, Kac, Wintner, and others.

Another point to be stressed concerns the abundance of open problems. The fact that we now have necessary and sufficient conditions both for the central limit theorem and the iterated logarithm, and that we are in a position to make a series of statements of the "best result" type, seems to have created the impression that "nothing remains to be done." Actually we have just succeeded in producing good working tools and in opening the gate to a multitude of new problems both of theoretical interest and of practical importance. (This is true even for the classical field of so-called independent variables. The much wider domain of dependent variables, excepting only the theory of Markov chains, remains practically untouched despite the excellent pioneer work by P. Lévy and S. Bernstein.)

It must be understood that the following exposition is concerned only with *one* aspect of the limit theorems and is not intended as a survey of modern tendencies in probability. This theory has developed rapidly (thanks in particular to the famous Moscow School in probability) and many new channels have been opened which link the theory to many branches of mathematics. Thus the true role of the Gaussian distribution can be understood only in connection with stochastic processes. The foundations of this new branch of probability have been laid in a well known paper by Kolmogoroff [1931]. It leads to partial integrodifferential equations of a special kind, but it throws new light even on the classical equation of diffusion and puts new interesting problems concerning this and other parabolic equations [Fortet 1941; Feller 1936]. More generally, this theory seems to lead to a new type of functional equations which has not yet been investigated. Another aspect of the Gaussian distribution leads to the modern theory of infinitely divisible laws [Gnedenko, Khintchine, Kolmogoroff, P. Lévy] and to the so-called arithmetic of distribution functions, inaugurated by P. Lévy [Cramér, Khintchine, Raikov]. A third approach is that from the classical time series problem or, in modern language, from the measure theory in functional spaces [Wiener, Doob]: this approach would lead to the theory of random noises which now occupies so many minds [cf. Doob, 1944]. In order not to get lost in a jungle of general remarks we shall have to restrict the considerations to the well defined case of *independent variables*; we shall not even pause to consider S. Bernstein's well known generalization of the central limit theorem to certain classes of dependent variables or the important application, due to Kolmogoroff and W. Doblin, of that theorem to a more precise study of the ergodic properties of Markov chains.

2. Random variables associated with the dyadic case. For the convenience of the uninitiated reader we shall explain the modern terminology and the type of our problems on the trivial special case of random variables associated with "spinning a coin." It will be seen that this case is still considerably more general than the number-theoretical case to which we have alluded before.

It is simplest to consider only infinite sequences of tossings of a coin: each trial results in a symbol H (head) or T (tail), and the sequence of trials will be represented by an infinite sequence like $HHHTHT \cdots$. The aggregate of all such sequences (all thinkable results of our "experiment") forms the *label space* $\mathfrak{S}$ and each sequence is called a *point*. The mapping $H \rightarrow 1$, $T \rightarrow 0$ makes to each point of $\mathfrak{S}$ correspond a dyadic fraction like $.111010 \cdots$, that is to say, a number x $(0 \leqq x \leqq 1)$ in its dyadic representation; the label space $\mathfrak{S}$ is in this way mapped onto the unit interval, which will also be denoted by $\mathfrak{S}$. It is true that the mapping is not unique for numbers like $.011111 \cdots$ which contain only a finite number of zeros or of ones; but this ambiguity will be seen to be of no consequence. In the usual way we shall associate with the symbols H and T probabilities $1/2$ each, which is equivalent to saying that we introduce the ordinary Lebesgue measure on the unit interval. In this manner the latter becomes the analytical model of a real "experiment"; every picturesque expression concerning tossings of a coin is automatically translated into a statement concerning certain subsets of the unit interval: and the words "event" and "probability" become synonymous with "set" and "measure," respectively. The mathematician can from here on forget all about coins, while the classical probability student would from the beginning refuse to translate his picturesque statements into equivalent statements referring to the unit interval.

Consider now "a player who at the kth trial receives or loses an amount a_k, depending on whether the kth trial results in H or T." The kth trial stands for the kth dyadic digit of the number x representing our particular sequence of trials. Our picturesque description therefore associates with every x, given by the dyadic expansion

$$x = \sum_{k=1}^{\infty} \frac{\epsilon_k}{2^k} \qquad (\epsilon_k = 0 \text{ or } 1), \tag{1}$$

a sequence of functions $X_k(x)$ defined by

$$X_k(x) = \begin{cases} a_k & \text{if } \epsilon_k = 1, \\ -a_k & \phantom{\text{if }} \epsilon_k = 0. \end{cases} \tag{2}$$

Accordingly, $X_k(x)$ is a real function defined in $\mathfrak{S}$ assuming only two values, each with probability 1/2 (on a set of measure 1/2). The word *random variable* is synonymous with "measurable real function defined in the label space." Having defined the individual gains $X_k(x)$, "the total gain in n trials" is a new random variable given by

$$S_n(x) = \sum_{k=1}^{n} X_k(x). \tag{3}$$

The number-theoretical case is included herein as the simplest special case

$$a_k \equiv 1. \tag{4}$$

It concerns a player who loses or wins always the same amount, and can also be interpreted physically as a random walk in a one-dimensional lattice. Of course, $S_n(x)$ is simply the excess (positive or negative) of the number of occurrences of the digit 1 over the number of occurrences of the digit 0 among the first n digits in the dyadic expansion of x.

We shall put

$$s_n^2 = \sum_{k=1}^{n} a_k^2. \tag{5}$$

In the sequel we shall assume that $\{a_k\}$ is an arbitrarily prescribed sequence such that

$$s_n \to \infty \quad \text{and} \quad a_n = o(s_n); \tag{6}$$

if one of the conditions (6) is dropped, definite conclusions can still be stated, but the considerations become rather trivial.

The classical or Laplace's problem may be formulated as follows. Let n be large, but fixed, and consider $S_n(x)$ as a function of x (which means essentially that we compare the total gain after n trials for all thinkable results of the experiment). The problem is to determine the distribution of values of $S_n(x)$. Now the central limit theorem states in our special case that asymptotically for all real ξ and η $(\xi<\eta)$

$$\Pr\left\{\xi s_n < S_n(x) < \eta s_n\right\} \sim \Phi(\eta) - \Phi(\xi); \tag{7}$$

here the left side stands for the measure of those x for which the inequality within the braces is satisfied, and $\Phi(\xi)$ for the Gaussian distribution

$$\Phi(\xi) = \int_{-\infty}^{\xi} \exp\left(-\frac{v^2}{2}\right) dv. \tag{8}$$

The analytic form of (8) is here of no interest: what matters is that for the "reduced" variable

$$S_n^*(x) = S_n(x)/s_n \tag{9}$$

the measure of those x for which $S_n(x)$ lies in the interval (ξ, η) is, asymptotically, independent of the structure of the particular sequence $\{a_k\}$. We mention only in passim that much more precise estimates of the asymptotic behavior (7) are available; they will be discussed together with the central limit theorem for arbitrary random variables in arbitrary spaces. The importance of the relation (7) for many applications is so well known that it needs no amplification. Less known is that the central limit theorem is sometimes wrongly used in situations where it can not be applied. To this category belong all cases of so-called optional stopping, where the number n of trials is not a constant but itself a random variable. Thus a player (or the subject in psychological card-guessing experiments) does not necessarily decide in advance on the number of trials but will stop at an opportune moment. He embarks on *one* (potentially infinite) sequence of trials which is represented by *one* number x. He is not interested in comparing his gain after n trials with other sequences but rather in the fluctuation of his gain $S_n(x)$ for one particular point x and as a function of n.

The central limit theorem (7) teaches us only that, on the average, $S_n(x)$ will be of the order of magnitude of s_n. However, it leaves theoretically open the possibility that for every x the gain $S_n(x)$ will *occasionally* reach the magnitude s_n^{17}, and in many cases it is only such an occasional maximum that actually counts.[3] The *maxima* of $S_n(x)$ will indeed for almost all x be larger than the probable values given by (7). The *Khintchine-Kolmogoroff law of the iterated logarithm states in our case that with probability* 1 (*for almost all* x)

$$\limsup_{n\to\infty} \frac{S_n(x)}{s_n\{2 \log \log s_n\}^{1/2}} = 1, \tag{10}$$

provided only that

$$a_n = o(s_n\{\log \log s_n\}^{-1/2}). \tag{11}$$

To explain the meaning of (10) consider the Hardy-Littlewood case (4) in which $S_n(x)$ is the excess of digits one over digits zero among the n first digits in the dyadic expansion (2). Condition (11) is here trivially satisfied, and the law of the iterated logarithm states in that particular case that for every positive ϵ and almost all x the following statements hold: (i) There are infinitely many n such that $S_n(x) > \{(2-\epsilon)n \log \log n\}^{1/2}$; (ii) for all n sufficiently large $S_n(x) < \{(2+\epsilon)n \log \log n\}^{1/2}$. This is Khintchine's [1924] refinement of Borel's well known theorem that "*almost all numbers are normal,*" which in our notation means that $S_n(x) = o(n)$ for almost all x. Many intermediary steps have led from Borel's theorem to Khintchine's result. Hausdorff [1913] proved that $S_n(x) = o(n^{1/2+\epsilon})$, Hardy-Littlewood [1914] that $S_n(x) = O((n \log n)^{1/2})$, Steinhaus [1922] that $\limsup S_n(x)/(2n \log n)^{1/2} \leq 1$ (note the log n instead of the iterated logarithm); finally Khintchine himself [1923] had proved that $S_n(x) = O((n \log \log n)^{1/2})$. It is well to remember the tremendous computational effort which was necessary for the investigation of such a simple special case: only against this background can one fully appreciate the strength and value of the general arguments which permitted Kolmogoroff [1929] to prove the law of the iterated logarithm (10) not only for all sequences $\{a_n\}$ but for perfectly arbitrary random variables in arbitrary spaces (cf. §8). Moreover, the computational part of the argument in the general case is considerably simpler than in the special number-theoretical case.

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[3] In the theory of diffusion the central limit theorem corresponds to the statement that the random position of the particle is subject to a Gaussian distribution with variance propertional to the time parameter t. It leaves unanswered questions of the following type. What is the probability that a particle under diffusion, starting at $t=0$ from $x=0$, will forever remain within the domain, say, $|x| < t$? The significance of this, and similar more refined problems, stands to reason; they are of the category related to the law of the iterated logarithm.

Looking back at the modest beginnings of the theory, Kolmogoroff's result (10) would seem as complete as one could desire. That it nevertheless is not the final word was first made clear by an exciting discovery due to Marcinkiewicz and Zygmund [1937]. They constructed an example showing that the law of the iterated logarithm (10) does not necessarily hold if the condition (11) is replaced by the only slightly weaker hypothesis

$$a_n < \epsilon s_n \{\log \log s_n\}^{-1/2} \tag{12}$$

with ϵ an arbitrarily small constant. To make things more puzzling, the equality sign in (10) is, in the Marcinkiewicz-Zygmund example, replaced by the sign "smaller than," contrary to all expectations. To understand the inner mechanism of the phenomenon we must embark on the more ambitious undertaking and investigate, not only the upper limit (10), but also the manner in which it is approached. Again, for a real understanding and in order to find natural tools, such a problem must be considered in its greatest generality and not only for our very special random variables. However, we shall here describe the results for our particular case (but for arbitrary $\{a_n\}$).

Our results can best be described by means of a convenient terminology due to P. Lévy: *A nondecreasing sequence* $\{\phi_n\}$ *of positive numbers will be said to belong to the lower class, or in symbols*

$$\{\phi_n\} \in \mathcal{L}, \tag{13}$$

if, for almost all x (that is, with probability 1), *there exist infinitely many* n *such that*

$$S_n(x) > s_n\phi_n; \tag{14}$$

the sequence $\{\phi_n\}$ *belongs to the upper class* ($\in \mathcal{U}$), *if for almost all* x *there exist at most finitely many* n *such that* (14) *holds.* Every sequence $\{\phi_n\}$ belongs either to $\mathcal{L}$ or to $\mathcal{U}$.[4] With this notation Kolmogoroff's result (10) states that

$$\phi_n = \{(2+\epsilon)\log\log s_n\}^{1/2} \begin{matrix} \in \mathcal{L} & \text{if } \epsilon < 0, \\ \in \mathcal{U} & \text{if } \epsilon > 0. \end{matrix} \tag{15}$$

Thus here the gap between the two classes is of the same order of magnitude as ϕ_n. We shall not pause to describe certain improvements for special cases which are due to P. Lévy [4, 1931], Cantelli [1933] and Cramér [1934], but pass to the illustration of the complete result. To begin with, we shall replace (11) by the slightly stronger condition

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[4] This is a special case of the "null-or-one-law." It will be noticed that the statement is by no means obvious: a priori one might expect that a sequence $\{\phi_n\}$ could satisfy the criterion for each class on a set of measure 1/2.

$$a_n = O(s_n\{\log \log s_n\}^{-3/2}). \tag{16}$$

Of course, this condition is certainly satisfied if the a_k remain bounded or increase slowly; it holds in particular for the number-theoretical case (4). Now the following *criterion holds* (Feller [7, 1943]): *if the constants* $\{a_n\}$ *satisfy* (16), *then the necessary and sufficient condition that* $\{\phi_n\} \in \mathcal{U}$ $(\mathcal{L})$ *is that*[5]

$$\sum \frac{a_n^2}{s_n^2} \phi_n \exp\{-\phi_n^2/2\} \in \mathcal{C}(\mathcal{D}). \tag{17}$$

The law of the iterated logarithm is, of course, contained in this criterion and follows from (17) and the Abel-Dini theorem on infinite series. More generally the latter theorem and the conventional logarithmic scales show that *the sequence*

$$\begin{aligned}\phi_n = \{2 \log_2 s_n + 3 \log_3 s_n + 2 \log_4 s_n + \cdots \\ + 2 \log_{r-1} s_n + (2 + \delta) \log_r s_n\}^{1/2}\end{aligned} \tag{18}$$

belongs to $\mathcal{L}$ $(\mathcal{U})$ *if, and only if,* $\delta \leqq 0$ $(\delta > 0)$.

In the special case (4) we have $s_n^2 = n$, and (17) reduces to

$$\sum \frac{\phi_n}{n} \exp\{-\phi_n^2/2\} \in \mathcal{C}\ (\mathcal{D}). \tag{19}$$

The special result (19) (in an equivalent integral form) has been stated by Kolmogoroff (communicated without proof in P. Lévy's book of 1937) and confirmed by Erdös [1942]. The most general conditions under which a similar result holds will be indicated later.

Several interesting corollaries can be deduced from (17). Thus it follows that *for any constant M the sequence*

$$\phi_n^* = \phi_n + M/\phi_n \tag{20}$$

belongs to the same class as $\{\phi_n\}$. This is in a certain sense a "best" result and holds also in the general case [Feller, 1943]. Moreover, *if* $\{\phi_n\} \in \mathcal{L}$ *then there are* (for almost all x and every positive δ) *infinitely many n such that*

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[5] Here and in the following $\mathcal{C}$ and $\mathcal{D}$ stand for "converges" and "diverges," respectively.

(21) $$\phi_n < S_n(x)/s_n < \phi_n + \delta/\phi_n.$$

The criterion (17) is only a special case of a more general theorem. The next simplest case arises when the condition (16) is replaced by the weaker one

(22) $$a_n = O(s_n\{\log\log s_n\}^{5/6}).$$

If (22) *holds, we shall have* $\{\phi_n\}\in\mathcal{U}$ $(\mathcal{L})$ *if, and only if,*

(23) $$\sum \frac{a_n^2}{s_n^2}\phi_n \exp\{-\phi_n^2/2 + M_4\phi_n^4/12\}$$

where

(24) $$M_4 = \sum_{k=1}^{n} a_k^4.$$

It is readily seen that under the stronger condition (16) the second term in the exponent in (23) has no influence on the convergence of the series, so that (17) is, in the strict sense, a special case of (23). We can now proceed to relax the hypothesis of our criterion. If in (22) the exponent 5/6 is replaced by 7/10, an additional term containing ϕ_n^6 will appear in the exponent of (23), and its coefficient will depend on $\sum_{k=1}^n a_k^6$. Generally, if the exponent in (22) is replaced by $(2r+1)/(4r-2)$, the exponent in the criterion will contain r terms and be a polynomial of degree $2r$ in ϕ_n. Letting $r\rightarrow\infty$ we obtain the following final form of our criterion, which contains all others as a special case:

Define a number $\zeta=\zeta(n)$ *by the identity*

(25) $$\phi_n = s_n^{-1}\sum_{k=1}^{n} a_k \tanh(a_k\zeta).$$

There exists a numerical constant $\eta > 1/100$ *such that for all sequences satisfying the condition*

(26) $$a_n < \eta\frac{s_n}{\{\log\log s_n\}^{1/2}}$$

the criterion holds: $\{\phi_n\}\in\mathcal{U}$ $(\mathcal{L})$ *if, and only if,*

(27) $$\sum \frac{a_n^2}{s_n^2}\phi_n \exp\left\{-\zeta\sum_{k=1}^{n} a_k\tanh(a_k\zeta) - \sum_{k=1}^{n}\log\operatorname{ch}(a_k\zeta)\right\}.$$

In this general form the criterion is somewhat unhandy, but it explains completely the breaking down of the law of the iterated logarithm when the condition (11) is not satisfied.

3. Some unsolved problems. What happens when not even (26) is satisfied? The answer is unknown. Admittedly the question is somewhat artificial, since all "reasonable" sequences will satisfy even the strongest of our conditions, namely (16). However, considered as a special case of the general theory, the answer to the question would be of interest, for it would lead into new and unexplored domains. From special cases treated by P. Lévy [1931], Marcinkiewicz [1939], Hartman [1941], and Feller (unpublished) we know that the asymptotic behavior of sums of independent random variables changes its nature completely when (26) (or its analog in the general case) is dropped. Our present tools are not sufficient to treat such cases. Natural methods would probably be applicable also to quite different problems linking the theory of probability to certain unexplored functional equations.

Our theorems give precise theoretical information concerning the probable *amplitude* of the oscillations of $S_n(x)$ as a function of n. It would be of considerable theoretical and practical interest to have more information as to the *frequency* or wave-length of these oscillations. What can be said concerning the frequency with which $S_n(x)$ changes sign? Many similar questions can be raised, but again very little is known in this direction. (However, in the special case (4) some interesting results were recently obtained by Erdös; they are not yet published.) These questions are related to the iterated logarithm, that is to say, they are of the measure-theoretic or "strong" type. However, there are many open questions of the "weak" type, which are really problems concerning distribution functions and can be formulated in terms of Fourier analysis.

The central limit theorem gives us information concerning the average of $S_n(x)$ for a fixed n. This information is useful, but in many practical cases (for example, in all cases with "optional stopping") we are interested not so much in $S_n(x)$ as in the function

$$S_n^*(x) = \max_{k=1,\cdots,n} S_k(x). \tag{28}$$

It would be of great theoretical importance to have a theorem analogous to the central limit theorem and relative to $S_n^*(x)$. Once such a theorem is obtained, one will proceed to obtain estimates of the asymptotic error and statistical tests of significance of the same type as are now available for $S_n(x)$. These remarks apply to the general theory of random variables even more than to the present special case.[6]

In the special case (4) the problem can be easily solved but, as frequently happens, the solution is of little use in the general theory. For the particular case (4) our problem is essentially equivalent to the classical problem of ruin, or physically, to the problem of the symmetric restricted random walk in one dimension with one absorbing wall. The simplest method here seems to be that of difference equations, which also permits us to treat the more interesting problem of a random walk with two absorbing walls, say at $k=0$ and $k=r$. The problem arises classically in connection with a player who wins or looses in each trial a unit amount and starts with a capital less than r: the game ends when the player has either lost his capital ($S_n=0$) or if he succeeds in increasing it to r. Analytically one is led to a difference equation corresponding to the differential equation of diffusion

$$u_t = u_{xx} \tag{29}$$

with a boundary value problem $u(0, t)=u(r, t)=0$ and appropriate initial values. As in the case of the differential equation, our problem can be solved by two methods (method of images or trigonometric interpolation). Identifying the two solutions leads to curious identities between certain sums of binomial coefficients on one hand and trigonometric polynomials on the other. Passing to the limit one obtains the familiar transformation formula for theta functions which is usually proved operating in the same way on the equation of diffusion.[7] In the general case the difference equations are of no use, but Petrowski [1934] (cf. Khintchine's [1933] exposition) has shown that it is possible at least to obtain some limit theorems by the direct use of differential equations.

4. **The general notion of random variables.** In the simple example of §2, there were only two possible results of each individual trial, and therefore the random variables X_k could assume only two distinct values. In the general theory we shall consider perfectly arbitrary spaces. We consider first some empirical examples: throwing dice leads to a label space consisting of six points; in the case of the roulette each trial will result in a certain angle, and the label space consists of an interval $0 \leqq \psi < 2\pi$; in the theory of diffusion the result

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[6] For recent results connected with this problem, cf. Wald [1944].

[7] These facts are doubtless known in the classical literature. Related arguments have been used by P. Lévy [1931].

of an observation is, theoretically, a point, and the label space consists of a portion of the ordinary space; in statistical mechanics each observation is represented by a point in the phase-space, and the latter is the label space. Frequently a certain function will be associated with the points of the label space: in the case of games this function may represent the gain associated with the possible results of the experiment; in the theory of diffusion it may be the distance of the particle from the origin; in statistical mechanics it may be the kinetic energy or the entropy. This is the picturesque empirical background for our abstract definitions; purely mathematical illustrations will be supplied in the next section.

A *label space will in the sequel mean an arbitrary space in which a probability measure is defined* (that is to say an absolutely additive, non-negative measure such that the label space itself has measure 1). *A random variable is a real-valued measurable function in the label space.* How the probability measure has been obtained (or defined) is of no interest for our present purposes.

We have now to define the fundamental notion of *independent random variables*. For that purpose we can again use the language of "repeated trials," although the notion can actually be defined in a more general way. As in §2 we shall consider an infinite sequence of trials and suppose that to the kth trial there corresponds the label space $\mathfrak{S}_k$. In §2 all these spaces were congruent, each consisting of only two points. This, however, is by no means necessary, and the simplest applications actually lead to *variable* sequences of label spaces. The result of an infinite sequence of trials will now be represented by a sequence $P_1, P_2, \cdots$, where P_k is a point of $\mathfrak{S}_k$. Thus to our infinitely repeated trials there corresponds a label space $\mathfrak{S}$ whose points are represented by symbols $P=(P_1, P_2, \cdots)$ with $P_k \in \mathfrak{S}_k$. In modern terminology, $\mathfrak{S}$ is simply the combinatorial product of the spaces $\mathfrak{S}_k$,

$$\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2 \times \mathfrak{S}_3 \times \cdots . \tag{30}$$

Since in each $\mathfrak{S}_k$ a probability-measure has been defined, we can define probabilities in the product space $\mathfrak{S}$ in the usual way by taking the product measures. This definition of the product measure is the mathematical equivalent of the empirical notion of *independent trials*. We shall from now on suppose that the $\mathfrak{S}_k$ are arbitrary, and that in $\mathfrak{S}$ (given by (30)) the product measure has been defined. Every random variable $X_k(P_k)$ defined in $\mathfrak{S}_k$ then automatically becomes a random variable in $\mathfrak{S}$; any such sequence of random variables will be called a sequence of *mutually independent random vari-*

ables. We shall again be concerned in particular with sums

$$S_n(P) = \sum_{k=1}^{n} X_k(P_k). \tag{31}$$

The consideration of infinite product spaces (30) is in many cases most natural, but it means an unnecessary restriction. For example, the consideration of infinite product spaces led us in §2 to the unit interval in its dyadic representation. Clearly such a representation will not always be desirable, and it may be more convenient for many purposes to consider the unit interval itself as the directly given label space. A closer survey shows that the essential feature of statistical independence consists in a certain property of multiplicativity, and this consideration leads us to the following general definition (which contains the construction in product spaces as a special case). *A sequence of random variables X_k defined in an arbitrary label space $\mathfrak{S}$ will be called mutually independent, if for any choice of N, b_i and c_i*

$$\Pr\left\{b_k < X_k < c_k;\ k = 1, 2, \cdots, N\right\} = \prod_{k=1}^{N} \Pr\left\{b_k < X_k < c_k\right\}; \tag{32}$$

here the letters Pr denote the measure of the set in $\mathfrak{S}$ in which the inequalities within the braces are satisfied.

The *distribution function* $V_k(x)$ of X_k is defined by

$$V_k(x) = \Pr\left\{X_k \leqq x\right\}. \tag{33}$$

(In the case where $X_k = f(t)$ is a function defined on the unit interval, the inverse of the distribution function can be interpreted as a reordering of the values of $f(t)$ with preserved measure; in this way it has been used in real variable theory by Hardy, Littlewood, and other authors.)

Now let $F_n(x)$ denote the distribution function of the random variable $S_n(x)$ defined by (31). If the X_k are mutually independent, *the distribution function $F_n(x)$ of the sum $S_n(x)$ is given by the recurrence formulas*

$$F_1(x) = V_1(x), \qquad F_{n+1}(x) = \int_{-\infty}^{+\infty} F_n(x - y)\,dV_{n+1}(y). \tag{34}$$

The characteristic function (Fourier-Stieltjes transform) of a distribution function $V(x)$ is

$$v(s) = \int_{-\infty}^{+\infty} e^{isx}\,dV(x).$$

With this definition the characteristic function of $F_n(x)$ is simply the product of the characteristic functions of $V_1(x), \cdots, V_n(x)$.

The central limit theorem, the ordinary (weak) law of large numbers, and some other limit theorems are of the *weak type*, that is to say, they describe only the asymptotic behavior of the distribution functions $F_n(x)$. They can therefore be formulated in terms of an arbitrarily given sequence $V_k(x)$ of distribution functions and the formulas (34); *the notion of random variable and measure in the label space is, for the weak theorems, irrelevant.* However, the *strong* theorems, like the law of the iterated logarithm and the strong law of large numbers, are strictly speaking measure theoretic.

Before passing to examples, let us introduce a few definitions. The *expectation* of the random variable X_k is, by definition,

$$\mu_k = \int_{-\infty}^{+\infty} x dV_k(x), \tag{35}$$

provided the integral exists (is absolutely convergent). A simple computation shows that the expectation of the sum S_n is given by

$$m_n = \mu_1 + \cdots + \mu_n. \tag{36}$$

The *variance* of X_k is defined by

$$\sigma_k^2 = \int_{-\infty}^{+\infty} (x - \mu_k)^2 dV_k(x), \tag{37}$$

again provided that the integral converges. It is readily seen that the variance

$$s_n^2 = \int_{-\infty}^{+\infty} (x - m_n)^2 dF_n(x) \tag{38}$$

of S_n is given by

$$s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2. \tag{39}$$

5. **Examples.** The following examples are intended to illustrate the notion of independent random variables and their sums by means of some of the simplest purely mathematical applications. At the same time they will show that our limit theorems can sometimes be applied with greatest ease to problems which are frequently treated in a much more complicated manner; in such cases the refinements of the limit theorems are apt to give much more precise results than other methods.

(a) *Inversions. Among the $n!$ permutations of the elements* 1, 2, $\cdots$, n, *how many will exhibit exactly r inversions?*

Here the label space $\mathfrak{S}$ consists of the $n!$ permutations; with each point of $\mathfrak{S}$ we associate probability $1/n!$. Consider now an arbitrary, but fixed, permutation P. For $k=1, 2, \cdots, n$ we define the value of $X_k(P)$ to be the number of elements among $1, 2, \cdots, k-1$ which *succeed* the element k; in other words, $X_k(P)$ is the number of inversions in P which are produced by the element k with regard to the smaller elements. The sum

$$S_n(P) = X_1(P) + \cdots + X_n(P) \tag{40}$$

is the *total number of inversions* exhibited by P.

It is also possible to describe the situation in the terminology of independent trials; for that purpose we have to represent the label space $\mathfrak{S}$ as an n-tuple product space, which means that we have to build the permutations step by step. We first write down the element 1; for the next element, 2, we have two possibilities, namely to put it ahead or behind: accordingly, the label space $\mathfrak{S}_2$ corresponding to the second trial consists of two points, and we have to attribute to each the probability 1/2. For the third element we have three possibilities (places), and in general, the label space $\mathfrak{S}_k$ $(k=1, 2, \cdots, n)$ will consist of k points, each having probability $1/k$. The space $\mathfrak{S}$ of all permutations is clearly the combinatorial product of these $\mathfrak{S}_k$, and the original measure ($1/n!$ for each point) is the corresponding product measure. It is also seen that the random variable $X_k(P)$, the number of inversions produced by k, is defined in $\mathfrak{S}_k$ without regard to the other trials. Our X_k are therefore independent random variables, but this could also be checked directly in $\mathfrak{S}$ using the definition (32).

According to construction, X_k assumes the values $0, 1, \cdots, k-1$, each with probability $1/k$. Therefore (cf. (35)–(39))

$$\mu_k = (k-1)/2, \qquad \sigma_k^2 = (k^2-1)/12 \tag{41}$$

whence

$$m_n = n(n-1)/4 \sim n^2/4, \quad s_n^2 = (2n^3+3n^2-5n)/72 \sim n^3/36. \tag{42}$$

The central limit theorem (cf. the next section) shows *that the average number of inversions is m_n; that the number of permutations for which the number r of inversions satisfies*

$$(n^2/4) + (x_1/6)n^{3/2} < r < (n^2/4) + (x_2/6)n^{3/2} \tag{43}$$

is asymptotically given by $n!$ multiplied by $\Phi(x_2)-\Phi(x_1)$ (cf. (8)). Incidentally, known estimates of the asymptotic error in the central limit theorem permit us to give also more precise estimates for (43). Here we wish only to point out that the result is a trivial consequence of the central limit theorem.

(b) *Cycles. How many permutations of the elements* $1, 2, \cdots, n$ *exhibit exactly r cycles?*

In a self-explanatory way we begin to write the permutations in the form $1 \rightarrow e_1 \rightarrow e_2 \rightarrow \cdots$; the first cycle is completed when $e_s=1$; then we start with the smallest of the remaining elements and continue in the same way. Thus the permutation $1\rightarrow 3\rightarrow 4\rightarrow 1$; $2\rightarrow 2$; $5\rightarrow 7\rightarrow 8\rightarrow 6\rightarrow 5$ is a permutation of the elements $1, \cdots, 8$ with three cycles. Now for any permutation P we define $X_k(P)$ to equal 1 if a cycle is completed at the kth place, and $X_k(P)=0$ otherwise $(k=1, \cdots, n)$. Again the $X_k(P)$ form a set of mutually independent random variables, and their sum (40) now gives the total number of cycles in P. From the way in which the permutations are built it follows trivially that

$$\begin{aligned} &\Pr\{X_k = 1\} = 1/(n-k+1), \\ &\Pr\{X_k = 0\} = (n-k)/(n-k+1), \end{aligned} \tag{44}$$

so that

$$\mu_k = 1/(n-k+1), \qquad \sigma_k^2 = (n-k)/(n-k+1)^2 \tag{45}$$

and

$$\begin{aligned} m_n &= 1 + 1/2 + 1/3 + \cdots + 1/n \sim \log n, \\ s_n^2 &= \sum_{k=1}^{n} \frac{n-k}{(n-k+1)^2} \sim \log n. \end{aligned} \tag{46}$$

Therefore, according to the central limit theorem, *the average number of cycles is m_n; asymptotically, the number of permutations with r cycles, where*

$$\log n + x_1(\log n)^{1/2} < r < \log n + x_2(\log n)^{1/2}, \tag{47}$$

tends to $\Phi(x_2)-\Phi(x_1)$. Again more precise estimates are readily available.

If, instead of considering *all* cycles, only cycles of a prescribed length are taken into account, the method does not apply without modification: it is then necessary to introduce certain dependent random variables.

(c) *Runs.* We shall now consider what is known as Bernoulli trials. The label space is the same as in §2, that is to say each point is represented by an infinite sequence of symbols H or T; only this time we shall associate with H an arbitrary probability p (not necessarily 1/2), and with T the complementary probability $q=1-p$. For example, we may consider the ordinary decimal representation of the unit interval with its natural measure, calling the digit zero H; then $p=1/10$. By definition, *an H-run of length r is an uninterrupted sequence of at least r symbols H.* A run of length $r+1$ is then automatically a run of length r; and we shall agree to say that a run of length $2r, 3r, \cdots$ contains $2, 3, \cdots$ runs of length r. This is not strictly the usual nomenclature, but it is for our purposes by far the most convenient one. Moreover, the numerical differences arising from the change in the usual definition are perfectly negligible. Also, it requires only trivial modifications to adapt our developments to the standard definitions.

For any point P of the label space let $X_1(P)$ denote the number of trials before the completion of the first run of length r. Then X_1 is a random variable which may assume any of the values $r, r+1, \cdots$ (the probability that no run of length r occurs is zero). The distribution function $V_1(x)$ of X_1 is well known.[8] Next let $X_2(P)$ denote the number of trials from the completion of the first run to that of the second run. Clearly X_2 is a random variable which is independent of X_1 but has the same distribution function. Proceeding in the same way, we may define n independent random variables $X_1, \cdots, X_n$, all having the same distribution function and such that their sum S_n (cf. (40)) will give the number of trials from the beginning to the completion of the nth run.

The usual problem (gaining new importance in applied statistics) is to find *the probability of having in N trials k runs or more* (each of length r). In our notations this probability is

$$\Pr\{S_k \leqq N\}, \tag{48}$$

and the central limit theorem permits to evaluate this probability in the most trivial manner and with all desirable acuracy. Thus we have obtained, as a perfectly trivial corollary, the famous theorem to the effect that *the number of runs is approximately normally distributed.*[9]

(d) *Partial sums of the exponential series.* This example is, even formally, independent of the notion of random variable. Let $a>0$ be a constant, and define a distribution function $V(x; a)$ depending on the parameter a by

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[8] Cf., for example, Uspensky [1937].

[9] More interesting and much deeper results concerning the asymptotic distribution of runs have been obtained by Levene and Wolfowitz [1944].

$$V(x; a) = e^{-a}\sum_{k=0}^{[x]} \frac{a^k}{k!}. \tag{49}$$

This is the familiar *Poisson distribution*. Let, for every n, $V_n(x; a) = V(x; a)$ and define new distribution functions $F_n(x; a)$ by (34). A trivial computation shows that $F_n(x; a) = V(x; na)$. Accordingly, by the central limit theorem,

$$F_n(x; a) \sim \Phi\left(\frac{x - na}{(na)^{1/2}}\right). \tag{50}$$

With a slight change in notation this result can be restated by saying that, *for large values of* λ,

$$e^{-\lambda}\sum_{[\lambda+t_1\lambda^{1/2}]}^{[\lambda+t_2\lambda^{1/2}]} \frac{\lambda^k}{k!} \sim \Phi(t_2) - \Phi(t_1). \tag{51}$$

Again, known asymptotic estimates for the central limit theorem permit us to give, without computation, more precise estimates (cf. Kac [1942]).

6. **The central limit theorem.** Returning to the general theory we shall now consider an arbitrary sequence of mutually independent random variables X_k and their partial sums

$$S_n = X_1 + \cdots + X_n. \tag{52}$$

Using the notations defined in (36), (39) and (8), we may say that *the problem of the central limit theorem in its classical version is to establish the conditions under which the "reduced" random variable*

$$S_n^* = (S_n - m_n)/s_n \tag{53}$$

will, in the limit, be normally distributed,[10] by which is meant that

$$\Pr\{\xi < S_n^* < \eta\} \to \Phi(\eta) - \Phi(\xi). \tag{54}$$

Needless to say that the purely analytical content of the problem has not been understood from the beginning. For more than one hundred years a great many mathematicians have been working on the problem discovering many special cases to which the theorem applies and gradually establishing, and relaxing step for step, sufficient conditions under which the theorem holds. To less critical minds the law appeared as a universal law or, occasionally, as a law of nature. The first special case where the law *does not* apply has been discovered by Cauchy, but less than ten years ago a respectable mathematical journal contained a proof that the central limit theorem applies without restrictions. Great analytical progress has been made by Ljapunov, but the modern era in the theory may be said to date from the discovery by the Finnish mathematician Lindeberg [1922] that *the central limit theorem certainly holds if* $s_n \to \infty$ *and if for every* $\epsilon > 0$

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[10] This formulation presupposes the existence of second moments (34).

$$\lim_{n\to\infty} s_n^{-2} \sum_{k=1}^{n} \int_{|x-\mu_k|>\epsilon s_n} (x-\mu_k)^2 dV_k(x) = 0. \tag{55}$$

This is the famous *Lindeberg condition*; it is of importance not only because it is of remarkable generality, but still more because it turned out to be a useful tool for many purposes. For example, it has been used by Kolmogoroff in the theory of stochastic processes [**3**, 1931]. As for its generality it may be stated that, somewhat surprisingly, it turned out later (Feller, [1935])[11] that *Lindeberg's condition* (55) *is not only sufficient, but also necessary for the validity of the central limit theorem in its classical version* (which was the only one considered at that time).

At first sight this theorem would appear to state that Lindeberg's condition completely solves the problem. This, however, is far from being true. Why are the random variables S_n in (53) normed just by means of the numerical sequences $\{m_n\}$ and $\{s_n\}$? This is a tradition which goes back to pre-Laplacian times, and has gradually taken root so firmly that it has tacitly been looked upon as the only possibility. In reality this convention has done much harm. Admittedly it is natural and useful in most standard applications. Nevertheless its indiscriminate use has obscured the true content of the laws of large numbers. Worse than that, the failure to understand the arbitrariness of the special norming (53), and of the use of moments in general, has lead to many misunderstandings and to lamentable useless discussions (like the endless controversy connected with the so-called St. Petersbourg paradox). For real progress it is necessary to develop the most natural tools, which in turn is possible only by finding the most general formulation of the problem and freeing it from all artificial restrictions

Accordingly, we shall say that *a sequence* $\{V_k(x)\}$ *of distribution*

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[11] The proof will be found also in Cramér's booklet [1937].

functions obeys the (generalized) central limit law if there exist two sequences of constants $\{a_n\}$ *and* $\{b_n\}$ *such that the convolutions* $F_n(x)$ *defined by* (34) *satisfy the relation*

$$(56)\qquad \lim_{n\to\infty} F_n(a_n x + b_n) = \Phi(x);$$

in terms of the corresponding random variables X_k this means that the random variable

$$(57)\qquad S_n^* = (S_n - b_n)/a_n$$

is, in the limit, normally distributed (satisfies (54)). In this way, we have freed ourselves not only from the arbitrary classical norming (53), but also from the restriction that the moments (37) should exist: in the present formulation the definition applies to an arbitrary distribution function. Only the case where

$$(58)\qquad a_n \to \infty, \qquad a_n/a_{n+1} \to 1$$

presents interest,[12] and to avoid trivialities and clumsy formulations, we shall restrict our considerations to that case. The generalized central limit problem has been completely solved by the following

THEOREM. *The sequence* $\{V_k(x)\}$ *obeys the central limit law if, and only if, there exists a sequence* $q_n \to \infty$ *such that simultaneously*

$$(59)\qquad q_n^{-2}\sum_{k=1}^{n}\int_{|x|<q_n} x^2 dV_k(x) \to \infty, \qquad \sum_{k=1}^{n}\int_{|x|<q_n} dV_k(x) \to 0.$$

In that case one may put

$$(60)\qquad a_n^2 = \sum_{k=1}^{n}\left\{\int_{|x|<q_n} x^2 dV_k(x) - \left\{\int_{|x|<q_n} x dV_k(x)\right\}^2\right\}$$

and define b_n *by*

$$(61)\qquad \int_{|x|<a_k} (x - \beta_n) dV_k(x) = 0; \qquad b_n = \sum_{k=1}^{n} \beta_k.$$

With the constants so defined (56) *will hold* (Feller [1935]).[13]

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[12] According to a theorem of Cramér [1936], the central limit theorem can not hold with any *bounded* sequence $\{a_n\}$ unless all the variables $\{X_n\}$ are themselves normally distributed. The case where the other relation in (58) does not hold similarly leads to a trivial situation.

[13] Here and in the sequel it is supposed that the origin has been chosen in such a way that $V_k(+0) \geq q$, $V_k(-0) \leq 1 - q$ where q is some constant (arbitrarily small). This will almost always be the case and represents no restriction whatsoever. If, in particular, $q = 1/2$, the origin is the so-called median.

Whether or not a sequence $\{q_n\}$ satisfying (58) and (59) exists can be decided by means of a simple criterion, and the constants q_n, if they exist, can be computed explicitly in a trivial way.

It follows, in particular, that *for given sequences* $\{V_n(x)\}$, $\{a_n\}$, $\{b_n \equiv 0\}$ *the relation* (56) *will hold if, and only if, simultaneously*[14] *for every* $\epsilon > 0$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \int_{|x|>\epsilon a_n} dV_k(x) = 0, \tag{62}$$

$$\lim_{n\to\infty} a_n^{-2} \sum_{k=1}^{n} \left\{ \int_{|x|<a_n} x^2 dV_k(x) - \left\{ \int_{|x|<a_n} x dV_k(x) \right\}^2 \right\} = 1, \tag{63}$$

$$\lim_{n\to\infty} a_n^{-1} \sum_{k=1}^{n} \int_{|x|<a_n} x dV_k(x) = 0. \tag{64}$$

(In most practical cases the quadratic term in (63) is too small to be of influence.)

As the simplest example where the generalized, but not the classical, central limit theorem applies we may consider the special case where all $V_k(x)$ are identical:

$$V_k(x) \equiv V(x). \tag{65}$$

In this case the central limit theorem applies if, and only if,

$$\int_{|x|>z} dV(x) = o\left\{ z^{-2} \int_{|x|\leq z} x^2 dV(x) \right\} \tag{66}$$

as $z \to \infty$.[15] This criterion shows in particular that the central limit theorem may apply even if the second moment does not exist (so that the classical formulation of the theorem would break down). In such cases the norms a_n^2 will increase more rapidly than n. It was the failure to observe the possibility of similar phenomena in connection with the law of large numbers which led to the discussions of the St. Petersbourg "paradox"; within the analytical theory of limit theorems the latter does not present the slightest difficulty. In the example (65) the Lindeberg condition (55) loses sense if the moments σ^2 are infinite. However, it is easy to construct examples which look perfectly classical in the sense that moments of arbitrarily high orders exist, and for which the classical central limit theorem breaks down simply because the norming (53) is unnatural and must be replaced by another one. (For examples cf. Feller, loc. cit.)

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[14] Cf. Feller [1935]. Alternative proofs have subsequently been given by Marcinkiewicz [1938], Gnedenko [1, 1939] and Doblin [2, 1939]. If the terms of the series in (64) are replaced by their absolute values, the quadratic terms in (63) may be omitted. The proof of the *sufficiency* of this set of (slightly stronger) conditions will be found in Cramér's booklet [1937]. Simultaneously with Feller and in more probabilistic terms, P. Lévy [6, 1935, and chap. 5 of his book of 1937] has given the following solution of the central limit problem which, in a sense, should be equivalent to the second of Feller's theorems of the text: *Si chacune des variables X_k est individuellement négligeable devant la dispersion de la somme S_n, la condition nécessaire et suffisante pour que S_n dépende d'une loi d'un type généralisé peu different de celui de Gauss est que le plus grand de $|X_k|$ soit négligeable (négligeable veut dire négligeable en probabilité, c'est-à-dire très petit, sauf dans des cas très peu probable).* For further results cf. Raikov [2, 1938].

[15] This particular case has been discovered simultaneously by Khintchine [6, 1935], P. Lévy [6, 1935] and Feller [example (a) in 1, 1935].

7. **Generalizations.** The central limit problem was the starting point of many investigations. To begin with the simplest, the conditions can be generalized to various cases of convergence to distributions other than the normal (Bawly [1936], Gnedenko [**1**, **2**, 1939], Gnedenko and Groshev [1939], P. Lévy [**8**, 1936], Marcinkiewicz [**1**, 1938]); such questions are related to the nature of boundary values of analytic functions. More interesting and deeper results concern cases in which the central limit theorem does *not* hold. The classical example is furnished by the *Cauchy-distribution*

$$V_k(x) = V(x) = 1/2 + \pi^{-1} \text{arc tan } x; \tag{67}$$

here

$$F_n(nx) = V(x). \tag{68}$$

The Gaussian and the Cauchy-distributions are the oldest examples of a *stable distribution*, that is to say of a distribution function satisfying a functional relation of the form $V(x) * V(x) = V(ax)$,[16] where a is a constant. The systematic study of stable distributions has been initiated by Pólya [1923]; the most general form of the Fourier-transform of stable distributions has been obtained by Khintchine and P. Lévy [1936] (cf. also P. Lévy's book of 1937). More generally, Khintchine has obtained the totality of solutions of the functional equation $V(x) * V(x) = V(ax+b)$. Such distribution functions are the only ones which occur as limits of distribution functions of random variables of the form (57). The stable distributions are a subclass of a much wider and much more important class of distribution functions, the so-called *infinitely divisible laws.* A distribution function $F(x)$ is said to be infinitely divisible if it is the convolution of an arbitrarily large number, n, of distribution functions: $F(x) = F_1(x) * \cdots * F_r(x)$ where with increasing n the components $F_k(x)$ $(k = 1, \cdots, n)$ tend to the unitary distribution (which equals one for $x > 0$ and zero for $x < 0$).[17] Roughly speaking, the infinitely divisible distributions represent integrals of random variables and are therefore intimately connected with the theory of stochastic processes. The most general form of infinitely divisible distributions has been established by Kolmogoroff [1932] and P. Lévy [1934].[18] It is unfortunately impossible to describe the ties which link the infinitely divisible laws to many other topics: partial differential and more general functional equations, the theory of semi-groups, the arithmetic of distribution functions (inaugurated by P. Lévy and studied by Khintchine, Raikov, Gnedenko and others), and so on. Here it must suffice to mention the most direct connection with the central limit theorem which is furnished by the following beautiful theorem of Khintchine [8, 1937]: In order that $V(x)$ be the limiting distribution of a *subsequence* S_{n_k} of a sequence of form (57) it is necessary and sufficient that $V(x)$ be infinitely divisible.[19]

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[16] The star denotes the operation of convolution, defined in (34).

Returning to the central limit theorem itself, we may briefly touch on the important question concerning more precise estimates of the asymptotic error involved in (54). That this question is by no means simple is seen from the great number of papers treating the accuracy of the Gaussian distribution in the trivial special case of the binomial distribution.[20] The most general and most satisfactory asymptotic estimates now available are furnished by the well known asymptotic expansion due to Cramér [1928, 1937]. Quite recently P. L. Hsu [1945] has shown that Cramér's proof can be considerably simplified by introducing a Cesàro type kernel instead of M. Riesz' singular kernels.[21] At the same time it is seen that the same method can be used to obtain asymptotic expansions in much more general cases. It follows from these theorems that the difference between the two sides in (54) is usually of the order of magnitude of $n^{-1/2}$. Clearly such a statement has practical meaning only if ξ and η are of moderate magnitude. For large values of ξ obviously $\Phi(\xi) \sim 1$, and the relation (54) expresses a triviality. Actually the case of large ξ (or of ξ's increasing with n) is of great importance in statistics; moreover, more precise estimates for this case are essential for many theoretical investigations, for example, in connection with the iterated logarithm. Several special sequences $V_k(x)$ have been investigated from this point of view. General results for the case of equal components (65) have been obtained by Cramér [1938]; they have been generalized by Feller [6, 1943], but much more remains to be done in that direction.

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[17] For an account of the theory of infinitely divisible laws cf. P. Lévy's book [1937].

[18] Alternative proofs by Khintchine [9, 1937] and Feller [5, 1937].

[19] For further results concerning subsequences cf. Doblin [1, 3, 1938–1939], Gnedenko [1, 2, 1938].

[20] For quite recent results in this special case cf. S. Bernstein [1943].

[21] The use of Cesàro kernels has actually been suggested by Berry [1941], who has used this method to obtain very precise numerical estimates for the maximum of the difference between the distribution F_n and the corresponding Gaussian distribution.

A quite different type of open question connected with the central limit theorem has been discussed in §3.

8. **The iterated logarithm.** We consider again an arbitrary infinite sequence of mutually independent random variables X_k defined in arbitrary space. For example, the space may be the unit interval, and[22]

$$X_k = \operatorname{sign} \sin(2^k \pi x); \tag{69}$$

in that case the distribution function $V_k(x)$ defined by (33) is a step function with jumps of magnitude 1/2 at $x = \pm 1$. For simplicity we shall consider only individually bounded variables X_k; standard methods of truncation permit us to generalize all results, but the essence of our theorems will become clearer in the more restricted formulation. For bounded variables the moment (35) exists. Now the variable $X_k - \mu_k$ has a vanishing first moment, and therefore we can always by a simple change of notation achieve that

$$\mu_k \equiv 0. \tag{70}$$

Accordingly, we do not lose any generality by assuming from now on that (70) holds.

As before, the study of the partial sums

$$S_n = X_1 + \cdots + X_n \tag{71}$$

depends on the numbers σ_k and s_n defined by (37) and (39), respectively. With these notations, *Kolmogoroff's law of the iterated logarithm states in the most general case that with probability* 1

$$\limsup \frac{S_n}{s_n \{2 \log \log s_n\}^{1/2}} = 1, \tag{72}$$

provided only that

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[22] Cf. Rademacher [1922].

$$(73) \qquad \text{L.U.B. } |X_k| = o(s_n\{\log\log s_n\}^{-1/2}).$$

Again we can obtain more general and more precise results using the terminology of *upper and lower classes* introduced in §2. The definition given there applies without change if the numbers s_n in (14) are interpreted according to the general definition (39). To obtain the first generalization of Kolmogoroff's law we have simply to replace (16) by[23]

$$(74) \qquad \text{L.U.B. } |X_k| = O(s_n\{\log\log s_n\}^{-3/2}).$$

If the variables X_k satisfy (74), *then the necessary and sufficient condition that $\phi_n \in \mathcal{U}$ $(\mathcal{L})$ is that*

$$(75) \qquad \sum \frac{\sigma_n^2}{s_n^2}\phi_n \exp\{-\phi_n^2/2\} \in \mathcal{C}\,(\mathfrak{D}).$$

In particular, if all X_k have the same distribution function (as in the case (69)), the criterion (75) assumes the form (19). It can be shown that condition (74) is the *best possible* of its type. As in §2 we can proceed to relax it step for step: the exponent in (75) will again be replaced by increasingly more complex expressions. Now the exponents encountered in §2 were *even polynomials* in ϕ_n, but this is not true in general. It remains true only in the case of symmetric variables, that is to say in the case where the function $-X_k$ has the same distribution function as X_k. In §2 we passed directly from (16) to (22). In the general case, if the exponent 3/2 in (74) is replaced by 1, the exponent in (75) must be replaced by a certain polynomial of third degree. In the same way, if the exponent in (74) is replaced by $(m+2)/2m$, we shall have a polynomial of degree $(m+1)$ in ϕ_n figuring in the criterion analogous to (75). Passing to the limit we obtain the most general and most precise statement in the following form:

There exists a universal contant $\eta > 1/100$ such that for all sequences $\{X_k\}$ with

$$(76) \qquad \text{L.U.B. } |X_k| < \eta s_n\{\log\log s_n\}^{-1/2}$$

the criterion holds: $\{\phi_n\} \in \mathcal{U}$ $(\mathcal{L})$ *if, and only if,*

$$(77) \qquad \sum \frac{\sigma_n^2}{s_n^2}\phi_n \exp \mathfrak{P}_n(\phi_n) \in \mathcal{C}\,(\mathfrak{D});$$

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[23] The iterated logarithms in conditions (74) and (76) can be replaced by other functions, and the conditions restated in a slightly more general form; the above special form has been chosen only for comparison with Kolmogoroff's condition.

here $\mathfrak{P}_n(x)$ *is a power series whose mth coefficient depends only on the moments of* $X_1, \cdots, X_n$ *up to the order* m.

It can be shown that $\mathfrak{P}_n(x)$ is majorated by a certain geometric series, and other more precise estimates are available; we refer for such further results to Feller [7, 1943].

It seems that only quite artificial sequences $\{X_n\}$ will satisfy (76) but not (74). *Thus the rather complicated criterion* (77) *can, for all practical purposes, be replaced by the exceedingly simple and elegant criterion* (75). The theoretical importance of the general criterion lies in the fact that it helps us understand the actual mechanism which makes so amazingly general classes of functions exhibit an asymptotic behavior in accordance with the simple scheme given by (75). The individual terms in the power series in (77) are closely related to the asymptotic error terms for the *tails* of the Gaussian in the central limit theorem. In this sense (77) reveals the increasing complexity as we approach the outer boundaries of the domain in which the central limit theorem holds.

We are naturally led to the question of what happens if (76) does not hold. This problem is not answered and presents a challenge for the very reason that it leads beyond the central limit theorem into a domain where we still lack natural tools. Its solution would automatically give necessary and sufficient conditions for the strong law of large numbers (cf. §9), an elusive problem which has been many times attacked without success. Also, our problem would apply to several interesting stochastic processes exactly as our generalization applies to ordinary diffusion. Some interesting results leading beyond (76) have been obtained by Hartman [1941] for the case where all the random variables X_k are normally distributed (not necessarily with the same variance). A very special case studied both by P. Lévy [1, 1931] and Marcinkiewicz [1939] shows that the asymptotic behavior of sequences not obeying (76) is very different from that which we have considered so far. This is also borne out by the precise criterion obtained by the writer for the case where all the X_k have the same distribution function (not yet published).

Time and space unfortunately do not permit more than brief reference to Gnedenko's recent investigations related to our problems but pertaining to the case of continuous stochastic processes (like homogeneous diffusion).

9. **The laws of large numbers.** The implications of the so-called laws of large numbers are considerably weaker than the statements of the central limit theorem or the law of the iterated logarithm, but

these weaker implications naturally hold under much more general conditions. The distinction between *weak* and *strong* laws of large numbers is the same as between the central limit theorem and the iterated logarithm: the weak laws concern only the asymptotic behavior of convolutions of distribution functions and can be formulated without appeal to random variables. The strong laws, on the contrary, are of measure-theoretical nature. In classical textbooks only the weak law (in a special form) is proved; nevertheless the strong laws are of much greater importance in statistics, games, and other applications. In many books the weak law is proved, the strong one used.

Let $\{X_k\}$ be a sequence of mutually independent random variables with distribution functions $V_k(x)$ and expectations μ_k (cf. (33) and (35)). Let again S_n denote the nth partial sum of $\{X_k\}$, and $m_n=\mu_1+\cdots+\mu_n$ its expectation (if it exists). *In the restricted classical form one would say that the weak law of large numbers holds if for every positive* ϵ

$$\Pr\{|S_n - m_n| > \epsilon n\} \to 0. \tag{78}$$

In terms of the distribution functions (34) this means that $F_n(nx+m_n)$ tends to the unitary distribution function. Necessary and sufficient conditions for (78) to hold have been established by Kolmogoroff [1929]. It must be understood that the formulation (78) originates with the classical theory of games, where all X_k have the same distribution function. In that particular case the special norming by the factor n makes sense, since then m_n and "the total amount at stake" are proportional to n. In general, however, the factor n is perfectly arbitrary. The failure to understand this arbitrariness has led to many unnecessary discussions and "paradoxes." Also, the law of large numbers has frequently been regarded either as a law of nature or as a consequence of the definition of probability. In this sense its universal validity has been assumed and it has been applied to cases where it can easily be shown not to hold. For example, in the so-called St. Petersbourg problem the moments μ_k do not exist: by a series of misinterpretations and treating μ_k as a number (assuming, in particular, that $\infty-\infty=0$ and that all passages to limits may be interchanged), the classical theory managed to "prove" a mathematical theorem which contradicts common sense. Actually there is no problem to analyze the asymptotic behavior of the sums S_n in the St. Petersbourg case and thus to determine what used to be called "fair price." To analyze the real mathematical problem we must proceed as in the case of the central limit theorem and free the theory from all artificial restrictions. Accordingly we shall say that *the sequence* $\{X_k\}$ *obeys the (generalized) weak law of large numbers if there exist two sequences of constants* $\{c_n\}$ *and* $\{p_n\}$ *such that for every positive* ϵ

$$\Pr \{ | S_n - c_n | > \epsilon p_n \} \to 0. \tag{79}$$

The necessary[24] *and sufficient condition for* (79) *to hold is that*

$$\sum_{k=1}^{n} \int_{|x|>p_n} dV_k(x) = o(1), \tag{80}$$

$$\sum_{k=1}^{n} \int_{|x|<p_n} x^2 dV_k(x) = o(p_n^2); \tag{81}$$

the constants c_n *can then be defined by*

$$c_n = \sum_{k=1}^{n} \int_{|x|<p_n} x dV_k(x). \tag{82}$$

This theorem[25] completely solves the problem of the weak law of large numbers. Despite several attempts the problem of the strong law still remains open. We say that *the sequence* $\{X_k\}$ *obeys the strong law of large numbers if there exists a sequence of constants* $\{c_n\}$ *such that with probability one*

$$\Pr \{ | S_n - c_n | / n \} \to 0. \tag{83}$$

There exists a famous *sufficient* condition for this law which is due to Kolmogoroff [1928]. For its formulation we shall suppose that the origin has been chosen as described in footnote 13. *Kolmogoroff's condition then consists in the simultaneous covergence of the two series*

$$\sum_{k=1}^{n} \int_{|x|>k} dV_k(x) \tag{84}$$

and

$$\sum \frac{b_n}{n^2} \tag{85}$$

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[24] The condition is necessary only if the origin has been chosen as described in footnote 13, but sufficient even without this restriction.

[25] Feller [3, 1937]. For alternative proofs cf. Marcinkiewicz [1938], Gnedenko [1939], Doblin [3, 1939]. Several special cases (in particular for non-negative random variables) have been treated previously by Khintchine [7, 1936], Bawly [1936], Plessner [1936], and, perhaps, others.

where

$$b_n = \int_{|x|<a} x^2 dV_n(x). \tag{86}$$

The condition (84) is clearly also necessary. Other conditions, in part necessary and in part sufficient, can easily be obtained from the law of the iterated logarithm. However, up to now all attempts have failed to replace (85) by a weaker condition which is also necessary.[26]

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[26] Generalizing a result of Halmos [1939], Bobroff has just published the following theorem: If ψ_n and $\sum \psi_n^{-1}$ converges, then the convergence of $\sum b_n \psi_n^{-2}$ is a necessary condition for the strong law of large numbers. Unfortunately this theorem contains nothing new and is misleading. Actually Bobroff's condition is much weaker than the condition (81) (with $p_n = n$) which is necessary for the *weak law*. It is trivial that not only Bobroff's series, but also its majorating series $\sum b_n/(n\psi_n)$ must converge, even in the case of the weak law.

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5. *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Ergebnisse der Mathematik, vol. 2, no. 3, Berlin, 1933.

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1. *The covariance matrix of runs up and down*, Ann. Math. Statist. vol. 15 (1944) pp. 58–69.

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2. *Sulla legge forte dei grandi numeri*, Giornale dell'Istituto Italiano degli Attuari vol. 2 (1931) pp. 1–21.
3. *Nuove formule relative al giuco di testa e croce*, Giornale dell'Istituto Italiano degli Attuari vol. 2 (1931) pp. 127–160.
4. *Sur un théorème de M. Khintchine*, Bull. Sci. Math. vol. 55 (1931) pp. 145–160.
5. *Sur les intégrales dont les éléments sont des variables aléatoires indépendantes*, Annali della Scuola Normale Superiore di Pisà (2) vol. 3 (1934) pp. 337–366.
6. *Propriétés asymptotiques des sommes de variables aléatoires indépendantes ou enchainées*, J. Math. Pures Appl. vol. 14 (1935) pp. 347–402.
7. *La loi forte des grands nombres pour les variables aléatoires enchainées*, J. Math. Pures Appl. vol. 15 (1936) pp. 11–24.
8. *Detérmination générale des lois limites*, C. R. Acad. Sci. Paris vol. 203 (1936) pp. 698–700.
9. *Théorie de l'addition de variables aléatoires*, Paris, 1937.
10. *Compléments à théorème sur la loi de Gauss*, Bull. Sci. Math. vol. 61 (1937) pp. 115–128.
11. *Sur les exponentielles des polynomes et sur l'arithmétique des produits de lois de Poisson*, Ann. École Norm. vol. 54 (1937) pp. 231–292.
12. *L'arithmétique des lois de probabilité*, J. Math. Pures Appl. vol. 17 (1938) pp. 17–39.

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1. *Eine neue Herleitung des Exponentialgesetzes in der Wahrscheinlichkeitsrechnung*, Math. Zeit. vol. 15 (1922) pp. 211–225.

J. MARCINKIEWICZ

1. *Sur les fonctions indépendantes* I–III, Fund. Math. vol. 30 (1938) pp. 202–214, 349–364, vol. 31 (1938) pp. 86–102.
2. *Quelques théorèmes de la théorie des probabilités*, Travaux de la Société des Sciences et des Lettres de Wilno, Classe des Sciences Mathématiques et Naturelles vol. 13 (1939) pp. 1–13.

J. MARCINKIEWICZ AND A. ZYGMUND

1. *Remarque sur la loi du logarithme itéré*, Fund. Math. vol. 29 (1937) pp. 215–222.

I. PETROWSKI

1. *Ueber das Irrfahrtproblem*, Math. Ann. vol. 109 (1934) pp. 425–444.

A. Plessner
1. *Ueber das Gesetz der grossen Zahlen*, Rec. Math. (Mat. Sbornik) N.S. vol. 1 (1936) pp. 165–168.
G. Pólya
1. *Ueber den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung und das Momentenproblem*, Math. Zeit. vol. 8 (1920) pp. 171–181.
2. *Herleitung des Gausschen Fehlergesetzes aus einer Funtionalgleichung*, Math. Zeit. vol. 18 (1923) pp. 96–108.
H. Rademacher
1. *Einige Sätze über Reihen von Allgemeinen Orthogonalfunktionen*, Math. Ann. vol. 87 (1922) pp. 112–138.
D. Raikov
1. *On the decomposition of Gauss and Poisson laws*, Bull. Acad. Sci. URSS. Sér. Math. (1938) pp. 91–120 (Russian, English summary pp. 120–124).
2. *On a connection between the central limit theorem of the theory of probability and the law of large numbers*, Bull. Acad. Sci. URSS. Sér. Math. (1938) pp. 323–336 (Russian, English summary pp. 337–338).
H. Steinhaus
1. *Les probabilités dénombrables et leur rapport à la théorie de la mesure*, Fund. Math. vol. 4 (1922) pp. 286–310.
J. V. Uspensky
1. *Introduction to mathematical probability*, New York, McGraw-Hill, 1937.
A. Wald
1. *On cumulative sums of random variables*, Ann. Math. Statist. vol. 15 (1944) pp. 283–296.

Brown University

THE SUMMER MEETING IN NEW BRUNSWICK

The fifty-first Summer Meeting of the Society was held at New Jersey College for Women of Rutgers University, New Brunswick, New Jersey, Saturday, Sunday, and Monday, September 15–17, 1945. The Institute of Mathematical Statistics met on September 16. Four hundred persons attended the meeting among whom were the following three hundred twenty members of the Society:

C. R. Adams, R. B. Adams, Leonidas Alaoglu, C. B. Allendoerfer, Warren Ambrose, R. L. Anderson, T. W. Anderson, R. F. Arens, H. A. Arnold, Silvio Aurora, E. G. Baker, Valentine Bargmann, Ruth Bari, R. C. F. Bartels, P. T. Bateman, I. L. Battin, E. G. Begle, T. J. Benac, Stefan Bergman, J. S. Biggerstaff, Archie Blake, Gertrude Blanch, R. P. Boas, Salomon Bochner, H. W. Bode, H. F. Bohnenblust, G. L. Bolton, T. A. Botts, A. H. Bowker, J. L. Brenner, G. W. Brown, R. H. Brown, T. H. Brown, J. A. Bullard, R. S. Burington, L. H. Bunyan, S. S. Cairns, B. H. Camp, Abraham Charnes, S. S. Chern, Alonzo Church, R. F. Clippinger, I. S. Cohen, L. W. Cohen, J. B. Coleman, Richard Courant, A. P. Cowgill, E. L. Crow, Nancy Cole, T. F. Cope, Byron Cosby, N. A. Court, M. J. Cox, H. B. Curry, J. H. Curtiss, J. F. Daly, Norman Davids, M. M. Day, L. S. Dederick, J. B. Díaz, C. L. Dolph, M. D. Donsker, Arnold Dresden, R. J. Duffin, Nelson Dunford, W. D. Duthie, Jacques Dutka, P. S. Dwyer, W. F. Eberlein, Samuel Eilenberg, Churchill Eisenhart, L. P. Eisenhart, Wade Ellis, Benjamin Epstein, Paul Erdös, Herbert Federer, F. G. Fender, F. A. Ficken, C. H. Fischer, M. M. Flood, G. E. Forsythe, R. M. Foster, F. H. Fowler, R. H. Fox, J. S. Frame, Evelyn Frank, M. R. Freundlich, Hans Fried, Bernard Friedman, K. O. Friedrichs, R. E. Fullerton, G. N. Garrison, Abe Gelbart, H. H. Germond, J. H. Giese, B. P. Gill, H. F. Gingerich, M. A. Girshick, Casper Goffman, Michael Goldberg, A. W. Goodman, R. O. Goodman, W. H. Gottschalk, H. S. Grant, J. W. Green, Bernard Greenspan, Leonard Greenstone, Lewis Greenwald, T. N. E. Greville, V. G. Grove, P. E. Guenther, E. J. Gumbel, Margaret Gurney, K. W. Halbert, D. W. Hall, Marshall Hall, P. R. Halmos, G. E. Hay, K. E. Hazard, C. M. Hebbert, G. A. Hedlund, A. E. Heins, M. H. Heins, Erik Hemmingsen, M. R. Hestenes, Edwin Hewitt, T. H. Hildebrandt, Einar Hille, Abraham Hillman, William Hodgkinson, T. R. Hollcroft, Harold Hotelling, S. E. Hotelling, A. S. Householder, E. M. Hull, Witold Hurewicz, H. D. Huskey, (L.) C. Hutchinson, Rufus Isaacs, Nathan Jacobson, Herbert Jehle, Fritz John, G. K. Kalisch, Aida Kalish, Wilfred Kaplan, Irving Kaplansky, William Karush, M. E. Kellar, J. L. Kelley, L. M. Kells, C. J. Kirchen, S. C. Kleene, J. R. Kline, E. R. Kolchin, H. L. Krall, Jack Laderman, Rafael Laguardia, H. G. Landau, J. A. Larrivee, J. P. LaSalle, Solomon Lefschetz, D. H. Lehmer, Joseph Lehner, A. M. M. Lehr, Max LeLeiko, Howard Levene, D. C. Lewis, J. V. Lewis, Charles Loewner, A. T. Lonseth, E. R. Lorch, A. N. Lowan, Janet McDonald, Brockway McMillan, E. J. McShane, L. A. MacColl, J. K. L. MacDonald, G. W. Mackey, Saunders MacLane, H. F. MacNeish, Ingo Maddaus, W. G. Madow, M. H. Martin, R. M. Martin, W. T. Martin, D. C. May, Walther Mayer, A. E. Meder, H. L. Meyer, E. B. Mode, Deane Montgomery, C. B. Morrey, Richard Morris, D. J. Morrow, Marston Morse, J. E. Morton, F. J. Murray, C. A. Nelson, O. E. Neugebauer, A. B. Newton, Ivan Niven, P. B. Norman, C. O. Oakley, L. R. Norwood, P. S. Olmstead, L. F. Ollmann, A. F. O'Neill, E. R. Ott, F. W. Owens, H. B. Owens, J. C. Oxtoby, J. S. Oxtoby, R. S. Pate, W. H. Pell,

W. F. Penney, R. S. Phillips, Everett Pitcher, Walter Pitts, M. H. Protter, Hans Rademacher, John Raleigh, L. L. Rauch, Maxwell Reade, Mina Rees, R. W. Rempfer, R. G. D. Richardson, C. E. Rickart, John Riordan, E. K. Ritter, H. E. Robbins, M. S. Robertson, P. C. Rosenbloom, J. B. Rosser, H. E. Salzer, Hans Samelson, Arthur Sard, F. E. Satterthwaite, L. J. Savage, S. A. Schaaf, A. T. Schafer, Robert Schatten, Henry Scheffé, I. J. Schoenberg, Pincus Schub, G. E. Schweigert, Lowell Schoenfeld, Abraham Schwartz, I. E. Segal, Abraham Seidenberg, M. E. Shanks, C. E. Shannon, Seymour Sherman, Max Shiffman, S. S. Shü, D. T. Sigley, M. M. Slotnick, M. F. Smiley, P. A. Smith, Ernst Snapper, Herbert Solomon, R. D. Specht, A. H. Sprague, E. R. Stabler, E. P. Starke, N. E. Steenrod, F. F. Stephan, W. J. Sternberg, J. J. Stoker, C. N. Stokes, A. H. Stone, E. G. Straus, L. M. Straus, M. M. Sullivan, R. L. Swain, J. L. Synge, Otto Szász, A. H. Taub, Alfred Tarski, R. M. Thrall, Leonard Tornheim, W. R. Transue, W. J. Trjitzinsky, C. A. Truesdell, A. W. Tucker, Bryant Tuckerman, J. W. Tukey, H. S. Vandiver, V. J. Varineau, Andrew Vazsonyi, Oswald Veblen, R. J. Walker, A. D. Wallace, Henry Wallman, R. M. Walter, J. B. Walton, J. A. Ward, W. R. Wasow, J. V. Wehausen, E. T. Welmers, Hermann Weyl, George Whaples, A. P. Wheeler, G. W. Whitehead, P. A. White, Hassler Whitney, G. T. Whyburn, D. V. Widder, Norbert Wiener, S. S. Wilks, Jacob Wolfowitz, W. D. Wray, H. A. Wood, Alexander Wundheiler, G. S. Young, J. W. T. Youngs, Daniel Zelinsky, J. A. Zilber, H. J. Zimmerberg, Leo Zippin, Antoni Zygmund.

On Saturday at 9:30 A.M. Professor S. S. Chern of the National Tsing Hua University and the Institute for Advanced Study gave an address entitled *Some new viewpoints in differential geometry in the large.* At 2:00 P.M. Professor Samuel Eilenberg of the University of Michigan gave an address entitled *Topological methods in abstract algebra.* Professor Marston Morse presided in the morning and Professor Saunders MacLane in the afternoon.

Vice President G. T. Whyburn presided at the Sunday afternoon symposium on *Recent developments in numerical methods*, which consisted of these addresses: *Interpolation, smoothing and curve fitting* by Professor I. J. Schoenberg of the University of Pennsylvania; *Laurent expansions of algebraic functions* by Professor H. A. Rademacher of the University of Pennsylvania; and *Numerical solutions of integral equations* by Professor A. T. Lonseth of Northwestern University.

There was no session scheduled for Sunday morning in order that members might attend the symposium of the Institute of Mathematical Statistics on *The theory and application of sequential analysis.*

At 9:30 A.M. Monday, Professor Witold Hurewicz of the University of North Carolina gave an address *On some aspects of ergodic theory.* Vice President Einar Hille presided.

Presiding officers for the sessions of short papers were: Analysis, Saturday morning, Professor Norbert Wiener; Algebra and Geometry, Saturday morning, Professor V. G. Grove; Applied Mathematics, Saturday afternoon, Professor J. L. Synge; Analysis, Saturday afternoon, Lieutenant Marshall Hall; Analysis, Monday morning, Professor E. R. Lorch; and Topology, Monday morning, Professor S. S. Cairns.

There was a business meeting at 9:15 A.M. on Monday at which President T. H. Hildebrandt presided.

Headquarters for the meeting were in the Little Arcade on Jameson Campus. A photograph was taken at 1:15 P.M. Sunday.

On Saturday afternoon Dean Margaret T. Corwin of the New Jersey College for Women of Rutgers University received at her home the members attending the meeting.

The members of the Department of Music of the New Jersey College for Women gave an excellent musicale on Saturday evening.

At dinner Sunday evening Professor Saunders MacLane was toastmaster. Dean Corwin extended welcome on behalf of President R. C. Clothier of Rutgers University. President Hildebrandt spoke about the Canadian Mathematical Congress held in Montreal in June and on the immediate future needs of mathematics in America. Professor M. R. Hestenes of the University of Chicago presented a resolution of thanks for the cordial hospitality and excellent arrangements.

The Council met at 8:00 P.M. on September 16, 1945, in Agora.

The Secretary announced the election of the following twenty persons to ordinary membership in the Society:

Professor Alberto Barajas, Universidad Nacional de Mexico;
Dr. George Francis Carrier, Gordon McKay Laboratory, Harvard University;
Mr. Franco Croci, New York City;
Mr. Robert A. Deutsch, Kellex Corporation, Knoxville, Tenn.;
Mr. Nicholas A. Draim, Captain, U.S.N.;
Mr. John Arthur Healey Duffie, Catholic University of America;
Reverend Edward Henry Gallagher, University of Cincinnati;
Professor Carlos Federico Graef, Universidad Nacional de Mexico;
Professor Ralph Craig Huffer, Beloit College;
Professor Zeke Lowe Loflin, Southwestern Louisiana Institute;
Mr. George William MacCheyne, Stromberg Carlson Company, Rochester, N. Y.;
Dr. Ethel C. Muggli, St. Benedict's College, St. Joseph, Minn.;
Miss Virginia Macky Noyes, Cambridge, Mass.;
Mr. Nicanor Parra, Brown University;
Mr. Joel Pitcairn, Corporal, U.S.A.;
Professor B. S. Madhava Rao, Central College, University of Mysore, Bangalore, India;
Mr. Carlos Federico Secada, Lieutenant Commander, Peruvian Navy;
Mr. Frank Morris Steadman, Banning, Calif.;
Mr. Charles Joseph Titus, Brown University;
Mr. Ronald B. Wiley, Owens-Corning Fiberglas Corporation, Newark, Ohio.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

University of Manitoba: Assistant Professor Joseph William Lawson;
University of Oregon: Mr. Alfred Ellsworth Halteman.

Wheaton College, Norton, Massachusetts, was elected to institutional contributing membership.

The following appointments by President T. H. Hildebrandt were reported: as a Committee on Arrangements for the 1945 Summer Meeting at Rutgers University, Professors C. A. Nelson (Chairman), R. M. Foster, T. R. Hollcroft, A. E. Meder, and E. P. Starke; as a Committee on Sponsorship of Mathematical Reviews, Dean M. H. Ingraham (Chairman), Professors O. E. Neugebauer and M. H. Stone; as a Committee on Aid to Libraries Devastated by the War, Professors Arnold Dresden (Chairman), Garrett Birkhoff, H. E. Bray, Will Feller, and Gabor Szegö.

In order to aid the work of the Committee on Aid to Libraries Devastated by the War, members of the Society are requested to send to Professor Dresden information concerning the state of libraries in various countries which may come to them from various sources. The existence of this committee does not prevent any member of the Society from making a personal contribution to any library in which he may be particularly interested.

It was reported that Professor Hans Lewy had been invited to deliver an address at the November, 1945, meeting in Pasadena, California. Times and places of meetings during 1946 were set as follows: February 23, April 26–27, and October 26 in New York City; April 26–27 in Chicago; April 27 in the far west.

On recommendation of the Colloquium Editorial Committee, the Council voted to invite Professor Oscar Zariski to deliver a series of Colloquium Lectures in 1947. The Editorial Committee also reported that a volume, entitled *Foundations of algebraic geometry*, by Professor André Weil would be published under its auspices in the near future.

It was voted to invite Professor Subrahmanyan Chandrasekhar of the Yerkes Observatory, University of Chicago, to deliver the twentieth Josiah Willard Gibbs Lecture at the 1946 Annual Meeting.

President T. H. Hildebrandt was added to the Emergency Committee of the International Congress of Mathematicians, as an ex officio member, and the committee was empowered to elect its own chairman. (These changes were necessitated by the deaths of Professors G. D. Birkhoff and W. C. Graustein, former members of the Emergency Committee.)

Professors C. B. Morrey and C. V. Newsom were appointed representatives of the Society on the Council of the American Association for the Advancement of Science for the year 1946.

The Council accepted a report of the Committee on the Role of the Society in Mathematical Publication (Professor R. L. Wilder, Chairman), recommending that the Society continue its subsidy of the American Journal of Mathematics. Since 1927 the Journal has been a joint enterprise of The Johns Hopkins University and the American Mathematical Society.

The committee appointed to study the problem of a suitable memorial to the late Professor G. D. Birkhoff recommended the publication of the collected works of Professor Birkhoff and the dedication to him of an early volume of the Transactions. The President was authorized to appoint a committee to study the implementation of this committee report.

It was reported that, in the future, mathematics would have eight representatives in the Division of Physical Sciences of the National Research Council, six of these to be appointed by the American Mathematical Society, one by the Mathematical Association of America, and one by the Institute of Mathematical Statistics. (At present the Society has three representatives and the Association has one.)

A report on Universal Military Service in Peace Time was presented by a subcommittee of the War Policy Committee and approved by the Council.

The Council voted to discharge the War Policy Committee, appointed in December, 1942. The President and Secretary of the Society are to study the problem of establishing a policy committee to handle general problems affecting mathematicians in the postwar era.

Titles and cross references to the abstracts of papers read are given below. The papers were read as follows: papers 1–6 in the section for Analysis, Saturday morning; papers 7–12 in the section for Algebra and Geometry, Saturday morning; papers 13–17 in the section for Applied Mathematics, Saturday afternoon; papers 18–22 in the section for Analysis, Saturday afternoon; papers 23–29 in the section for Analysis, Monday morning; papers 30–35 in the section for Topology, Monday morning; and papers 36–131, whose abstract numbers are followed by the letter *t*, were read by title. Paper 3 was read by Professor Martin, paper 7 by Dr. Seidenberg, paper 8 by Dr. Cohen, paper 10 by Professor Tarski, paper 14 by Professor Wiener. Mr. De Baggis was introduced by Professor Karl Menger.

1. H. J. Zimmerberg: *A class of definite boundary value problems.* (Abstract 51-5-94.)

2. R. P. Boas: *Some complete sets of functions.* (Abstract 51-11-211.)

3. R. H. Cameron and W. T. Martin: *Fourier-Wiener transforms of analytic functionals.* (Abstract 51-7-111.)

4. Nelson Dunford: *Boolean algebras of projections.* (Abstract 51-11-214.)

5. W. J. Trjitzinsky: *Theory of functions of intervals and applications to functions of a complex variable.* (Abstract 51-9-164.)

6. Bernard Friedman: *Two theorems on simple functions.* (Abstract 51-11-215.)

7. I. S. Cohen and Abraham Seidenberg: *Prime ideals and integral dependence.* (Abstract 51-11-203.)

8. I. S. Cohen and Irving Kaplansky: *Rings with a finite number of prime elements.* (Abstract 51-11-202.)

9. Warren Ambrose: *Remark on measures on locally compact topological groups.* (Abstract 51-11-194.)

10. Bjarni Jónsson and Alfred Tarski: *On direct products of algebras.* (Abstract 51-9-149.)

11. N. A. Court: *On a skew quartic associated with a tetrahedron.* (Abstract 51-9-174.)

12. M. M. Day: *Polygons circumscribed about closed convex curves.* (Abstract 51-11-237.)

13. Herbert Jehle: *A new approach to stellar statistics.* (Abstract 51-9-170.)

14. Arturo Rosenbluth and Norbert Wiener: *Mathematics of fibrillation and flutter in the heart.* (Abstract 51-9-171.)

15. F. J. Murray: *Linear equation solvers.* (Abstract 51-11-231.)

16. Stefan Bergman: *The integration of equations of fluid dynamics in the three-dimensional case.* (Abstract 51-9-169.)

17. R. J. Duffin: *Nonlinear networks.* I. (Abstract 51-11-230.)

18. Leonard Greenstone: *Some properties of conformal mappings of multiply connected domains.* Preliminary report. (Abstract 51-11-216.)

19. Arnold Dresden: *An extension of the equation of Lagrange,* $y=a+xf(y)$. Preliminary report. (Abstract 51-9-156.)

20. Hans Rademacher: *On a function related to Riemann's zetafunction.* (Abstract 51-11-220.)

21. C. E. Rickart: *Banach algebras with an adjoint operation.* (Abstract 51-11-224.)

22. P. C. Rosenbloom: *Convex bodies in Banach spaces.* (Abstract 51-11-228.)

23. J. B. Díaz: *On a class of partial differential equations of even order.* (Abstract 51-11-213.)

24. C. A. Truesdell: *On the functional equation* $\partial F(z, \alpha)/\partial z = F(z, \alpha+1)$. (Abstract 51-11-229.)

25. Otto Szász: *On the absolute convergence of trigonometric series.* (Abstract 51-7-125.)

26. M. S. Robertson: *Univalent power series with multiply monotonic sequences of coefficients.* (Abstract 51-11-227.)

27. Robert Schatten: *The cross-space of linear transformations.* (Abstract 51-9-162.)

28. Ivan Niven: *A remark on Gaussian integers.* (Abstract 51-11-205.)

29. Einar Hille: *On uniform ergodic theorems.* (Abstract 51-11-218.)

30. W. H. Gottschalk: *Topological characterizations of almost periodicity.* (Abstract 51-11-248.)

31. J. C. Oxtoby: *Invariant measures in groups which are not locally compact.* (Abstract 51-9-187.)

32. G. T. Whyburn: *On monotone retractability into simple arcs.* (Abstract 51-9-192.)

33. R. H. Fox: *Knots in 3-dimensional manifolds.* (Abstract 51-7-140.)

34. A. D. Wallace: *Dimensional types.* (Abstract 51-9-190.)

35. G. S. Young: *The introduction of local connectivity by change of topology.* (Abstract 51-9-193.)

36. R. P. Agnew: *Characterization of methods of summability effective for power series inside circles of convergence.* (Abstract 51-11-209-*t*.)

37. R. P. Agnew: *Tauberian theorems for Nörlund summability.* (Abstract 51-7-109-*t*.)

38. R. F. Arens: *The linear homogeneous continua of G. D. Birkhoff.* (Abstract 51-7-137-*t*.)

39. Reinhold Baer: *Null systems in projective space.* (Abstract 51-11-195-*t*.)

40. Reinhold Baer: *Representations of groups as quotient groups.* I. (Abstract 51-11-196-*t*.)

41. Reinhold Baer: *Representations of groups as quotient groups.* II. *Minimal central chains of a group.* (Abstract 51-11-197-*t*.)

42. Reinhold Baer: *Representations of groups as quotient groups.* III. *Invariants of classes of related representations.* (Abstract 51-11-198-*t*.)

43. Reinhold Baer: *Sampling from a changing population.* (Abstract 51-11-241-*t*.)

44. Reinhold Baer: *The homomorphism theorems for loops.* (Abstract 51-11-199-*t*.)

45. Joshua Barlaz: *On some triangular summability methods.* (Abstract 51-11-210-*t*.)

46. Stefan Bergman: *Pseudo harmonic vectors and their properties.* (Abstract 51-9-155-*t*.)

47. R. H. Bing: *Collections filling up a simple plane web.* (Abstract 51-9-185-*t.*)

48. R. H. Bing: *Concerning simple plane webs.* (Abstract 51-7-138-*t.*)

49. R. H. Bing: *Collections cutting the plane.* (Abstract 51-11-245-*t.*)

50. R. H. Bing: *Generalizations of two theorems of Janiszewski.* (Abstract 51-9-186-*t.*)

51. Garrett Birkhoff: *Universal algebra.* (Abstract 51-9-182-*t.*)

52. L. M. Blumenthal: *Characterization of ϕ-spherical subsets and pseudo sets.* (Abstract 51-7-130-*t.*)

53. L. M. Blumenthal: *Metric study of elliptic spaces.* (Abstract 51-7-131-*t.*)

54. Salomon Bochner and Deane Montgomery: *Groups of differentiable and real or complex analytic transformations.* (Abstract 51-7-139-*t.*)

55. A. T. Brauer: *A problem of additive number theory and its application in electrical engineering.* (Abstract 51-9-145-*t.*)

56. Richard Brauer: *A note on systems of homogeneous algebraic equations.* (Abstract 51-11-200-*t.*)

57. Richard Brauer and H. F. Tuan: *On simple groups of finite order.* I. (Abstract 51-7-105-*t.*)

58. F. L. Brown: *A simplification of the postulates of tri-operational algebra.* (Abstract 51-9-146-*t.*)

59. E. T. Browne: *Concerning a certain ring of homographies.* (Abstract 51-7-106-*t.*)

60. R. H. Cameron: *Some examples of Fourier-Wiener transforms of analytic functionals.* (Abstract 51-7-110-*t.*)

61. S. S. Chern: *Characteristic classes of Hermitian manifolds.* I. (Abstract 51-7-132-*t.*)

62. S. S. Chern: *Characteristic classes of Hermitian manifolds.* II. (Abstract 51-7-133-*t.*)

63. A. H. Copeland and Paul Erdös: *Note on normal numbers.* (Abstract 51-9-147-*t.*)

64. M. M. Day: *Note on the billiard ball problem.* (Abstract 51-11-236-*t.*)

65. H. F. DeBaggis: *A reduction of the postulational basis of non-euclidean geometry.* (Abstract 51-7-134-*t.*)

66. John DeCicco: *Equilong maps of the ∞^3 circles.* (Abstract 51-9-175-*t.*)

67. Nelson Dunford and Robert Schatten: *On the associate and conjugate space for the direct product of Banach spaces.* (Abstract 51-7-112-*t.*)

68. C. J. Everett: *Two representations for real numbers.* (Abstract 51-9-148-*t*.)

69. Evelyn Frank: *On the zeros of polynomials with complex coefficients.* (Abstract 51-9-157-*t*.)

70. W. H. Gottschalk: *Properties of minimal sets.* (Abstract 51-11-247-*t*.)

71. J. D. Hill: *Summability of sequences of* 0's *and* 1's. (Abstract 51-7-116-*t*.)

72. Einar Hille: *A note on semi-groups analytic in a sector.* (Abstract 51-11-217-*t*.)

73. Abraham Hillman: *A note on differential polynomials.* II. (Abstract 51-7-107-*t*.)

74. Abraham Hillman: *Complex zeros of the Bessel-Weber functions* Y_n. (Abstract 51-7-117-*t*.)

75. W. H. Ingram: *A boundary value problem and its Stieltjes integral equations.* (Abstract 51-11-219-*t*.)

76. Bjarni Jónsson and Alfred Tarski: *A generalization of Wedderburn's theorem.* (Abstract 51-9-150-*t*.)

77. Mark Kac: *On the average of a certain Wiener functional.* (Abstrct 51-11-243-*t*.)

78. Irving Kaplansky: *A contribution to von Neumann's theory of games.* (Abstract 51-5-77-*t*.)

79. Edward Kasner: *Neo-Pythagorean triangles.* (Abstract 51-9-176-*t*.)

80. Edward Kasner: *Null hexagons.* (Abstract 51-9-177-*t*.)

81. Edward Kasner and John DeCicco: *Multi-isothermal systems.* (Abstract 51-9-178-*t*.)

82. J. W. Lasley: *On the classification of collineations in the plane.* (Abstract 51-9-179-*t*.)

83. J. W. Lasley: *On the equations of certain osculants.* (Abstract 51-9-180-*t*.)

84. Jakob Levitski: *Chains of multipla and semi-primarity.* (Abstract 51-11-204-*t*.)

85. A. N. Milgram: *Cyclotomically saturated polynomials and trioperational algebra.* (Abstract 51-9-151-*t*.)

86. Josephine M. Mitchell: *Some properties of solutions of partial differential equations given by their series development.* (Abstract 51-9-158-*t*.)

87. Deane Montgomery: *Measure preserving homeomorphisms.* (Abstract 51-11-250-*t*.)

88. Deane Montgomery: *Topological groups of differentiable transformations.* (Abstract 51-7-141-*t*.)

89. C. N. Moore: *Convergence factors in general analysis.* II. (Abstract 51-7-118-*t*.)

90. A. F. Moursund: *Non-summability of the conjugate series of the Fourier series.* (Abstract 51-9-159-*t*.)

91. M. E. Munroe: *A note on weak differentiability of Pettis integrals.* (Abstract 51-9-160-*t*.)

92. N. D. Nelson: *Recursive functions and intuitionistic number theory.* Preliminary report. (Abstract 51-11-240-*t*.)

93. J. M. H. Olmsted: *Transfinite rationals.* (Abstract 51-9-152-*t*.)

94. Isaac Opatowski: *Direct and reverse transitions in Markoff chains.* (Abstract 51-9-184-*t*.)

95. Gordon Pall: *Hermitian quadratic forms in a quasi-field.* (Abstract 51-9-153-*t*.)

96. W. V. Parker: *The characteristic roots of matrices.* (Abstract 51-11-206-*t*.)

97. H. P. Pettit: *On the generation of certain algebraic surfaces.* (Abstract 51-9-181-*t*.)

98. H. P. Pettit: *The tangents at certain multiple points on a curve* C_{2mn}. (Abstract 51-11-238-*t*.)

99. H. P. Pettit: *The tangents at certain ordinary points on a curve* C_{2mn}. (Abstract 51-11-239-*t*.)

100. V. C. Poor: *Complex functions possessing differentials.* (Abstract 51-7-120-*t*.)

101. Tibor Radó: *On continuous mappings of Peano spaces.* (Abstract 51-11-221-*t*.)

102. W. T. Reid: *Definitely self-adjoint differential systems.* (Abstract 51-11-222-*t*.)

103. W. T. Reid: *Integral criteria for solutions of linear differential equations.* (Abstract 51-11-223-*t*.)

104. Moses Richardson: *On weakly ordered systems.* Preliminary report. (Abstract 51-9-188-*t*.)

105. P. R. Rider: *A new use for tables of the incomplete beta function.* (Abstract 51-3-66-*t*.)

106. H. E. Robbins: *Cesàro convergence of certain random sequences.* (Abstract 51-11-225-*t*.)

107. M. S. Robertson: *The coefficients of univalent functions.* (Abstract 51-11-226-*t*.)

108. R. M. Robinson: *Finite sequences of classes.* (Abstract 51-9-183-*t*.)

109. Edward Rosenthall: *On the sums of two squares and the sum of cubes.* (Abstract 51-11-207-*t*.)

110. Raphael Salem and Antoni Zygmund: *Capacity of sets and Fourier series.* (Abstract 51-7-122-*t*.)

111. Raphael Salem and Antoni Zygmund: *Lacunary power series and Peano curves.* (Abstract 51-9-161-*t.*)

112. Raphael Salem and Antoni Zygmund: *The approximation by partial sums of Fourier series.* (Abstract 51-7-123-*t.*)

113. H. E. Salzer: *Formulas for direct and inverse interpolation of a complex function tabulated along equidistant circular arcs.* (Abstract 51-7-127-*t.*)

114. H. E. Salzer: *Note on coefficients for numerical integration with differences.* (Abstract 51-11-232-*t.*)

115. H. E. Salzer: *Table of coefficients for double quadrature without differences, for integrating second order differential equations.* (Abstract 51-7-128-*t.*)

116. H. E. Salzer: *Table of coefficients for repeated integration with differences.* (Abstract 51-9-172-*t.*)

117. Peter Scherk: *On differentiable arcs and curves.* V. Preliminary report. (Abstract 51-5-100-*t.*)

118. A. R. Schweitzer: *A theory of congruence in the foundations of geometry.* II. (Abstract 51-7-135-*t.*)

119. A. R. Schweitzer: *A theory of congruence in the foundations of geometry.* III. (Abstract 51-7-136-*t.*)

120. I. M. Sheffer: *Note on Appell polynomials.* (Abstract 51-7-124-*t.*)

121. C. F. Stephens: *Systems of linear difference equations containing a parameter.* (Abstract 51-9-163-*t.*)

122. W. J. Trjitzinsky: *Integral equations in problems of representation of functions of a complex variable.* (Abstract 51-7-126-*t.*)

123. S. M. Ulam and John von Neumann: *Random ergodic theorems.* (Abstract 51-9-165-*t.*)

124. C. W. Vickery: *Concerning the notion of measure in metric spaces.* (Abstract 51-9-166-*t.*)

125. H. S. Wall: *Analytic functions with positive real parts.* (Abstract 51-9-167-*t.*)

126. H. S. Wall: *Theorems on arbitrary J-fractions.* (Abstract 51-9-168-*t.*)

127. A. D. Wallace: *Another fixed point theorem.* (Abstract 51-9-189-*t.*)

128. A. D. Wallace: *Extension sets.* I. (Abstract 51-7-142-*t.*)

129. A. D. Wallace: *Extension sets.* II. (Abstract 51-7-143-*t.*)

130. A. D. Wallace: *Extension sets.* III. (Abstract 51-7-144-*t.*)

131. G. T. Whyburn: *Boundary alternation of monotone mappings.* (Abstract 51-9-191-*t.*)

T. R. HOLLCROFT,
Associate Secretary

UNIVERSAL MILITARY SERVICE IN PEACE TIME

REPORT OF A SUBCOMMITTEE OF THE WAR POLICY COMMITTEE[1] OF THE AMERICAN MATHEMATICAL SOCIETY AND THE MATHEMATICAL ASSOCIATION OF AMERICA

Foreword (prepared by Professor M. H. Stone, Chairman of the War Policy Committee): The War Policy Committee of the American Mathematical Society and the Mathematical Association of America was formed to study the many questions of professional and scientific policy arising out of the war. No subject has been of greater interest or more vital concern to the Committee than the relations between scientific effectiveness on the one hand and the military requirements of the nation on the other. A most important aspect of this subject is treated in the report on Universal Military Service in Peace Time which is now made public. This report, prepared some time ago by a special subcommittee, is directed in the main at points upon which mathematicians as such are particularly qualified to express informed opinions. Whatever view may ultimately prevail concerning universal military training in peace time—and it should be emphasized that there are many citizens, mathematicians included, who doubt the wisdom of introducing such a peace-time military program—it is clearly of the first importance that no program deleterious to the scientific and technological vigor of the nation should be adopted. The report deals frankly and in detail with this vital segment of the problem now before Congress. In offering the recommendations of the report as a professional contribution to the current discussion, the War Policy Committee hopes to render a modest public service strictly within the natural sphere of its activity.

1. *Introduction.* In the Congress of the United States, bills are under consideration (S. 188 and H.R. 515, companion bills) which would prescribe a year of military service in peace time for all young men of suitable physical qualifications. These bills present the nation with a proposal for a fundamental new departure in our national life and it is appropriate that the widest discussion should occur concerning all sides of the situation which would result. It is natural that the greater part of the arguments pro or con about S. 188 in the public press should deal with military, internal political, and international

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[1] Prepared, in July, by a subcommittee consisting of Professor W. L. Hart, Chairman, Saunders MacLane, and C. B. Morrey, and approved by the War Policy Committee and the Council of the American Mathematical Society.

aspects of the matter. Various spokesmen of the general field of education, particularly at the college level, as well as of specific fields of learning have likewise discussed the measure under consideration. It is the main purpose of this report to state opinions concerning the impact of compulsory military service in peace time on those phases of our national life which are of primary significance to mathematicians and which lie in areas where we may speak with special authority. In carrying out this aim, it is natural that at times we shall phrase remarks from a general educational viewpoint not primarily associated with the field of mathematics. A summary of the opinions advanced in the report is to be found in Section 11.

2. *Orientation of the report.* In the discussion of any measure like S. 188, we may conceive of a division of interest between two major questions. First, one may ask, "Should the United States have universal military service in peace time?" Second, "If Congress is to pass such an act, what provisions should it contain in order that the greatest possible good should be obtained for the nation as a whole and for the young men who will perform the service?" We realize that these questions are considerably interrelated but, for our purposes, we shall act as if they are relatively independent. It is probable that the decision on the first question should and will be reached mainly on the basis of significant national and international considerations of a social, economic, political, and military nature. In this connection it is likely that only small weight will be given to those points where our opinions as mathematicians or educators would be of any special importance as compared to those ideas we may express individually as mere citizens. Hence we have decided that, in regard to the first question and the most intimately connected phases of the second question, we shall make only a single statement.

We believe that a decision about introducing universal military service in peace time should be reached only after an investigation by a carefully selected commission, preferably appointed by the President of the United States. We recommend that this commission should include not only representatives of the armed forces but also individuals from all other important sections of our population, with a strong representation from the fields of industry, science, and technology which have been of such great importance during the present war. This commission should have the wide objective of considering what combination of universal military service, scientific development work on weapons of war, and associated industrial and educational measures would be best adapted to insure the safety of the nation.

The balance of this report will be concerned with matters particularly germane to a discussion of the second of the two major questions which we have mentioned, although we have no intention of considering all its aspects. Our remarks will be phrased under the explicit assumption that *universal military service in peace time will be adopted.* However, this viewpoint is not to be taken as evidence that we desire or expect a bill such as S. 188 to become law.

For future reference, we note that the present bill S. 188 includes the following features.

2.1. The bill omits specification of all subsidiary values as justification for military training and is based only on reasons relating to national security and the prevention of dissipation of training resources and experience.

2.2. The bill contains provision for a period of four years (ages 18 to 22) in which the individual may select the year for military training. A high school graduate with his parents' consent may volunteer to start his year of training while he is still 17 years of age.

2.3. The year of training will be used by the Army and Navy as a basis for selecting non-commissioned and commissioned officers.

3. *Background relating to the present war.* The importance of technical developments of new weapons and tactics during the military operations of the last few years has emphatically shown that science has made a fundamental contribution to the military power of the nation. More particularly, the abstract science of mathematics has been put to effective application in a surprising number of fields, including even the extensive use of mathematicians in operational analysis work overseas. These facts are important in the formulation of an intelligent long term policy for maintaining the nation's military strength. It can safely be said that the military potential of a country does not depend solely upon the size and character of its armed forces. This potential also depends largely upon the technical and industrial talent available and upon the vigor, ability, and training of its scientific and technical experts.

Long range military planning, therefore, requires that able young men be encouraged both in preparing themselves in basic scientific and mathematical knowledge and in acquainting themselves with the ways in which this knowledge can be put to use in time of war or national emergency.

With the preceding viewpoint in mind, we deeply regret to note that the manpower policies of the United States during the war have not been well formulated with respect to the utilization of men trained in science and technology or potentially able to assimilate such training. Draft deferments for the purpose of obtaining certain varieties of technical training have been at times difficult and often impossible to obtain. Some valuable technical men have been wasted in non-technical activities and others would have been wasted had it not been for the energetic efforts of representatives of science and technology in prodding reluctant or unenthusiastic government bureaus into action. During the war the United States has followed a sadly shortsighted and unintelligent policy with regard to science and technology by not providing for appropriate annual increments of undergraduate men to be trained in those fields. There are facts at hand which indicate that England and probably Russia have not made a similar error during the war. We agree strongly with the many opinions which have recently been expressed concerning the necessity for prompt resumption of training for the scientific professions, even before the end of the war with Japan. We believe that any additional delay in provisions for such training may have fatal effects not only on national health and the technological side of our economy but, also, on our means for future national defense. These remarks are well justified by the established facts about the present and potential shortages of physicians, dentists, Ph.D.'s in the various fields of the physical sciences, and technologists. The preceding background leads us to the following conclusion.

We consider it very important that universal military service in peace time should not be planned in such a way as to interfere with efforts, first, to eliminate as quickly as possible the present shortage of scientists and technologists and, second, to provide for a continuous generous supply of such essential categories of trained citizens in the future.

4. *An attitude about exemptions from military service.* It has been the traditional American viewpoint that all men should be treated as nearly alike as possible in fundamental respects whenever a call for universal military service is issued by our Government. It could be argued that this was a major cause for some of the shortsighted man-power policies of the present war period which have been referred to previously. However, we believe that it would have been possible for Selective Service to have been administered during the war without violating the specified tradition and, also, without introducing various major evils. This democratic viewpoint is so deeply seated that, regardless of whatever arguments might be presented against it, neither the general public nor the affected young men could be

expected to give friendly support to measures for creating numerous exempt classes of the population in case universal military service in peace time should be adopted. Realistically, we appreciate that any request for the exemption of mathematicians or physical scientists, for instance, could be expected to induce requests for the exemption of social scientists, and scholars in almost all other fields of learning, as well as representatives designated by innumerable pressure groups outside the field of education. The result of such requests for exemptions would undoubtedly be the creation of an ironclad policy of *no* exemptions. The requests might even develop an unsympathetic attitude among the directors of military service with respect to rational plans for differentiation of training, which we propose to emphasize in place of exemptions.

We shall proceed, then, with the explicit premise that it would be undesirable to suggest the exemption of men with special mathematical talent and that each man, regardless of his abilities, will have to spend one year of his life in military service.

5. *A liberal interpretation of the words "military service."* The present rate of increase in the military uses of science and technology convinces one that success in any future war will be dependent on continued progress in science and on the existence of a large reservoir of trained technologists. Thus it is essential that any program for universal military service should be consistent with associated plans for the development of our scientific and technological potential. It follows that the final program should not be organized with the narrow objectives of merely disciplining masses of men and preparing them for handling existing weapons of war. Hence, the term "military service" should be interpreted in a liberal fashion.

"Military service" should include not only the usual routine service in the Armed Forces but, also, various other highly technical forms of training which are just as essential for enhancing the military strength of the nation. These types of service should be made available for properly qualified young men under some studied system for differentiation of training in accordance with ability. In particular, the system should provide appropriate advanced types of service in the case of the small but important group having special aptitude for and training in mathematics, the physical sciences, or technology.

6. *Candidates for the most highly technical military service.* During the present war, under Selective Service the Armed Forces have made intelligent efforts to place the inductees into various enlisted classifications in accordance with the abilities shown by the men in

civilian life and in various tests given by the Armed Forces. Also, similar efforts have been made to locate men with appropriate backgrounds as candidates for commissioned ranks in various branches of the Armed Forces. It should be anticipated that these commendable practices in the direction of differentiated training would continue under universal military service in peace time. The Committee then directs the main force of its recommendation for differentiated training at the cases of the exceptionally superior young men who, let us say, have the requisite mathematical and scientific aptitude and inclination to become Ph.D.'s in mathematics or a physical science, or to advance to the highest levels of attainment in an engineering field. Under the reasonable hypothesis that less than one third of the men in our population with such potentialities finally pursue careers in the indicated fields, we are led to estimate the number of men of this sort available in any annual age group at about 3000 men. If we should attempt to sift them out of the enormous annual group of high school graduates by any existing devices for the prediction of mathematical or scientific aptitude, the percentage of error in diagnosis would be very high. It is likely that we would have to earmark about 20,000 boys annually in order to be reasonably certain that the final group would include 90% of the 3,000 boys whom we have mentioned. Such an attempt at selection of 20,000 boys would entail severe difficulties in the field of educational measurements and also many later complications if the members of the earmarked group were to be given individual attention during the next four to seven years. Our final plan will indicate a procedure for obtaining such candidates which appears more simple than an attempt to earmark them at an early age. At present we offer merely the following conclusion. Among the boys who graduate from high school in any year there is a group of about 20,000 with such high mathematical and scientific aptitude that they deserve special consideration under any plan for universal military service. This group will hereafter be referred to as the *select technical group*.

7. *Advantages of deferment of military service for the select group.* It is likely that any boy in the group would graduate from high school by the time he is 18. Regardless of considerations related to military service, the rarity of the aptitudes present in the group makes it desirable that each member should be shielded from unnatural influences which might lead him away from the fields of science and technology where his abilities would be so valuable to the nation. We believe that he should be encouraged to carry out any latent

inclination to enter upon scientific or technical training. Immediately after graduation from high school, an interruption of a full year in the educational process might eliminate or dim his embryonic urge to scientific study. The interruption would also cause an undesirable break in the important preliminary stages of progress in mathematics and science. Moreover, a year of military service at age 17 or 18 might expose the boy to certain military, social, or industrial activities which, although appealing to a youth, could lead him after a few years to an undesirably low terminal point outside of advanced science or technology, with a corresponding loss to the nation.

If our attention is now fastened primarily on differentiated training at a high level under military service, we can likewise argue that a delay beyond age 18 in this service may be desirable for a boy in the select group. When he graduates from high school, he will not yet possess the requisite mathematical and scientific background to be eligible for or to obtain maximum good from many of the high grade types of differentiated service which can be made available. Before he receives college training, his possibilities might be merely those of an exceptionally intelligent young man with latent scientific abilities and interests but no particularly useful background knowledge. He might even gain a permanently wrong orientation with respect to various aspects of military service where he should eventually become most useful to the nation in case of war.

The preceding remarks lead us to advocate the following flexible provision, similar to Item 2.2, in any final regulations governing universal military service in peace time.

The directors of the program should have the power to approve a request for deferment of the year of service for any young man when this is deemed best for the interest of the nation. In particular, continued efficient progress with high achievement in the study of mathematics, a physical science, or technology, even through the stage of a Ph.D. degree, should be considered sufficient cause for such deferment.

Conceivable complications in the administration of the rules for deferment should not be considered a deterrent to the adoption of the preceding recommendation. It should be possible to develop a smooth routine for such a process in peace time which would be superior to the method employed with respect to draft deferments of scientists and scientific students during the war. In this connection it is pertinent to observe that a proposal for special treatment immediately under Selective Service for a select group of 20,000 was presented to Congress recently.

8. *Suggestions concerning highly differentiated service.* We proceed under the assumption that the deferment recommendation of Section 7 could be adopted. Then, we propose the following rough outlines of a plan for implementing the suggestions about differentiated service.

8.1. A board should be created to canvass the possibilities for in service training on intermediate and high technical levels for properly qualified applicants and to supervise plans for instituting such training. This board should involve civilian representatives of the fields of mathematics, the physical sciences, and engineering, together with representatives of the armed forces.

8.2. The search for training locations should cover the various laboratories, research divisions, arsenals, and other technical activities of the Army and Navy. Also, the search should extend into associated private industries and related curricula and research in universities. As a mere sample of possibilities we mention the following types of training which might be given to properly selected groups each year.

8.21. An advanced course in meteorology for the future use of the Army or Navy, along the lines of the curriculum presented at five major universities in the United States during the early years of the war, and associated field experience.

8.22. In service training at the Ballistic Research Laboratory of the Army Ordnance Department at the Aberdeen Proving Ground.

8.23. In service training in the artillery fire control division at an ordnance plant.

8.24. In service training at a laboratory of the Signal Corps, or at an aeronautical research laboratory of either the Army, the Navy, private industry, or a university.

8.25. Study of the mathematical and operation phases of cryptanalysis in the intelligence division of the Army or Navy.

8.3. For each of the training possibilities, the Board should decide upon appropriate academic prerequisites to be demanded of any man who is to qualify for the opportunity. These prerequisites should probably be much more extensive than for any similar training during the war, because in peace time interested men would have ample opportunity to obtain the necessary background. A pamphlet should then be published listing the available varieties of technical training which might be taken as "military service" so that interested boys, even early in their high school life, could begin to orient their school work with respect to the future opportunities in military service. The pamphlet should urge boys of proper aptitudes to defer their entrance to military service until they gain the prerequisites for the type or types of service which appeal to them.

8.4. If administratively possible, assignments to any one of the most exacting and desirable varieties of training should be made on the basis of applications asserting definite interest in the field involved, as well as possession of the necessary prerequisites.

8.5. In addition to the technical training experience mentioned previously, an appropriate amount of routine "boot" training should be given either simultaneously with the technical work or in a separate period of time during the year of service.

8.6. The directing Board should examine the possibility of granting commissions in the Reserve Corps of the Army or Navy for satisfactory completion of certain types of technical training which would be available to the most highly qualified young men during their year of service. Acceptance by a young man of the opportunity of such training should not commit him to accepting such later commission as might be offered in this connection.

9. *Civilian educational by-products of universal military service.* If the interesting nature of the various technical types of differentiated military service, at low or intermediate as well as at high levels, and the corresponding academic prerequisites are properly advertised, strong repercussions might occur in the field of secondary education. It is probable that boys of ability and their parents would then demand maximum mathematical and scientific opportunities in the secondary curriculum for students of proper ability. Also, regardless of the possibility of differentiated training, it is our belief that a requirement of a year of military service, with the attendant delay in the study of the professions, would place an added demand for educational efficiency on both the high schools and colleges. The knowledge resulting from a brief exposure to substantial secondary mathematics or science is easily forgotten. On the other hand, a long exposure to these fields gives the student a body of content which, even though it should remain dormant during a year of military service, would be effective in the next year after a brief review.

Hence, if universal military service is adopted, added efforts should be made in the secondary field to expose the better students to as much substantial mathematics and physical science as possible.

10. *Educational responsibilities of the Armed Forces resulting from universal military service.* Even without the delay which would be caused by military service, the time for completing professional training encroaches on the age of greatest productivity and ingenuity as well as on the normal age for assumption of family responsibilities.

Hence, under universal military service, the Armed Forces will be presented with a challenge to compensate as much as possible for the year of delay which would be introduced. It is obvious that the handicaps suffered by a young man due to an interruption of a year in his formal education would be considerably lessened if he should systematically carry on correspondence study during the year.

If universal military service is adopted, the Armed Forces should offer to the men in training the fullest cooperation and encouragement in the extension of their formal education by means of correspondence courses. At the college level, we recommend that this be done by paying the registration fees for men who take and diligently pursue courses offered by regular college correspondence departments, rather than through such an agency as the present Armed Forces Institute.

11. *Summary*. In the development of this report we have emphasized the following points and recommendations.

11.1. A joint civilian and military commission should be appointed by the President of the United States to study the problem of universal military service in peace time before final action is taken on the matter.

11.2. An expression of opinions by the American Mathematical Society and the Mathematical Association of America should be focused on constructive suggestions about the administration of universal military service if it should be adopted and about the solution of resulting educational problems.

11.3. No outright exemptions from universal military service should be requested.

11.4. The required military service should be highly differentiated in accordance with the aptitudes and training of the young men involved, with emphasis on exceptional differentiation for those few with the greatest technical abilities.

11.5. A system for deferring the year of military service should be instituted so that gifted young men might prepare themselves for advanced varieties of differentiated service before entering the Armed Forces.

11.6. The possibilities for technical varieties of differentiated service should be canvassed by a joint civilian and military board. The resulting training programs with the corresponding academic prerequisites should be well advertised among high school boys, their parents, and the teachers and administrators in the secondary field.

11.7. If universal military service is adopted, the field of secondary education will have the added responsibility of increasing the efficiency and quantity of instruction given in mathematics and physical science to the students of better than average ability.

11.8. The Armed Forces should encourage young men to continue their education during military service by taking correspondence work through regular school channels, and the Armed Forces should pay the costs of such study if a man carries it through diligently.

PREDOCTORAL FELLOWSHIPS IN THE NATURAL SCIENCES

The National Research Council announces that it is now ready to receive nominations and applications for the predoctoral fellowships in the natural (that is mathematical, physical, and biological) sciences which it is administering under a grant from the Rockefeller Foundation. These fellowships are intended to assist young men and women, whose graduate study has been prevented or interrupted by the war, to complete their work for the doctorate. It is hoped that these fellowships will do much to accelerate the recovery of the scientific vigor and competence of the country which is so seriously threatened by the loss of almost two graduate school generations of scientifically trained men and women.

This program will be administered by a Committee on Predoctoral Fellowships of the National Research Council whose members are Henry A. Barton, Charles W. Bray, Detlev W. Bronk, Luther P. Eisenhart, Ross G. Harrison (Chairman—National Research Council, ex officio), W. A. Noyes, Jr., and John T. Tate, chairman; Enid Hannaford, secretary.

The annual stipend will be $1200 for single persons and $1800 for married men. In general it is expected that each recipient will spend at least eleven months per year on academic work. An additional allowance up to $500 per year will be made for tuition fees. Fellowships granted to individuals who are eligible for educational support from the "G.I. Bill of Rights" will be at such stipends as to bring the total income from these two sources to that which would be received at the above rates.

Each fellow, before entering on his graduate studies, will submit for review by the Committee on Predoctoral Fellowships a schedule, approved by the dean of his graduate school, for the completion of his work for the doctorate. This schedule, as approved by the committee, will constitute an informal agreement upon the basis of which stipend payments will be made. At the discretion of the university concerned the fellowship stipend may be supplemented by university grants. All such supplementary sources of income should be made a matter of record with the committee. The progress of the fellows will be subject to periodic review by the committee which reserves the right to cancel fellowships when in their judgment satisfactory progress is not being maintained.

Prospective candidates for these fellowships are urged to apply at once even though they may be unable to undertake their graduate study in the immediate future. Information concerning these fellowships and Nomination-Application blanks are being mailed out widely to graduate schools and wartime research laboratories. They may also be obtained by writing directly to the Secretary, Committee on Predoctoral Fellowships, National Research Council, 2101 Constitution Avenue N.W., Washington 25, D. C.

CANADIAN MATHEMATICAL CONGRESS

The first meeting of the Canadian Mathematical Congress was held in Montreal, June 18–23. It was bilingual and its sessions were held in various parts of Montreal. The Congress had the support and approval of all universities throughout Canada, and was made possible financially by substantial grants from the National Research Council of Canada, the Government of the Province of Quebec, the City of Montreal, and numerous insurance and industrial companies. Through the kindess of the British Council and the Royal Society, Professor L. J. Mordell (now of Cambridge University) attended as British delegate. A permanent organization was formed to meet about every four years and to operate in the meantime through committees and a permanent executive. The meeting was attended by about 200 mathematicians, some accompanied by their families, from all parts of Canada.

The sessions on Monday, Thursday forenoon, Friday and Saturday were held at McGill University, the session on Tuesday at Montreal University, the session on Wednesday at the Agricultural Institute, and the session on Thursday afternoon at the Ecole Polytechnique. The program of the meeting gives a good view of the scope of the discussions.

Secondary school mathematics from the university point of view, Norman Miller of Queen's University, D. C. Murdoch of the University of British Columbia, M. S. Macphail of Acadia University, and Herbert Tate of McGill University.

Comparaison entre les programmes des universités de France et des universités françaises de Québec, Emile Gérard of St. Mary's, Montreal.

On some developments of modern algebra. I, Richard Brauer of the University of Toronto.

Role of mathematics in a mathematics and physics course, Samuel Beatty of the University of Toronto and Arthur Léveillé of Montreal University.

Cosmic rays and the elementary particles of physics, F. Rasetti of Laval University.

Rôle de la géométrie dans la formation générale, J. Flahault of Montreal University and the Ecole Polytechnique.

Les nouvelles statistiques en physique, F. Rasetti of Laval University.

On some developments of modern algebra. II, Richard Brauer of the University of Toronto.

Engineering mathematics. I, V. G. Smith of the University of Toronto, Spencer Ball of the Nova Scotia Technical College, and M. Burke-Gaffney of St. Mary's College, Halifax.

Les mathématiques de l'astronomie, Frère Robert of Mont St. Louis, Montreal.

The problem of mathematical research in Canada. R. L. Jeffery of Queen's University.

Methods of fostering research and graduate work, W. L. G. Williams of McGill University.

Engineering mathematics. II, Adrien Pouliot of Laval University and F. M. Wood of McGill University.

Introduction aux mathématiques, A. Larue of Laval University.

Foundations of geometry, G. de B. Robinson of the National Research Council.

Postwar problems of the Canadian universities, F. S. Nowlan of the University of British Columbia.

The nine regular solids, H. S. M. Coxeter of the University of Toronto.

Le calcul vectorial avec applications à la géométrie analytique, Gaston Bertrand of the Ecole Polytechnique.

A national policy for applied mathematics, W. H. Watson of the University of Saskatchewan, L. Infeld of the University of Toronto and A. H. S. Gillson of McGill University.

The geometry of numbers, L. J. Mordell of the University of Manchester.

Some topological properties of disk and sphere, A. W. Tucker of Princeton University.

High-speed computing devices and mathematical analysis, John von Neumann of the Institute for Advanced Study.

Universal algebra, Garrett Birkhoff of Harvard University.

On the use of the differential analyzer in solving partial differential equations, Douglas R. Hartree of the University of Manchester.

La théorie des groupes de Lie, Claude Chevalley of Princeton University.

Conformal representation and hydrodynamics, A. Weinstein of the University of Toronto.

A symposium on statistics was held in two sessions with the following program:

The field for university students in the life insurance business, Gordon Beatty of the Canada Life Insurance Company.

The practical use of statistics and actuarial mathematics in the life insurance business, C. D. Rutherford of the Sun Life Assurance Company of Canada.

Prediction of students' success in a commerce course, L. A. H. Warren of the University of Manitoba.

Certain mathematical aspects of statistics, R. H. Cole of the University of Western Ontario, E. S. Keeping of the University of Alberta, N. Keyfetz of the Dominion Bureau of Statistics, and W. Kozakiewicz.

There was also a symposium on war research with the following program:

Mathematics in radio, W. J. Henderson of the National Research Council.

Mathematics in ballistics research, N. S. Mendelsohn of the Ballistics Laboratory, Valcartier, Quebec.

Mathematics in aeronautical research, W. F. Campbell of the National Research Council.

There were two sessions for short research papers with the following papers presented:

Isotropic solutions of Einstein's field equations, Max Wyman of the University of Alberta.

Lattice points in ovals, Douglas S. Derry of the University of Saskatchewan.

The 3-row, n-column Latin rectangle derangement problem, Lloyd Dulmage of the University of Manitoba.

Some problems of approximation associated with linear differential systems, W. H. McEwen of Mount Allison University.

The four vertex theorem and related topics, Peter Scherk of the University of Saskatchewan.

At the luncheon on Thursday addresses were given by C. J. Mackenzie, President of the National Research Council, and by L. J. Mordell.

Two mathematical films, *The isograph* and *A triple integral*, were shown through the kindness of the Bell Telephone Company and of E. A. Whitman of the Carnegie Institute of Technology.

Several sets of resolutions were adopted at the General Meeting on Friday evening and Saturday forenoon. To implement these the following permanent Committees were appointed:

Committee on Secondary School Mathematics, Norman Miller and Abbe G. Perras, joint chairmen.

Committee on Scholarships, Fellowships and Prizes, F. S. Nowlan, chairman.

Committee on Research, G. Pall and A. Pouliot, joint chairmen.

Committee for Exchange of Staffs, A. H. S. Gillson, chairman (American Representative, R. G. D. Richardson).

Committee for Journal, A. Léveillé and W. H. Watson, joint chairmen.

Committee for Publishing Proceedings of the First Meeting, H. S. M. Coxeter and A. Léveillé, joint chairmen.

Finance Committee for the Congress, W. L. G. Williams, chairman.

A. H. S. Gillson was appointed to prepare a pamphlet for educational authorities, business and industry.

The Executive of the Congress, appointed to function until the next general meeting, is as follows: Dean S. Beatty, President; Dean A. Pouliot, Vice President; G. Perras, French Secretary; R. E. O'Connor, English Secretary; W. L. G. Williams, Treasurer; and the Chairmen of the above committees.

R. ERIC O'CONNOR, S.J.,
Secretary

BOOK REVIEWS

The problem of moments. By J. A. Shohat and J. D. Tamarkin. (Mathematical Surveys, vol. 1.) New York, American Mathematical Society, 1943. 140 pp. $2.25.

This book is the first in a new series sponsored by the American Mathematical Society. Each number of the series is evidently designed to collect all important existing material on a single mathematical discipline and to expound it in a readable and understandable manner. If so, then the volume under review succeeds admirably for that branch of mathematics which has recently grown up about a problem posed and solved by T. J. Stieltjes and named by him "the moment problem." The authors have set a high standard of excellence in presentation and choice of material, which may well establish the tone for the series.

The moment problem may be stated as follows: Given a sequence of real numbers $\mu_0, \mu_1, \cdots$; determine a nondecreasing function $\psi(t)$ such that

$$\mu_n = \int_a^b t^n d\psi(t), \qquad n = 0, 1, 2, \cdots. \tag{1}$$

If $a=0$, $b=\infty$, the problem is precisely as Stieltjes set it and is known as the Stieltjes problem; it is known by the names of F. Hausdorff or H. Hamburger according as $a=0$, $b=1$ or $a=-\infty$, $b=+\infty$. Of course the first two are special cases of the last, but it is profitable to study them separately. The case in which $\psi(t)$ is to be the Riemann integral of a positive function was studied by earlier authors, notably E. Heine and P. Tchebycheff. In spite of this it seems proper that the name of Stieltjes should be attached to the problem, for the generalized integral which he introduced, now known as the Riemann-Stieltjes integral, was indeed a very happy idea for the development of the problem. Without it finite linear combinations and integrals would have to be studied separately, and nothing like the existing elegance in the theory could possibly be attained.

The reason for the term "moment" becomes evident if one interprets $\psi(t)$ as defining a distribution of mass along the interval $a \leqq t \leqq b$. Then μ_0 is the total mass; μ_1 is the statical moment which measures the tendency of the segment to turn about the origin if held horizontally; μ_2 is the moment of inertia about the origin. Stieltjes defines μ_n as the nth moment, and his fundamental question is whether the distribution of mass can be ascertained from a knowledge of all the moments. After this physical statement it is clear why one is particularly interested in nondecreasing functions $\psi(t)$. From the point of view of the pure mathematician the case when $\psi(t)$ is of bounded variation is equally interesting. Indeed this case has been studied in detail for the Hausdorff problem with suitable restrictions on the μ's. However, for the other two problems it turns out that the equations (1) are consistent for an *arbitrary* sequence of moments if $\psi(t)$ is only required to be of bounded variation. With so slight a relation between the μ's and $\psi(t)$ there remains little to study.

We return to the problems as originally set and state the fundamental conditions for consistency. The Hausdorff problem has a solution if and only if

$$(-1)^k\Delta^k\mu_n \geqq 0, \qquad k, n = 0, 1, 2, \cdots .$$

The sequences $(n+1)^{-1}$ and 2^{-n} have this property. Indeed they correspond to the nondecreasing functions $\psi(t)=t$ and $\psi(t)=0$, $t<1/2$, $\psi(t)=1$, $t>1/2$, respectively. The corresponding conditions for the Hamburger problem are that

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \mu_{i+j}x_ix_j \geqq 0, \qquad n = 0, 1, 2, \cdots , \tag{2}$$

for all real numbers x_i. Finally, for the Stieltjes problem one must add to the inequalities (2) that

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \mu_{i+j+1}x_ix_j \geqq 0, \qquad n = 0, 1, 2, \cdots .$$

Having established conditions for the existence of a solution we naturally ask if the solution is unique. At first sight we might perhaps not expect uniqueness under any circumstances, for we are trying to determine a function over an interval, a point set with the power of the continuum, by the countable set of moments. However, our more familiar experience with Fourier series, where the Fourier coefficients do determine the function from which the series derives, may serve to orient us here. In fact the solution of the Hausdorff problem is unique, even if $\psi(t)$ is of bounded variation. However, in the other two cases when the interval is infinite the solution is not in general unique, and it becomes desirable to know if further conditions can be imposed on the μ's to guarantee uniqueness. Many such sufficient conditions have been found. The most general one known, due to T. Carleman, is that the divergence of the series

$$\sum_{n=1}^{\infty} \mu_{2n}^{-1/(2n)}$$

implies the uniqueness of the solution of the Hamburger problem.

The book begins with a brief introductory chapter designed chiefly to collect the needed tools but also containing a historical sketch and a description of the main results. Here is proved a theorem on the extension of non-negative functionals which is fundamental for later work. There follow four chapters which we now describe briefly.

Chapter 1 is concerned with the fundamental theorems of consistency mentioned above. It also proves Carleman's uniqueness theorem and derives others therefrom.

Chapter 2 deals with the Hamburger problem in minute detail and contains perhaps the most profound material of the book. The approach is through the theory of continued fractions, which really was the origin of the investigation of Stieltjes. These fractions in turn are studied by use of a class of functions which are analytic in a half-plane and which have a negative imaginary part there. Necessary and sufficient conditions for uniqueness are found in terms of the continued fraction associated with the moment problem. Many details concerning the polynomials which appear in the successive approximants of this continued fraction are developed.

The third chapter discusses various modifications of the original problem, notably one mentioned above: the Hausdorff problem in which $\psi(t)$ is no longer required to be nondecreasing. Relations with the Laplace transform are considered. Explicit expansions of $\psi(t)$ in terms of the moments are obtained. This part of the book is replete with ideas and makes fascinating reading.

The fourth and final chapter concerns itself with methods of approximate quadrature. There is such a method, corresponding to each moment sequence, for approximating the integral

$$\int_{-\infty}^{\infty} f(t)d\psi(t). \tag{3}$$

Here $\psi(t)$ is any solution of the Hamburger problem. The method described may be considered as a generalization of the classical formula of mechanical quadrature due to Gauss. In fact it reduces thereto if $\psi(t)=t$ in the interval $(-1, 1)$ and is constant elsewhere. The chapter closes with a discussion of the convergence of the approximations to the integral (3).

Throughout the book the method of exposition is to proceed from the general to the particular. An effort is made to drive directly at the most general result known and to discuss all other results as special cases. This procedure is an ideal one for a survey of the present character. As a result the reader is able to get a clear picture of the whole subject and to see in proper perspective the interrelations of the various special parts. In view of the nature of the book the authors have not included proofs of all theorems. But enough of these have been given to give the reader a real understanding of the methods. The choice of what to include and what to omit is generally in excellent taste. The reviewer would prefer to have seen included more of the proofs of the results of N. I. Achyeser and M. Krein, for their treatise on the subject is written in Russian. There are very few misprints. Those discovered by the reviewer are of a trivial nature and can be corrected in an obvious way by the reader. The book is certainly a very valuable addition to mathematical literature. The American Mathematical Society is to be congratulated on this auspicious initiation of its series of surveys.

D. V. WIDDER

Introduction to mathematical logic. Part I. By Alonzo Church. (Annals of Mathematics Studies, no. 13.) Princeton University Press, 1944. 6+118 pp. $1.75.

This booklet contains the material of the first half of a graduate course given by the author at Princeton and is, in fact, the revision of a set of lecture notes of that course. It is hoped that Part II, covering the second half of the course, which exists now in rough draft in the form of lecture notes, will some day appear in print.

The title of the book is somewhat misleading. As the author says, this is a monograph rather than a text. Its aim is not to give a broad survey of recent developments in symbolic logic but to present formally and rigorously, and with all the latest improvements, the theory of one of the oldest branches of the subject, namely, the calculus of propositional functions. In this aim the author has succeeded admirably. The difficult points are emphasized instead of being avoided. The care and precision for which the author is noted are in evidence throughout. The end result is, for the qualified beginner, a compact presentation of an important theory which can serve also as a model of present-day standards of rigor. The book is of value to the specialist also in bringing together in one place and in one notation rigorous proofs of important basic theorems which are otherwise only to be found scattered throughout the literature.

There are four chapters, of which the first deals with the calculus of propositions, the second and third with two different formulations of the functional calculus of first order, and the fourth with functional calculi of higher order.

The first chapter presents various alternative but equivalent formulations of the propositional calculus as a formal system. The now familiar distinctions between object language and syntax language, between axioms and rules of procedure, and between theorems of the calculus and theorems about the calculus (syntactic theorems) are explained. The highlights of this chapter are the deduction theorem and the solution of the decision problem, both syntactic results.

The second chapter presents one formulation of the functional calculus of first order, called by the author the system F^1. By using infinite axiom schemata instead of finitely many axioms, the author avoids the necessity of including the complicated substitution rules for individual variables, propositional variables, and functional variables among the primitive assumptions. These substitution rules, however, are obtained as derived rules. Other important results of this chapter are the deduction theorem for the functional calculus, and the theorem on reduction to prenex normal form.

The third chapter takes up another formulation called the pure functional calculus of first order F^1_p. This system has only a finite number of axioms and no constants but it has complicated rules of substitution. This chapter contains some of the deepest results of the book. Among them are the Gödel completeness theorem and the Löwenheim-Skolem theorem. It is noted that the decision problem for the functional calculus is unsolvable in the general case. A solution of the decision problem for formulas in certain special forms is given and the problem of finding simple forms with only a few quantifiers to which every well formed formula is reducible is treated.

The last chapter is a short one. It discusses informally without proofs the extension of the previous theory to the functional calculi of second order F^2 and of infinite order F^ω. Also discussed are the axiom of infinity, the axioms of extensionality, the theory of types, and various forms of the axiom of choice. The difficulties of proving consistency when various combinations of these assumptions are made is pointed out. In this connection the famous Gödel incompleteness theorem is mentioned but discussion of it is postponed to Part II. However, Gödel's conditional consistency theorem is discussed.

It is interesting to note that, although the title of the book refers to mathematical logic, the subject matter is mostly logic with very little mathematics. This is because the applications to number theory and to set theory have been postponed to Part II. The book has no index or bibliography, but there are historical and bibliographical notes at the ends of the first two chapters. This is in every way a welcome and useful addition to the literature of logic.

ORRIN FRINK, JR.

Mathematical theory of optics. By R. K. Luneberg. (Supplementary notes by M. Herzberger.) Providence, Brown University, 1944. 17+6+401+93 pp. $4.00. (Mimeographed.)

In this book, geometrical optics and diffraction are both treated as special applications of the electromagnetic theory of light. This treatment has advantages in some respects and disadvantages in others. Some types of problems, for instance those of polarized light, are handled easily, while the development of the problems of geometrical optics proper becomes somewhat involved.

The author has made several ingenious contributions in this connection. The idea of treating geometrical optics as a special limiting case of wave optics has never been carried out before in so much detail. The author has nowhere restricted himself to homogeneous media, following in the path of W. R. Hamilton. He covers first-order theory in general systems, third-order image-error theory in rotationally symmetrical systems, and the diffraction theory of spherical and near-spherical waves. A special chapter is devoted to media with parallel layers of constant refractive index.

Luneberg's treatment of diffraction optics is an important step forward. His fundamental contribution may be described as follows:

The light distribution is assumed to be known in a plane of infinite extent, and represented by the function $f(x_0, y_0)$. The light distribution may be only sectionally continuous inside a finite area, but outside this area the function must be small and continuous, and grow smaller with increasing distance from the center of the plane. Specifically, outside the limited area, f, $\partial f/\partial x_0$, and $\partial f/\partial y_0$ must all be continuous and smaller than $B(x_0^2+y_0^2)^{-1/2}$, where B is a constant. A further condition is that the resulting light distribution in space, represented by the function $u(x, y, z)$, shall be indistinguishable from that of a spherical wave at great distances from the plane, since at great distances the finite area of the plane is indistinguishable from a point. Mathematically, this condition is that, beyond a certain large distance from the center of the plane represented by $R=(x^2+y^2+z^2)^{1/2}$, the absolute values of u and $\partial u/\partial R$ shall be smaller than C/R, and the absolute value of the expression $\partial u/\partial R - iku$ shall be smaller than D/R^2, where D and C are constants.

The solution for the light distribution in space under these conditions is given as

$$u(x, y, z) = -\frac{z}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_0y_0)\left(\frac{ik}{r^2}-\frac{1}{r^3}\right)e^{ikr}dx_0dy_0$$

where $r=((x-x_0)^2+(y-y_0)^2+z^2)^{1/2}$. The solution is found by the application of a mathematical technique which is reminiscent of a method used by A. Sommerfeld (Math. Ann. vol. 47 (1896)) for the solution of another problem of diffraction.

Luneberg proves that this solution fulfills all the conditions imposed, and he proves furthermore that all of these conditions are necessary to guarantee that the solution shall be unique.

It should be noted that this solution demands only the knowledge of u in the plane $z=0$, and not its normal component. Thus are avoided the difficulties encountered by Kirchhoff and Abbe in their early attempts to solve the problem.

The book has some of the advantages and also some of the shortcomings of lecture notes. A new edition might be more helpful to the reader if certain formal changes were made. For instance, the reviewer found it very disturbing that the author does not distinguish between the symbols for scalars and those for vectors, especially in view of the fact that the author uses vector integration.

At many points the author might with advantage have stated the physical reasons for his assumptions before proceeding with the mathematical argument. It is also regrettable that the book quotes little of the recent literature in the fields of geometrical optics and diffraction.

Nevertheless, we may say in conclusion that the book contains important new material, interestingly presented. It is highly recommended as a source of valuable information.

M. HERZBERGER

It was very fortunate that Herzberger cared to add some supplementary notes to Luneberg's book. The first supplement deals with so-called electron optics and gives an excellent account of this theory in close analogy to ordinary optics. Starting from a properly generalized Fermat principle and the associated canonical equations, the author leads up to the set of propositions that correspond to the Gaussian approximation and to the third-order error theory in the usual treatment of optical instruments. Greatly at variance with many papers and even textbooks, the presentation is logically complete and self-consistent, not leaving out necessary intermediate steps. At some places the reviewer would have liked more explicit references to plain geometrical facts. But a reader well versed in the language of analysis and familiar with the elements of differential geometry will never be at a loss. This is probably the best succinct presentation of electronic optics so far published in any language.

In the second supplement Herzberger discusses the physical properties of optical glass. Based on his long standing experience in all parts of optical practice, he supplies a most useful collection of facts and numerical data. In a brief note headed Mathematics and geometrical optics, he examines the relationship between different starting points in the presentation of the usual material of geometrical optics, comparing Hamilton's approach with Lagrange's theory, and so on.

Of greatest interest is the last supplement on Symmetry and asymmetry in optical images. Here Herzberger gives some very brief hints of a new error theory of which he is the author. This theory is based on the notion of "diapoint," that is, the point in which the image ray intersects the meridian plane of the object point. By studying the manifolds of diapoints, one arrives at a new classification and thus at a new treatment of optical errors. Although it cannot be expected that this conception will finally supersede the classical Seidel theory, it presents new, surprising, and very useful aspects which might become of great value in the design of instruments. It is to be hoped that Herzberger will soon find the opportunity to give a full account of his results in a book of his own.

R. v. MISES

NOTES

Additional information concerning certain mathematicians in Holland has become available. Professor J. C. van der Corput of the University of Groningen is safe as is also Professor B. L. van der Waerden of the University of Leipzig who has recently come to Holland. Professor J. A. Schouten and Professor F. Schuh of the Technical University of Delft and Professor J. A. Barrau of the University of Utrecht have retired. Professor Otto Blumenthal of the Technical School of Aachen, who emigrated to Holland in 1939, was deported to Theresienstadt in 1944 and is now reported to have died.

The following news concerning Polish mathematicians has been received from various sources.

The following mathematicians were killed during the German occupation of Poland: Stanislaw Ruziewicz, Professor at the University of Lwów, in July, 1941; Antoni Lomnicki, Professor and Pro-rector at the Polytechnical School in Lwów, in July, 1941; Casimir Bartel, Professor and former Rector at the Polytechnical School in Lwów (several times Prime Minister of Poland), in July, 1941; Stanislaw Saks, Docent at the University of Warsaw, in November, 1943; Julius Schauder, Docent at the University of Lwów, in July, 1941; Herman Auerbach, Docent at the University of Lwów, in the summer of 1943; Moses Jacob, Docent at the University of Lwów, in 1943; Stefan Kaczmarz, Docent at the University of Lwów, officer in the Polish army, in the fall of 1939; Adolf Lindenbaum, Docent at the University of Warsaw, in the summer of 1941.

The following died in concentration camps: Antoni Hoborski, Professor at the University of Cracow, former Rector at the Mining Academy in Cracow, in 1940; Zygmunt Zalcwasser, Professor at the Free University in Warsaw, in January, 1943; Aleksander Rajchman, Professor at the Free University in Warsaw, in 1940.

The following were taken prisoner and disappeared without trace: Józef Marcinkiewicz, Docent at the University of Wilno; Konstanty Sokół-Sokołowski, Instructor at the University of Wilno.

Stefan Kempisty, Professor at the University of Wilno, was deported and died in prison in 1940.

Samuel Dickstein, Professor at the University of Warsaw, died in 1939 during an air raid.

The following persons died of natural causes during the war: Witold Wilkosz, Professor at the University of Cracow, in March, 1941; Antoni Przeborski, Professor at the University of Warsaw, in 1941; Stefan Kwietniewski, Lecturer at the University of Warsaw, in 1940; Andrew Kozniewski, in December, 1939; Stanislaw Zaremba, Professor at the University of Cracow, in 1943; Stefan Banach, Professor at the University of Lwów, in September, 1945.

The following Polish mathematicians are known to have survived: W. Sierpiński, K. Kuratowski, K. Borsuk, E. Szpilrajn, B. Knaster, O. Nikodym, S. Straszewicz, S. Otto, W. Pogorzelski, H. Steinhaus, W. Mazur, W. Orlicz, W. Nikliborc, W. Zylinski, J. Rudnicki, J. Biernacki, Mrs. Gruzewski, E. Leja, J. Wazewski, S. Gołab, W. Slebodzinski.

A very valuable library of the Mathematical Seminar of the University of Warsaw was completely destroyed by fire during an air raid on September 1, 1942. All private libraries of Warsaw mathematicians were destroyed during the Warsaw uprising in the fall of 1944, or immediately after.

Volume 33 of Fundamenta Mathematicae was set in type in the fall of 1939. All except the first one hundred pages, which were already printed, were destroyed, together with the manuscripts. All copies of the first ten volumes of Monografje Matematyczne were destroyed by fire.

Among the mathematicians who are teaching in the Army University Centers in Shrivenham, England, Biarritz, France, and Florence, Italy, are: V. W. Adkisson, P. L. Armstrong, Frank Ayres, J. P. Ballantine, T. A. Bancroft, I. A. Barnett, T. C. Benton, E. E. Blanche, E. T. Browne, F. J. H. Burkett, H. C. Christoferson, F. K. Danforth, D. R. Davis, W. M. Davis, P. D. Edwards, W. H. Fagerstrom, H. M. Gehman, Olaf Helmer, L. S. Hill, C. G. Jaeger, C. G. Latimer, L. L. Lowenstein, R. B. McClenon, E. R. C. Miles, J. H. Neelley, A. L. Nelson, B. P. Reinsch, P. R. Rider, N. E. Rutt, C. G. Stipe, Gabor Szegö, H. S. Thurston, J. C. Tolley, J. I. Tracy, Charles Vehse, R. M. Winger.

Professor Emeritus W. F. Durand of Stanford University has received the American Society of Mechanical Engineers Medal for his work in hydrodynamic and aerodynamic science.

Professor R. P. Stephens of the University of Georgia has retired with the title professor emeritus.

Professor H. S. Vandiver of the University of Texas has received an honorary doctorate of science from the University of Pennsylvania.

Dr. A. C. Offord of King's College, Newcastle-upon-Tyne, has been appointed to the chair of mathematics in the Newcastle division of the University of Durham.

Assistant Professor F. T. Adler of the University of Colorado has been promoted to an associate professorship of physics.

Dr. H. H. Alden has been appointed to an associate professorship at the University of Wyoming.

Associate Professor C. K. Alexander of Occidental College has been promoted to a professorship.

Dr. Florence E. Allen of the University of Wisconsin has been promoted to an assistant professorship.

Assistant Professor H. G. Apostle of the University of Rochester has been appointed to an assistant professorship at Amherst College.

Dr. K. J. Arnold has been appointed to an assistant professorship at the University of Wisconsin.

Assistant Professor A. E. Basch of Rensselaer Polytechnic Institute has been appointed to an assistant professorship at Amherst College.

Assistant Professor R. A. Beaver of the New York State College for Teachers has been promoted to a professorship.

Associate Professor H. F. Bohnenblust of Princeton University has been appointed to a professorship at Indiana University.

Assistant Professor E. C. Brown of Worcester Polytechnic Institute has been promoted to a professorship.

Assistant Professor O. E. Brown of Case School of Applied Science has been promoted to a professorship.

Associate Professor R. E. Brown of Rhode Island State College has been promoted to a professorship.

Assistant Professor J. H. Butchart of Grinnell College, Grinnell, Iowa, has been appointed to a professorship at Arizona State Teachers College.

Miss Eleanor Calkins of the College of William and Mary has been promoted to an assistant professorship.

Dr. M. A. Coler of New York University has been promoted to an assistant professorship of chemical engineering.

Dr. W. J. Combellack of Northeastern University has been promoted to an assistant professorship.

Associate Professor A. T. Craig of the University of Iowa has been promoted to a professorship.

Dr. A. W. Davis of Iowa State College has been promoted to an assistant professorship.

Mr. James Edgar Davis has been appointed to an assistant professorship at the University of Illinois.

Assistant Professor D. B. DeLury of the University of Toronto has been appointed to an associate professorship at Virginia Polytechnic Institute.

Assistant Professor F. L. Dennis of Ursinus College, Collegeville, Pennsylvania, has been promoted to an associate professorship.

Assistant Professor R. P. Dilworth of the California Institute of Technology has been promoted to an associate professorship.

Professor C. E. Dimick of the U. S. Coast Guard Academy has retired.

Mr. N. E. Dodson of Newberry College, Newberry, South Carolina, has been promoted to an associate professorship.

Dr. T. C. Doyle of Stanford University has been appointed to an assistant professorship at Dartmouth College.

Assistant Professor W. C. Doyle of Rockhurst College, Kansas City, Missouri, has been promoted to an associate professorship.

Assistant Professor L. T. Dunlap of Pennsylvania State College has been promoted to an associate professorship.

Assistant Professor Samuel Eilenberg of the University of Michigan has been promoted to an associate professorship.

Assistant Professor L. M. Garrison of Louisiana Polytechnic Institute has been promoted to an associate professorship.

Professor G. D. Gore of Central Y.M.C.A. College has been appointed to a professorship at Roosevelt College, Chicago, Illinois.

Professor Maria D. Graham of the East Carolina Teachers College, Greenville, North Carolina, has retired.

Assistant Professor J. W. Green of the University of Rochester has been appointed to an assistant professorship at the University of California at Los Angeles.

Mr. W. C. Griffith of DePauw University has been appointed to a professorship at Marion Institute, Marion, Alabama.

Professor H. H. Hartzler of Goshen College will be at Bluffton College, Bluffton, Ohio, during the present academic year.

Dr. C. C. Hurd of the U. S. Coast Guard Academy has been appointed dean of Alleghany College.

Dr. Lois Kiefer of New Mexico State Teachers College has been appointed to an acting assistant professorship at the University of Tennessee.

Associate Professor Eugene Lukacs of Berea College has been appointed to a professorship at Our Lady of Cincinnati College.

Assistant Professor J. C. C. McKinsey of Montana State College has been appointed to an assistant professorship at the University of Nevada.

Professor Edith J. McKissock of Youngstown College, Youngstown, Ohio, has been appointed assistant dean of women at the University of Cincinnati.

Professor A. E. Meder, secretary of Rutgers University, has been appointed also dean of administration.

Dr. N. S. Mendelsohn has been appointed lecturer at Queen's University, Kingston, Ontario, Canada.

Dr. Peter Scherk of the University of Saskatchewan has been promoted to an assistant professorship.

Dr. D. R. Shreve has been appointed to an associate professorship at the University of Tulsa.

Assistant Professor C. C. Torrance of Case School of Applied Science has been promoted to an associate professorship.

Dr. H. S. Wall of the Illinois Institute of Technology has been promoted to a professorship.

The following appointments to instructorships are announced: Amherst College: Mr. F. G. Graff, Mr. R. L. Wine; University of California; Dr. O. G. Owens; University of Chicago: Dr. William Karush; University of Illinois: Miss Corinne Hattan; Louisiana State University: Mr. (T.) H. Lee; Vanderbilt University: Mr. S. K. Bright.

Mr. H. W. Curjel died in England on July 17, 1945. He had been a member of the Society for forty-one years.

Dr. Willis Whited of Harrisburg, Pennsylvania, died April 28, 1945, at the age of eighty-six years. He had been a member of the Society for twenty-three years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

194. Warren Ambrose: *Remark on measures on locally compact topological groups.*

Let G be a locally compact topological group with Haar measure m. Let m' be another left invariant measure which satisfies André Weil's condition: $(x, y)\rightarrow(x, xy)$ is measure preserving. The measure m' is a *sub-measure* of the Haar measure m if and only if every m-measurable set of positive m-measure contains an m'-measurable set of positive m'-measure. It is shown that every sub-measure m' gives rise in a natural way to a locally compact topology on G which is a refinement of the given locally compact topology, and in which G is a topological group. Strong use is made of André Weil's fundamental theorem about measure giving rise to a topology, but it is necessary to modify his topology, in general, to obtain local compactness. (Received August 8, 1945.)

195. Reinhold Baer: *Null systems in projective space.*

Extending a result by O. Veblen-Young, R. Brauer has characterized the null-systems over an n-dimensional projective space P, proving in particular that n must be odd, provided that P is the n-dimensional projective space over a commutative field of coordinates. It is the object of the present note to remove this last hypothesis. More precisely it is proved that the existence of a null-system in an n-dimensional projective space P with $1<n$ is equivalent to the fact that n is odd and that P is the n-dimensional projective space over a commutative field of coordinates. (Received August 22, 1945.)

196. Reinhold Baer: *Representations of groups as quotient groups.* I.

If N is a normal subgroup of the group H, and if the groups G and H/N are isomorphic, consider H/N a representation of the group G. In this first part of a systematic theory of such representations the author is concerned with three problems: the classification of representations; the derivation of invariants of classes of representations; the construction of "interesting" classes of representations. (Received August 22, 1945.)

197. Reinhold Baer: *Representations of groups as quotient groups.* II. *Minimal central chains of a group.*

If N is a normal subgroup of the group G, define the subgroups ${}_iN$, ${}^iG$ inductively by ${}^iG=(G, {}^{i-1}G)$, ${}_iN=(G, {}_{i-1}N)$ where (X, Y) is the subgroup generated by all the commutators $x^{-1}y^{-1}xy$ for x in X and y in Y. It is shown that ${}^iG/{}_iN$ is finite, if G/N is finite. For G a free group, a survey is given of the manifold of the groups ${}^iG/W$ where ${}_{i+1}N \leqq W$ and ${}_iN/{}_{i+1}N$ is the direct product of $W/{}_{i+1}N$ and of $({}_iN \cap {}^{i+1}G)/{}_{i+1}N$. These groups ${}^iG/W$ may be considered to be generalizations of I. Schur's "Darstellungsgruppen." (Received August 22, 1945.)

198. Reinhold Baer: *Representations of groups as quotient groups. III. Invariants of classes of related representations.*

In the first part of this investigation the representations of a group as quotient groups have been divided into classes of related representations; and it has been shown that these classes of related representations possess a vast system of structural invariants. The present note is devoted to the investigation of the interrelatedness of these invariants and to applications of these invariants, in particular for deriving criteria for a homomorphism to be an isomorphism. (Received August 22, 1945.)

199. Reinhold Baer: *The homomorphism theorems for loops.*

This note concerns itself with the isomorphism laws, including Zassenhaus' formula, which play a fundamental role in the proof of the Jordan-Hölder-Schreier-Zassenhaus Theorem. Proofs of these theorems, of (new) converses and extensions of them are given which apply not only to classical group theory, but also to loop theory. Preference is given to working with homomorphisms, instead of doing computations with elements. (Received August 22, 1945.)

200. Richard Brauer: *A note on systems of homogeneous algebraic equations.*

Assume that the field K has the following property (A): For every positive integer r there exists an integer $\psi(r)$ such that for $n \geqq \psi(r)$ every equation $a_1x_1^r + \cdots + a_nx_n^r = 0$ with coefficients a_i in K has a nontrivial solution $(x_1, \cdots, x_n)$ in K. It is shown that for every system of positive integers $r_1, r_2, \cdots, r_h$ and for every integer $m \geqq 0$ there exists an expression $\Omega(r_1, r_2, \cdots, r_h; m)$ such that for $n \geqq \Omega(r_1, \cdots, r_h; m)$ every system of h homogeneous algebraic equations of degrees $r_1, r_2, \cdots, r_h$ respectively with coefficients in K is satisfied by all points of an m-dimensional linear manifold M, defined in K. If K does not have the property (A), a similar statement holds, if one allows that M is not defined in K but in a soluble extension field of K. The p-adic fields have the property (A). As a further application, it is shown that if l_n denotes the smallest number such that the general algebraic equation $F(x) = 0$ of degree n can be solved by means of algebraic functions of at most l_n arguments (Hilbert's resolvent problem), then $\lim_{n\to\infty} (n - l_n) = \infty$. (Received August 11, 1945.)

201. Richard Brauer: *On the representation of a group of order g in the field of the gth roots of unity.*

It has long been surmised that every representation L of a group G of order g can be written in the field Ω of the gth roots of unity, that is, that L is similar to a representation with coefficients in Ω. In the present paper, this conjecture is proved for the first time. The proof is obtained by combining the methods of I. Schur and of H. Hasse with results concerning the modular representations of groups. (Received August 21, 1945.)

202. I. S. Cohen and Irving Kaplansky: *Rings with a finite number of prime elements.*

Let R be an integral domain in which every element is a product of prime elements, of which it is assumed there are but a finite number n. Every ideal in R has a finite basis (this is proved more generally for any integral domain having a finite number of prime ideals, each of which has a finite basis), and its prime ideals $M_1, \cdots, M_h$ are all maximal. Each prime element of R is contained in just one M_i, and if a_i and b_i are products of prime elements in M_i and $\prod_{i=1}^{h} a_i = \prod_{i=1}^{h} b_i$, then $(a_i) = (b_i)$, $i = 1, \cdots, h$. The factorization problem can then be reduced to the case of a local ring. If R is a local ring and factorization is not unique, then $n \geqq 3$ and the residue class field contains at most $n-1$ elements; the completion of R is free from zero-divisors and has exactly the same primes as R. The multiplicative structure of R is explicitly determined for $n=3, 4, 5$. (Received August 10, 1945.)

203. I. S. Cohen and Abraham Seidenberg: *Prime ideals and integral dependence.*

Let $\mathfrak{R}$ and $\mathfrak{S}$ be commutative rings, $\mathfrak{R} \subseteq \mathfrak{S}$, with the same identity element, and let $\mathfrak{S}$ be integrally dependent on $\mathfrak{R}$. The following theorems are proved: (1) If $\mathfrak{p}$ is a prime ideal in $\mathfrak{R}$, then there exists a prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ such that $\mathfrak{P} \cap \mathfrak{R} = \mathfrak{p}$. (2) If $\mathfrak{q}$ and $\mathfrak{p}$ are prime ideals in $\mathfrak{R}$, $\mathfrak{q} \subset \mathfrak{p}$, and if $\mathfrak{Q}$ is a prime ideal in $\mathfrak{S}$ such that $\mathfrak{Q} \cap \mathfrak{R} = \mathfrak{q}$, then there exists a prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ containing $\mathfrak{Q}$ such that $\mathfrak{P} \cap \mathfrak{R} = \mathfrak{p}$. (3) Assume now that no element of $\mathfrak{R}$ is a zero-divisor in $\mathfrak{S}$ and that $\mathfrak{R}$ is integrally closed. If $\mathfrak{q}$ and $\mathfrak{p}$ are as in (2) and if $\mathfrak{P}$ is a prime ideal in $\mathfrak{S}$ such that $\mathfrak{P} \cap \mathfrak{R} = \mathfrak{p}$, then there exists a prime ideal $\mathfrak{Q}$ in $\mathfrak{S}$ contained in $\mathfrak{P}$ such that $\mathfrak{Q} \cap \mathfrak{R} = \mathfrak{q}$. The present proofs are simpler than those of Krull (Math. Zeit. vol. 42 (1937) pp. 745–766), which in addition assume that $\mathfrak{R}$ and $\mathfrak{S}$ are integral domains. (Received August 10, 1945.)

204. Jakob Levitzki: *Chains of multipa and semi-primarity.*

A descending chain of right ideals $R_1 \supset R_2 \supset R_3 \supset \cdots$ of an arbitrary ring S is called a chain of multipla, if for each n there exists a right ideal R_n' for which $R_{n+1} = R_n' R_n$. As in the classical theory of ideals, each descending chain of right ideals in a semi-simple ring is a chain of multipla. A slight modification of the usual argument leads to the following statement: A ring S is semi-simple if and only if (1) S is without nilpotent ideals and (2) each chain of multipla in S is finite. The corresponding theorem of E. Noether assumes instead of (2) the finiteness of each descending chain of right ideals. It is further proved that from condition (2) alone follows that S is semi-primary (that is, the radical N of S is nilpotent, and S/N is semi-simple). A necessary and sufficient condition for semi-primarity is obtained by assuming condition (2) for the right ideals which either contain or are contained in the radical of S. For the truth of this last theorem it is irrelevant which of the various existing definitions of the radical is chosen. (Received September 1, 1945.)

205. Ivan Niven: *A remark on Gaussian integers.*

The author has shown (Trans. Amer. Math. Soc. vol. 48 (1940) p. 410) that every Gaussian integer of the form $a+2bi$ is expressible as a sum of three squares, and has given necessary and sufficient conditions for a sum of two squares. The question is now raised, how many representations are possible? It is shown that any $a+2bi$ has an infinite number of three square, but a finite number (with formula given) of two square, representations. (Received August 20, 1945.)

206. W. V. Parker: *The characteristic roots of matrices.*

If λ is a characteristic root of a matrix A, and if $\rho_1^2 \geqq \rho_2^2 \geqq \cdots \geqq \rho_n^2$ $(\rho_i \geqq 0)$ are the characteristic roots of $A\bar{A}'$, Browne has shown that $\rho_1 \geqq |\lambda| \geqq \rho_n$. In this paper it is shown that for any complex number μ such that $\rho_1 \geqq |\mu| \geqq \rho_n$ there exists a matrix B which has the characteristic root μ and is such that $B\bar{B}' = A\bar{A}'$. Consideration is also given to the distribution of the roots of B within the annular region. (Received August 10, 1945.)

207. Edward Rosenthall: *On the sums of two squares and the sum of cubes.*

In this paper the complete rational integer solution of each of the diophantine systems $x_1^2+y_1^2=x_2^2+y_2^2= \cdots =x_n^2+y_n^2$ and $(x_1^2+y_1^2)(x_2^2+y_2^2) \cdots (x_n^2+y_n^2)=(u_1^2+v_1^2) \cdots (u_m^2+v_m^2)$ is obtained in terms of integral parameters. Also, a method is given for finding all the sets of rational integers satisfying the diophantine equation $x_1^3+x_2^3 + \cdots +x_{2n}^3=0$. This method reduces the resolution of this equation to a system of linear homogeneous equations in which the number of unknowns always exceeds the number of equations. The above results are deduced from the complete integer solution in the quadratic fields $Ra(i)$, $Ra((-3)^{1/2})$ of certain connected simple and extended multiplicative equations. (Received September 4, 1945.)

ANALYSIS

208. R. P. Agnew: *A simple sufficient condition that a method of summability be stronger than convergence.*

Let $\sigma_n=\sum_{k=1}^{\infty} a_{nk}s_k$ be a regular Silverman-Toeplitz transformation by which a sequence $s_1, s_2, \cdots$ is summable to σ if $\sigma_n \to \sigma$ as $n \to \infty$. It is shown that if $a_{nk} \to 0$ as $n, k \to \infty$, then some divergent sequences of zeros and ones are summable. A more general theorem applies to transformations which are not necessarily regular. If (i) $\sum |a_{nk}| < \infty$ for each fixed $n=1, 2, \cdots$ and if (ii) as $n \to \infty$, the maximum for $k=1, 2, \cdots$ of $|a_{nk}|$ converges to 0, then some divergent series of zeros and ones are summable. The theorems furnish criteria for determination of relations among methods of summability. (Received August 20, 1945.)

209. R. P. Agnew: *Characterization of methods of summability effective for power series inside circles of convergence.*

A matrix b_{nk} of real or complex constants determines a series-to-sequence transformation $\sigma_n=\sum_{k=0}^{\infty} b_{nk}u_k$ by means of which a series $\sum u_n$ is summable B to σ if $\sigma_0, \sigma_1, \cdots$ exist and $\lim \sigma_n=\sigma$. In order that the matrix b_{nk} be such that $\sum u_n$ is summable B whenever $\sum u_n z^n$ has radius of convergence greater than 1, it is necessary and sufficient that (1) constants $\beta_0, \beta_1, \cdots$ exist such that $\lim_{n\to\infty} b_{nk}=\beta_k$ when $k=0, 1, 2, \cdots$ and (2) to each number θ in the interval $0<\theta<1$ corresponds a constand $M(\theta)$ such that $|b_{nk}\theta^k|<M(\theta)$ when $n, k=0, 1, 2, \cdots$. If (1) and (2) hold and $\sum u_n z^n$ has radius of convergence greater than 1, then $\sum \beta_n u_n$ converges absolutely and the number $B\ \{\sum u_n\}$ to which $\sum u_n$ is summable B is $\sum \beta_n u_n$. In order that $B\{\sum u_n\}=\sum u_n$ whenever $\sum u_n$ has radius of convergence greater than 1, it is necessary and sufficient that (1) and (2) hold with $\beta_k=1$ for each k. (Received August 3, 1945.)

210. Joshua Barlaz: *On some triangular summability methods.*

A class of triangular sequence-to-sequence summability methods is given by the transform $t_n=e^{-x_n}\sum_{\nu=0}^{\infty} s_\nu x_n^\nu/\nu!$, $n=0, 1, 2, \cdots$, where $\{x_n\}$ is a sequence of numbers tending to infinity. The transform is obtained by applying to Borel's exponential limit a process which Professor Szász applied to Abel summability in a recent paper (Ann. of Math. vol. 43). A necessary and sufficient condition for the regularity of the transform is that $(x_n-n)n^{-1/2}\rightarrow-\infty$ as $n\rightarrow\infty$. Imposing other conditions on the sequence $\{x_n\}$, the author makes an investigation of some of the properties of the class of summability methods. For instance if $x_n=o(n)$, it is shown that the transform is as powerful as Borel's method when applied to power series with a positive radius of convergence. It is also shown that the method sums the sequence $s_\nu=(-1)^\nu$ for all sequences $\{x_n\}$, $x_n\rightarrow\infty$. Finally it is shown that for no sequence $\{x_n\}$ does the transform include the method of arithmetic means. (Received August 10, 1945.)

211. R. P. Boas: *Some complete sets of functions.*

The following generalization of a theorem of W. H. J. Fuchs (Proc. Cambridge Philos. Soc. vol. 40 (1944) pp. 189–197) is proved. Let $\{\lambda_n\}$ be a sequence of positive numbers, and let $n(r)$ denote the number of $\lambda_n\leqq r$. Then the set $\{e^{-t}t^{\lambda_n}\}$ is complete in $L^p(0, \infty)$, $1\leqq p<\infty$, if $n(r)-r/2\geqq -r\delta(r)$, where, as $r\rightarrow\infty$, $\delta(r)\downarrow 0$, $r^{-1}\int_0^r\delta(u)du=0(\delta(r))$, and $\int^\infty r^{-1}\delta(r)dr<\infty$. Fuchs also proved that, if the set of all positive integers is divided into two complementary sets $\{\lambda_n\}$ and $\{\mu_n\}$, then at least one of $\{e^{-t}t^{\lambda_n}\}$ and $\{e^{-t}t^{\mu_n}\}$ is complete. This result is extended to subsets of a set $\{a_n\}$, whose function $n(r)$ satisfies $|n(r)-r|\leqq r\delta(r)$, with the same conditions on $\delta(r)$. Applications are given to singularities of power series with gaps. (Received August 9, 1945.)

212. Paul Civin: *Fourier coefficients of dominant functions.*

Let $f(x)$ be a complex integrable function of period 2π and let $F(x)$ be an integrable function of period 2π which dominates $f(x)$, that is, $F(x)\geqq|f(x)|$. If $h(x)\sim\sum_{-\infty}^{\infty}d_me^{mix}$, denote $(\sum_{-\infty}^{\infty}|d_m|^r)^{1/r}$ by $N_r(h)$ and $(\int_{-\pi}^{\pi}|h(x)|^rdx)^{1/r}$ by $A_r(h)$. The following theorem is proved. For $q=2, 4, 6, \cdots$, $N_q(f)\leqq N_q(F)$, while for $p=q/(q-1)$ there is some dominant function $F_p(x)$ of each $f(x)$ such that $N_p(F_p)\leqq N_p(f)$. This is the dual of a problem of Hardy and Littlewood (Quart. J. Math. Oxford Ser. vol. 6 (1935) pp. 304–315) who consider functions $F(x)$ whose Fourier series are majorants of the Fourier series of $f(x)$, and consider relations of inequality between $A_q(f)$ and $A_q(F)$. (Received September 18, 1945.)

213. J. B. Díaz: *On a class of partial differential equations of even order.*

Partial differential equations of the form $L^Nu=0$, where $u=u(x, y)$ and L is a certain linear partial differential operator of second order with variable coefficients (the most important special case being $Lu=\Delta u+\phi(x)u_x+\psi(y)u_y$), are studied by the method of hypercomplex variables, which involves replacing $L^Nu=0$ by a system of $2N$ partial differential equations of first order denoted by $E(\Sigma, N)$, where Σ is the matrix of the coefficients of L. $E(\Sigma, N)$ plays the role of the Cauchy-Riemann equations in the study of Laplace's equation. A function theory of $E(\Sigma, N)$ is constructed by introducing hypercomplex-valued functions, $f(x, y)=\sum_{i=0}^{2N-1}a_i(x, y)j_N^i$ (where the hypercomplex unit j_N satisfies $(1+j_N^2)^N=0$), such that the real functions a_i, $i=0, 1, \cdots, 2N-1$, satisfy $E(\Sigma, N)$. The processes of (Σ, N)-integration and (Σ, N)-differentiation are defined, and several analogues of known results for analytic functions are proved. (The system $E(\Sigma, 1)$ has been discussed by Bers and Gelbart, and L. Sobrero has studied $\Delta^2u=0$ employing hypercomplex variables.) There is given a sequence of particular solutions of $L^Nu=0$, obtained by quadratures from the coefficients of L, and such that every solution u of $L^N u = 0$ may be expanded in a uniformly and absolutely convergent series of these particular solutions. (Received August 10, 1945.)

214. Nelson Dunford: *Boolean algebras of projections.*

The problem of embedding a bounded Boolean algebra of projections on a Banach space X in a complete Boolean algebra of projections in X is investigated. In particular, it is shown that if X is reflexive, this can always be accomplished. (Received August 11, 1945.)

215. Bernard Friedman: *Two theorems on simple functions.*

Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be a function regular and simple ("schlicht") in the unit circle. It is well known that $|a_2| < 2$, $|a_3| < 3$. With the help of an inequality due to Prawitz (Arkiv för Matematik, Astronomi och Fysik vol. 20 (1927–1928)) it is proved that $|a_4| < 4.16$. It is also proved that if all the a_n are integers, then $f(z)$ must be one of the following: z, $z/(1 \pm z^2)$, $z/(1 \pm z + z^2)$, $z/(1 \pm z)$, $z/(1 \pm z)^2$. (Received August 10, 1945.)

216. Leonard Greenstone: *Some properties of conformal mappings of multiply connected domains.* Preliminary report.

The author investigates the Gaussian curvature of a non-Euclidean metric which is invariant with respect to conformal mappings of multiply connected domains. Employing Bergman's method of the minimum integral (*Partial differential equations, advanced topics*, Publication of Brown University), one can represent this curvature in terms of certain minima. Normal family methods, in conjunction with certain properties of these minima, are then employed in order to prove that if $\{B_n\}$ is a sequence of increasing non-null domains, all B_n lying in the unit circle, which sequence converges in the sense of Carathéodory to a domain B, then the curvature (of the metric) of B_n converges uniformly to the curvature (of the metric) of B. Let C be a circle which together with its boundary Γ is contained in a multiply connected domain B. The author obtains inequalities in C for the value of a function closely associated with the curvature (of the metric) of B, in terms of the values this function assumes on Γ. (Received August 11, 1945.)

217. Einar Hille: *A note on semi-groups analytic in a sector.*

Let $\mathfrak{S} = \{T(\zeta)\}$ be a semi-group of linear bounded transformations on a complex B-space $\mathfrak{X}$ to itself. Let $T(\zeta)$ be holomorphic in $-\pi/2 \leqq \Phi_1 < \arg \zeta < \Phi_2 \leqq \pi/2$ and $\|T(re^{i\phi})\| \leqq B(\phi)$, $0 < r < 1$, where $B(\phi)$ is bounded for $\Phi_1 + \epsilon \leqq \phi \leqq \Phi_2 - \epsilon$. Then $\sigma(\phi) = \lim_{r \to \infty} r^{-1} \log \|T(re^{i\phi})\|$ exists; either $\sigma(\phi) \equiv -\infty$ or $\sigma(\phi)$ is continuous and is the function of support of a closed convex unbounded region D. If $R(\lambda; A)$ is the resolvent of the infinitesimal generator of $\mathfrak{S}$, then $R(\lambda; A)$ is holomorphic outside of $\overline{D}$, the conjugate of D in the complex λ-plane, and all extremal points of $\overline{D}$ are spectral points of A. Conversely, if A is an unbounded linear operator on $\mathfrak{X}$ to itself with domain dense in $\mathfrak{X}$, if $R(\lambda; A)$ is holomorphic outside of $\overline{D}$, all the extremal points of which are spectral points of A, if $\delta(\lambda)$ is the distance from λ to D, and if $\delta(\lambda)\|R(\lambda; A)\|$ is bounded in a certain qualified sense outside of $\overline{D}$, then A is the infinitesimal generator of a semi-group $\mathfrak{S} = \{T(\zeta)\}$, holomorphic in a sector and bounded in interior sectors near the origin, and the growth function $\sigma(\phi)$ of $T(\zeta)$ is the function of support of D. (Received September 26, 1945.)

218. Einar Hille: *On uniform ergodic theorems.*

Let $\mathfrak{S} = \{T(\xi)\}$ be a semi-group of linear bounded transformations on a complex B-space $\mathfrak{X}$ to itself. Let $T(\xi)$ be uniformly measurable for $\xi > 0$, $\int_0^\infty e^{-\lambda\xi}\|T(\xi)\|d\xi$ exist for $\lambda > 0$, and $(1/\eta)\int_0^\eta T(\xi)x d\xi \to x$ for all x when $\eta \to 0$. Then $T(\xi)$ is uniformly continuous for $\xi > 0$ and $R(\lambda; A) = \int_0^\infty e^{-\lambda\xi}T(\xi)d\xi$ is holomorphic for $\Re(\lambda) > 0$. The following assertions are equivalent: (1) $T(\xi)$ is uniformly Abel ergodic for $\xi \to \infty$, that is, $\lambda R(\lambda; A) \to P$ when $\lambda \to +0$. (2) $\lambda = 0$ is a simple pole of $R(\lambda; A)$ with residue P. (3) $\lim_{\lambda \to +0} \lambda^2 R(\lambda; A)x = 0$ for all x and $\mathfrak{X} = \Re[A] \oplus \mathfrak{Z}[A]$, $\Re[A]$ being closed. Further $\Re[A^k] = \Re[A]$ for all positive integers k. Here A is the infinitesimal generator of $\mathfrak{S}$, $\Re[A]$ the range of A and $\mathfrak{Z}[A]$ the manifold annihilated by A. If for some $\alpha \geqq 0$, $T_\alpha(\xi)$ is bounded, $0 < \xi < \infty$, and (1) holds, then $T(\xi)$ is also uniformly (C, β) ergodic for $\beta > \alpha$, that is, $\lim_{\xi \to \infty} T_\beta(\xi) = P$. Here $T_0(\xi) = T(\xi)$, $T_\alpha(\xi) = \alpha\xi^{-\alpha}\int_0^\xi(\xi - \tau)^{\alpha-1}T(\xi)d\xi$, $\alpha > 0$. If $T_\alpha(\xi)$ is feebly oscillating when $\xi \to \infty$, the (C, α) limit also exists. The proof uses the fact that solutions of the resolvent equation, having an n-tuple pole, are characterized by n simple limit conditions. (Received September 26, 1945.)

219. W. H. Ingram: *A boundary value problem and its Stieltjes integral equations.*

The problem $y' = (A(x) + \mu B(x))y$, $y(x_i+) - y(x_i-) = (A^*(x_i) + \mu B^*(x_i))y(x_i)$, $a \leqq x \leqq b$, $a \leqq x_i \leqq b$, $i = 1; 2, \cdots$, $2y(x_i) = y(x_i+) + y(x_i-)$. $Ly(a) + Ry(b) = 0$, contains as a special case the problem treated by Birkhoff and Langer, Bliss, and Reid. The matrices $A(x)$, $B(x)$ are of bounded variation on $[ab]$ but need not be defined over the set $x_1, x_2, \cdots$. The matrices $A^*(x_i)$, $B^*(x_i)$ are zero everywhere except on the set $x_1, x_2, \cdots$ and $\sum|A^*(x_i)| < \infty$, $\sum|B^*(x_i)| < \infty$. With accommodations as to form and the substitution of the Riemann-Stieltjes for the Riemann integral, the theorems on the number, distribution and reality of the critical values of the parameter μ, for a self-adjoint system as defined after Bliss, the relation between the nullity of the characteristic matrix and the multiplicity of μ, and the expansion theorem are generally preserved under extensions of the various definitions of definiteness of Bliss and Reid. (Received September 17, 1945.)

220. Hans Rademacher: *On a function related to Riemann's zeta-function.*

The function in question is $\eta(w) = \sum \exp(\rho w)$, where the sum is extended over the complex roots of $\zeta(s)$ of positive imaginary part. The sum is convergent in the upper w-halfplane and defines there an analytic function. The function $\eta(w)$ can be continued analytically and turns out to be meromorphic on the logarithmic Riemann surface. All its poles are of the first order. Their locations and residues can be fully determined. (Received August 10, 1945.)

221. Tibor Radó: *On continuous mappings of Peano spaces.*

The purpose of this paper is to discuss certain topological questions that arise in the theory of length and area. The notion of a cyclic decomposition of a mapping on a general Peano space, corresponding to the cyclic element decomposition of the middle space in a monotone-light factorization of the mapping, is first introduced and studied. Particular results are then developed concerning mappings of 2-spheres and 2-cells, the cases of especial significance in the theory of area. Finally, a general theorem on cyclic additivity of a general function $F(T)$ of a mapping T, which includes both Lebesgue area $A(T)$ and lower area $\alpha(T)$ in the sense of Geöcze, is formulated and proved. (Received August 8, 1945.)

222. W. T. Reid: *Definitely self-adjoint differential systems.*

This paper contains an analysis of the conditions that have been used by Bliss (Trans. Amer. Math. Soc. vol. 44 (1938) pp. 413–428) in defining a definitely self-adjoint boundary value problem consisting of a system of first-order linear ordinary differential equations and two-point boundary conditions. In particular, it is shown that if a differential system satisfies the conditions of definite self-adjointness with the exception of the normality condition, then there exists an "equivalent" system which is definitely self-adjoint. (Received August 10, 1945.)

223. W. T. Reid: *Integral criteria for solutions of linear differential equations.*

The following theorem is one of a set of related results established for linear ordinary differential equations. Suppose that $a_i(x)$ $(i=0, 1, \cdots, n)$ is of class $C^{(i)}$, and $a_n(x)\neq 0$ on (a, b), while $g(x)$ is integrable on this interval. If $f(x)$ is integrable on each closed sub-interval (α, β) interior to (a, b), then there exists a solution $u(x)$ of the differential equation $\sum_{i=0}^{n} a_i(x)u^{(i)}+g(x)=0$ such that $f(x)=u(x)$ a.e. on (a, b) if and only if for each such subinterval (α, β), $\int_\alpha^\beta \left|\sum_{i=1}^{n} a_i(x)h^{n-i}\Delta_h^i f(x)+h^n[a_0(x)f(x)+g(x)]\right| dx=o(h^n)$, where $\Delta_h^i f(x)$ denotes the ith forward difference of $f(x)$ with interval h. There is also given a corresponding integral criterion for harmonic functions. (Received August 10, 1945.)

224. C. E. Rickart: *Banach algebras with an adjoint operation.*

A "B^*-algebra" $\mathcal{A}$ is a normed ring (Gelfand, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941)), not necessarily commutative, in which to each element $x\in\mathcal{A}$ corresponds a unique element $x^*\in\mathcal{A}$ with the properties: (1) $x^{**}=x$, (2) $(xy)^*=y^*x^*$, (3) $(\lambda x+\mu y)^*=\bar{\lambda}x^*+\bar{\mu}y^*$, (4) $\|xx^*\|=\|x\|^2$ (Gelfand and Neumark, ibid. vol. 12 (1943)). An element $u\in\mathcal{A}$ is called a "projection" if $u^2=u$ and $u^*=u$. A B^*-algebra is called a "B_p^*-algebra" if for every $x\in\mathcal{A}$ a projection $u\in\mathcal{A}$ exists such that $xu=x$ and $xy=0$ implies $yu=0$. Special examples of B_p^*-algebras are the factors of Murray and von Neumann (Ann. of Math. vol. 37 (1936)). The structure of a B_p^*-algebra $\mathcal{A}$ is studied here in terms of the projections in the algebra. $\mathcal{A}$ is generated by its projections. Notions of "equivalence" and "finiteness" of projections are used closely related to similar notions of Murray and von Neumann for factors. A B_p^*-algebra is simple if, and only if, all of its projections are finite. $\mathcal{A}$ is said to satisfy the "denumerability condition" if every family of orthogonal projections is denumerable. $\mathcal{A}$ is said to be "quasi-transitive" when $xAy=(0)$ implies either $x=0$ or $y=0$. If $\mathcal{A}$ is quasi-transitive, regular in the sense of von Neumann and satisfies the denumerability condition, then it is necessarily simple. Every quasi-transitive B_p^*-algebra is central. If the denumerability condition is satisfied, a central B_p^*-algebra is quasi-transitive. (Received August 11, 1945.)

225. H. E. Robbins: *Cesàro convergence of certain random sequences.*

Let $\{a_n\}$ be a sequence of numbers with $\sum a_n n^{-2}<\infty$ and which converges $(C, 1)$ to a limit α. It is shown that almost every sequence derived from $\{a_n\}$ by a random process of a certain general type, involving the dropping or repeating of terms of

$\{a_n\}$, will also converge $(C, 1)$ to α. This generalizes a theorem of Buck and Pollard (Bull. Amer. Math. Soc. vol. 49 (1943) p. 928, Theorem 2). It follows that with probability 1 certain random sequences of transformations exhibit ergodic behavior. (Received August 10, 1945.)

226. M. S. Robertson: *The coefficients of univalent functions.*

Let $f(z)=\sum_1^\infty c_n z^n$, $c_1=1$, be regular and convex in the direction of the imaginary axis for $|z|<1$ and be real on the real axis. Then the coefficients c_n satisfy the inequalities $c_{n-1}-c_{n+1}\leqq 4n(n^2-1)^{-1}\cdot(1-|c_n|)$, n even, $|c_{n-1}-c_{n+1}|\leqq 4n(n^2-1)^{-1}\cdot(1-c_n)$, n odd and greater than 1. The factor $4n\cdot(n^2-1)^{-1}$ cannot be replaced by a smaller one and the equality signs are attained by the function $z(1-z)^{-1}$. Let the function $F(z)=\sum_1^\infty a_n z^n$, $a_1=1$, be regular and univalent for $|z|<1$ and real on the real axis. Then $(n+1)a_{n-1}-(n-1)a_{n+1}\leqq 4(n-|a_n|)$, n even, $|(n+1)a_{n-1}-(n-1)a_{n+1}|\leqq 4(n-a_n)$, n odd and greater than 1, $4(1+a_2+a_3+\cdots+a_{n-1})-(n-2)a_n\leqq n^2$, $n>1$. The proofs depend upon the trigonometric inequalities $(n+1)\sin(n-1)\theta/\sin\theta-(n-1)\sin(n+1)\theta/\sin\theta\leqq 4(n-\sin n\theta/\sin\theta)$, $n^2+n\sin 2n\theta/\sin 2\theta-2(\sin n\theta/\sin\theta)^2\geqq 0$. (Received August 9, 1945.)

227. M. S. Robertson: *Univalent power series with multiply monotonic sequences of coefficients.*

The Cesàro partial sums of the first order $S_n^{(1)}(z)=\sum_{k=0}^{n}(n-k+1)z^k$ of the geometric series are univalent for $|z|\leqq 1$ (E. Egerváry, Math. Zeit. vol. 42 (1937) pp. 221–230). The author obtains a new proof which permits him to generalize this result to linear sums $\sum_{k=1}^{m}c_k S_{n+k-1}^{(1)}(z)$ with non-negative coefficients not all zero. These sums are univalent for $|z|\leqq 1$ provided that for all α, $\sum_{k=2}^{m-1}B_k^m\{(\sin k\alpha/\sin\alpha)^2-k\sin 2k\alpha/\sin 2\alpha\}\geqq 0$ where $B_k^m=\sum_{s=1}^{m-k}c_s c_{s+k}$, and in particular if the sequence $\{B_2^m, B_3^m, \cdots, B_{m-1}^m, 0, 0, \cdots\}$ is monotonic of order 2. If the sequence $\{a_n\}$ is monotonic of order 2, $a_n\rightarrow 0$, and if for each $n>n_0$ the sequence $\{A_2^n, A_3^n, \cdots, A_{n-1}^n, 0, 0, \cdots\}$ is monotonic of order 2 where $A_k^n=\Delta_n^{(2)}a_{n-k}\cdot\Delta^{(1)}a_n+\sum_{s=1}^{n-k-1}\Delta^{(2)}a_s\cdot\Delta^{(2)}a_{s+k}$ then $f(z)=\sum_0^\infty a_n z^n$ is regular and univalent for $|z|<1$. This result is an extension of a theorem of L. Fejér (Trans. Amer. Math. Soc. vol. 39 (1936) pp. 18–59) and the improvement on his result by G. Szegö (Duke Math. J. vol. 8 (1941) pp. 559–564). The method of proof yields a positive lower bound for $|f(z_1)-f(z_2)|$, $z_1\neq z_2$. (Received August 7, 1945.)

228. P. C. Rosenbloom: *Convex bodies in Banach spaces.*

The notions of Minkowski functional, supporting functional, and polar body of a convex body can be generalized from finite-dimensional spaces. If E is a separable Banach space, then a necessary and sufficient condition that a convex body K^* in its conjugate space E^* be regularly convex—which is equivalent to its being the polar body of a convex body K in E—is that its Minkowski functional $K^*(f)$ be lower semicontinuous in the weak topology of E^*. We say that $K^*(f)$ is uniformly differentiable at f_0 if $\lim_{h\rightarrow 0}\{K^*(f_0+hf)-K^*(f_0)\}/h$ exists uniformly in $\|f\|\leqq 1$. If K is a convex body in E, and $f_0(x)=1$ is a supporting hyperplane of K, and if $K^*(f)$ is uniformly differentiable at f_0, where K^* is the polar body of K, then $f_0(x)=1$ has a unique point in common with K and this point is an extreme point of K. We say, in this case, that f_0 is tangent to K. If E^* is separable, then the set of points on the boundary of K^* which are tangent to K is dense on the boundary of K. This work is related to that of Mazur, Yosida, Fukamiya, Krein, Šmulian, Gantmacher, Milman, and Mackey. (Received August 17, 1945.)

229. C. A. Truesdell: *On the functional equation* $\partial F(z, \alpha)/\partial z = F(z, \alpha+1)$.

It is attempted to provide a theory which motivates and verifies many seemingly special relations among various familiar special functions. The recurrence relation $\partial F(z, \alpha)/\partial z = A(z, \alpha)F(z, \alpha)+B(z, \alpha)F(z, \alpha+1)$ satisfied by many familiar functions furnishes common ground for study. In all cases of interest this equation is reducible to the form $\partial F(z, \alpha)/\partial z = F(z, \alpha+1)$. If $\phi(\alpha)$ is bounded in a right half-plane a unique solution $F(z, \alpha)$, an integral function of z, exists such that $F(z_0, \alpha) = \phi(\alpha)$. Two solutions which agree in a right half-plane of α when $z = z_0$ agree for all values of z whether or not they are bounded functions of α. On the basis of these theorems it is possible to establish many relationships satisfied by certain classes of solutions of the F-equation: (1) power series solutions, (2) factorial and Newton series solutions, (3) contour integral solutions, (4) generating expansions, (5) definite integrals, (6) relations among various different solutions of the F-equation. Methods of discovery are stressed because the discovery of a relationship satisfied by some special function or functions is almost always more difficult and more interesting than the construction of an ad hoc rigorous proof. A number of these special relations are shown to be obtainable by substitution in general formulas. (Received August 7, 1945.)

APPLIED MATHEMATICS

230. R. J. Duffin: *Nonlinear networks.* I.

A system of n nonlinear algebraic equations in n real variables is studied and shown to have a unique solution. A special case of this system is the equations which govern the distribution of current in a direct current electrical network when the conductors are *quasi-linear.* A quasi-linear conductor is one in which the potential drop across the conductor and the current through the conductor are nondecreasing functions of one another. It follows that the distribution of current among the conductors of a quasi-linear network is unique. (Received September 6, 1945.)

231. F. J. Murray: *Linear equation solvers.*

The theory and actual construction of certain devices for the solution of a system of linear equations $\sum_{j=1}^{n} a_{i,j}x_j = b_i$ is described. In these machines, the variables x_j are subject to the control of the operator and the machine indicates the value of $\mu = \sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ij}x_j - b_i)^2$. The operator varies each x_j in turn to minimize this expression. This is equivalent to the Gauss-Seidel method applied to the symmetric system obtained by multiplying the given set of equations by the adjoint matrix. The expressions $\sum_{j=1}^{n} a_{i,j}x_j - b_i$ are realized (except possibly in sign) as the amplitudes of alternating current voltages by means of a combination of bell transformers and variable resistances. This can be done in a number of ways. These alternating currents are then rectified by means of diode vacuum tubes and the combined currents measured by a microammeter. The result is essentially μ. Emphasis is placed upon the possibility of amateur construction and standard radio parts are used. The cost of the part of the device associated with the coefficients is proportional to n^2. The sensitivity of modern vacuum tubes is utilized to minimize the constant of proportionality. Received August 3, 1945.)

232. H. E. Salzer: *Note on coefficients for numerical integration with differences.*

In a recent paper (A. N. Lowan and H. E. Salzer, *Table of coefficients for numerical integration without differences*, Journal of Mathematics and Physics vol. 24 (1945) pp. 1–21) quantities $B_i^{(n)}(p)$ are tabulated to 10 decimal places, for continuous numerical integration (that is, integration to various points within an interval of tabulation) by a Lagrangian formula which uses the tabular entries only. The present note indicates how those same quantities $B_i^{(n)}(p)$ can be employed as they stand, for continuous numerical integration using differences, in formulas obtained by integrating the interpolation formulas of Gregory-Newton, Newton-Gauss (two forms), Everett, and Steffensen. (Received August 6, 1945.)

GEOMETRY

233. H. S. M. Coxeter: *Quaternions and reflections.*

Every quaternion $x=x_0+x_1i+x_2j+x_3k$ determines a point $P_x=(x_0, x_1, x_2, x_3)$ in Euclidean 4-space, and every quaternion a of unit norm determines a hyperplane $a_0x_0+a_1x_1+a_2x_2+a_3x_3=0$. The reflection in that hyperplane is found to be the transformation $x\to -a\bar{x}a$. This leads easily to the classical expression $x\to axb$ for the general displacement preserving the origin. If p and q are pure quaternions of unit norm, the transformation $x\to(\cos\alpha+p\sin\alpha)x(\cos\beta+q\sin\beta)$ represents the double rotation through angles $\alpha\pm\beta$ about the two completely orthogonal planes $P_0P_{p\mp q}P_{1\pm pq}$. (Received October 1, 1945.)

234. H. S. M. Coxeter: *The order of the symmetry group of the general regular hyper-solid.*

Schlafli defined $\{p, q, r\}$ as the regular four-dimensional polytope bounded by $\{p, q\}$'s, r at each edge; for example, $\{4, 3, 3\}$ is the hyper-cube. The order, g, of the symmetry group is found to be given by $16h/g=6/j_{p,q}+6/j_{q,r}+1/p+1/r-2$, where $\cos^2\pi/h$ is the greater root of the equation $x^2-(\cos^2\pi/p+\cos^2\pi/q+\cos^2\pi/r)x+\cos^2\pi/p\cos^2\pi/r=0$, and $j_{p,q}=[(2p+2q+7pq)/(2p+2q-pq)]^{1/2}+1$; for example, for the hyper-cube $\{4, 3, 3\}$, $128/g=6/8+6/6+1/4+1/3-2=1/3$. (Received October 1, 1945.)

235. H. S. M. Coxeter: *The Petrie polygon of a regular solid.*

Schlafli defined $\{p, q\}$ as the regular solid bounded by p-gons, q at each vertex; for example, $\{4, 3\}$ is the cube. The Petrie polygon of $\{p, q\}$ is a skew h-gon such that every two consecutive sides, but no three, belong to a face of the solid; for example, the Petrie polygon of the cube is a skew hexagon. It is found that $h=(g+1)^{1/2}-1$, where g is the order of the symmetry group (that is, four times the number of edges). Since $4/g=1/p+1/q-1/2$, we deduce an expression for h in terms of p and q. Moreover, the solid has $3h/2$ planes of symmetry. (Received October 1, 1945.)

236. M. M. Day: *Note on the billiard ball problem.*

In his book *Dynamical systems*, G. D. Birkhoff proves by the rather deep Poincaré ring theorem the fact that for each convex cornerless billiard table and each integer n there is a closed path around the table of precisely n sides such that a billiard ball will follow this path around and around if it satisfies the usual reflection law that the angle of incidence equals the angle of reflection whenever the ball hits a side. An elementary proof of this result is given by the following two statements: (1) Any n-sided convex polygon of stationary length inscribed in a convex cornerless curve satisfies the reflection law. (2) For each $n \geq 2$ there is an n-sided convex polygon of maximal length inscribed in such a curve. (Received August 9, 1945.)

237. M. M. Day: *Polygons circumscribed about closed convex curves.*

An earlier abstract (50-5-132) announced a proof without use of fixed point theorems of the following result. If C is a symmetric closed convex curve, there exists a parallelogram P circumscribed about C so that the midpoint of each side of C is on P. Using very elementary minimal area methods this result is extended to polygons of an arbitrary number of sides, and to polyhedra about convex bodies in Euclidean space of any finite dimension. In higher dimensions the phrase "midpoint of each side" is replaced by "centroid of each face." (Received August 9, 1945.)

238. H. P. Pettit: *The tangents at certain multiple points on a curve C_{2mn}.*

In a paper published in the Tôhoku Mathematical Journal in 1927, the author discussed the construction of a curve C_{2mn} of order $2mn$ by the use of two pencils of lines with centers A_1, A_3, related by means of two base curves C_n, C_m of orders n and m, respectively, and an auxiliary pencil with center A_2. It was shown that the line A_1A_2 meets C_m in m n-fold points of C_{2mn}. In the present paper, it is shown that certain projective relationships yield the following method of determining the tangents at such an n-fold point: Project the points in which A_1A_2 meets C_n from the point of intersection of A_1A_3 with the tangent to C_m at the point in question, thus determining n points on A_2A_3. These points are projected from the n-fold point in the desired tangents. (Received August 6, 1945.)

239. H. P. Pettit: *The tangents at certain ordinary points on a curve C_{2mn}.*

As shown by the author in a previous paper published in the Tôhoku Mathematical Journal in 1927, a curve of order $2mn$ is generated by means of two pencils of lines related by means of two base curves of orders m and n respectively and an auxiliary pencil. The generated curve was shown to pass through all common points of the base curves. This paper discusses a method of constructing the tangents to the generated curve at these common points. There is shown to be a projective relationship between the tangents to the base curves at the common point and the tangent to the generated curve, which results in the following. If A_1, A_2, A_3 are the vertices of the first, auxiliary, and second pencils, and P is the common point of the base curves C_1, C_3, determine K_1 in which the tangent to C_3 at P meets A_2A_3 and the point K_3 in which the tangent to C_1 at P meets A_1A_2. Draw the line K_1K_3 meeting A_1A_3 in T. The line PT is the required tangent to the generated curve. (Received May 26, 1945.)

LOGIC AND FOUNDATIONS

240. N. D. Nelson: *Recursive functions and intuitionistic number theory.* Preliminary report.

It is shown that the interpretation by the intuitionistic truth notion of realizability of Kleene (Bull. Amer. Math. Soc. abstract 48-1-85) satisfies certain formal systems of intuitionistic number theory. Further results are obtained which complete reasoning outlined by Kleene (loc. cit. and Trans. Amer. Math. Soc. vol. 53 (1943) pp. 41–73,

§16) to show that intuitionistic number theory admits, besides the extension which gives classical number theory, also an intuitionistic extension. Both extensions are simply consistent if the unextended system is, and the two extensions are contradictory to each other. The work involves formalization of the theory of certain primitive recursive predicates $R_k(e, x_1, \cdots, x_k, y)$ which in a formal intuitionistic system afford a representation of the theory of general and partial recursive functions and predicates. The present results combine with reasoning of Kleene (Abstract 48-1-85) to establish the independence of certain formulas of the intuitionistic predicate calculus, in particular of the formula $\neg\neg(x)(A(x) \vee \neg A(x))$. (Received August 8, 1945.)

STATISTICS AND PROBABILITY

241. Reinhold Baer: *Sampling from a changing population.*

The stochastic limits of certain functions of random samples are determined where the samples are taken from different distributions belonging to a continuous family of distributions. (Received August 22, 1945.)

242. J. L. Doob: *Markoff chains—denumerable case.*

Let $p_{ij}(t)$, $i, j=1, 2, \cdots, 0 \leqq t < \infty$, be the transition probability functions of a Markoff process. Let $x(t)$ be the (integral) value assumed by the probability system at time t. Necessary and sufficient conditions are found that the $p_{ij}(t)$ satisfy the systems of first order differential equations (*) $p'_{ik}(t) = -q_i p_{ik}(t) + \sum_{j\neq i} q_{ij} p_{jk}(t)$, $p'_{ik}(t) = -p_{ik}(t) q_k + \sum_{j\neq i} p_{ij}(t) q_{jk}$, where $q_i = -p'_{ii}(0)$, $q_{ik} = p'_{ik}(0)$ $(i \neq k)$. A detailed analysis is made of the processes for which the discontinuities of $x(t)$ are well ordered. It is shown that if q_i, q_{ik} are specified arbitrarily except that $q_{ik} \geqq 0$, $q_i = \sum_j q_{ij}$, there is always a corresponding set of functions $\{p_{ij}(t)\}$ determining a Markoff process, but that in general there will be infinitely many such sets of functions, and even infinitely many satisfying (*), such that the discontinuities of $x(t)$ are well ordered. The initial conditions $p_{ij}(0) = \delta_{ij}$ are thus insufficient to determine uniquely the solutions to (*). (Received September 22, 1945.)

243. Mark Kac: *On the average of a certain Wiener functional.*

Let $x(t)$ be an element of the Wiener space. It is shown that the average of the functional $\exp\left(-z\int_0^1 |x(t)|\,dt\right)$ $(z>0)$ is given by the formula $\sum_1^\infty \kappa_j \exp\left(-(\delta_j/2)z^{2/3}\right)$, where $\delta_1, \delta_2, \cdots$ are positive zeros of the derivative of $P(y) = (2y)^{1/2}\{J_{-1/3}((2^{3/2}/3)y^{3/2}) + J_{1/3}((2^{3/2}/3)y^{3/2})\}$ and $\kappa_j = (1+\int_0^{\delta_j} P(y)dy)/\delta_j P(\delta_j)$. A related limit theorem in calculus of probability is discussed. In the course of the proof the following seemingly new result was also obtained: If r_j is the jth positive root of $J_\nu(x)$ $(\nu \geqq 0)$ then $r_j^2 > \nu^2 + (2\pi j)^{2/3}\nu^{4/3}$. For j fixed the estimate is weaker than a known asymptotic formula. The value of the estimate is due to the fact that j can depend on ν. (Received August 2, 1945.)

244. Isaac Opatowski: *Calculation of Markoff chains by incomplete gamma and beta functions and by Charlier polynomials.*

Several types of stochastic processes consisting of successive transitions between n states $\{i\}_1^n$ are considered (cf. Bull. Amer. Math. Soc. vol. 51 (1945) p. 665). Call $P_{r,s}(t)$ the probability of being a system at the time t in the state s if it is at $t=0$ in the state r. Let the only transitions possible during dt be $(i-1 \rightarrow i)(i+1 \rightarrow i)$ and the corresponding probabilities be equal respectively to $k_i dt + o(dt)$ and $g_i dt + o(dt)$ where $k_{n+1} \geqq 0$, the remaining constants k_i being all greater than 0 and either independent of i or forming an arithmetic progression. Let g_i be either 0 or a constant $g > 0$. If $g_m = 0$ for a certain $m \leqq n-2$ then $P_{1,n}(t)$ is the convolution of $P_{1,m}(t)$ and $P_{m+1,n}(t)$. This reduces the calculation of $P_{1,n}(t)$ to the case when all g_i's are not equal to 0, and $P_{1,n}(t)$ is obtained in this case as a power series in g whose coefficients are expressed in terms of the functions specified in the title of this abstract. The paper is a part of two articles to be published in vol. 8 (1946) of the Bulletin of Mathematical Biophysics. (Received October 1, 1945.)

TOPOLOGY

245. R. H. Bing: *Collections cutting the plane.*

The following is proved for the plane: Suppose that K is a bounded set of n components which does not cut the point A from the point B, that G is a collection of compact continua with a closed sum, that the sum of no n elements of G cuts A from B in the complement of K and that no two elements of G intersect each other in the complement of $\bar{K}$ but each element of G intersects $\bar{K}$. Then the sum of the elements of G does not cut A from B in the complement of K. Also, there is a subset T of $\bar{K} - K$ irreducible with respect to $\bar{K} - T$ not cutting A from B and such that if the components of T are regarded as points, there is an arc (or a point) in the complement of $\bar{K} - T$ that contains A and B but no point of an element of G. (Received September 4, 1945.)

246. D. G. Bourgin: *Quadratic forms.*

This note gives a topological interpretation of the two numerical invariants characterizing a real quadratic form, namely the rank and the signature. Thus for Q (not definite) in $n+1$ essential variables the signature is $n+2-\sum_0^n R_j$ where R_j is the j-dimensional mod 2 Betti number of the configuration $Q=0$ in $(n+1)$-dimensional projective space. The case of definite forms is included by either identifying rank and signature here or by using $Q' = Q + x_{-1}^2 - x_{-2}^2$. The appearance of the sum rather than the alternating sum of the Betti numbers is interesting. The invariant above may be taken as a definition of the signature for more general forms, and presumably other numerical invariants needed for more general forms may be introduced in a natural way by taking account of the topological aspects of the manifolds corresponding to $Q=0$. (Received September 22, 1945.)

247. W. H. Gottschalk: *Properties of minimal sets.*

Let X be a topological space in which there operates a *flux*, that is, a homeomorphism, a mapping, a one-parameter group of transformations, or a one-parameter semi-group. A subset of X is *minimal* in the sense of G. D. Birkhoff provided that it is a smallest orbit-closure. It is shown that a minimal-set partition carries over from a flux to a sub-flux. Minimality is characterized. The existence of minimal sets is demonstrated in case the phase space X is compact. Finally, it is proved that if X is a compact Hausdorff space, if f is a continuous flux, and if X is minimal under f, then either X is minimal under every sub-flux or X is the cartesian product, with bases properly identified, of the unit interval and a set minimal under some sub-flux. It is to be noted that minimal sets and almost periodicity are intimately related. (Received August 8, 1945.)

248. W. H. Gottschalk: *Topological characterizations of almost periodicity.*

The author generalizes previous results of his (see *Orbit-closure decompositions and almost periodic properties*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 915–919) to include nonmetric spaces and to include homeomorphisms, mappings, one-parameter transformation groups and one-parameter semi-groups. It is noted that an almost periodic function may be considered as an almost periodic point in a function space. Bochner's characterization of almost periodicity is extended to the cases of a homeomorphism and a one-parameter group in a uniform space. Several uses of these theorems are made to obtain further results. (Received August 8, 1945.)

249. William Gustin: *Countable connected spaces.*

Let $\mathfrak{S}$ be the class of all countable and connected perfectly separable Hausdorff spaces containing more than one point. Urysohn, using a complicated identification of points, has constructed an $\mathfrak{S}$ space (Math. Ann. vol. 94 (1925) pp. 274–283). Two $\mathfrak{S}$ spaces, X and X^*, more simply constructed and not involving identifications, will be presented. The space X^* is a connected subspace of X and contains a dispersion point; that is, the subspace formed from X^* by removing this one point is totally disconnected. (Received July 17, 1945.)

250. Deane Montgomery: *Measure preserving homeomorphisms.*

Kerékjártó has obtained interesting results on a class of transformations which he named similitudes (C. R. Acad. Sci. Paris vol. 198 (1934) pp. 1345–1347). By modifying his method one can obtain certain results about the structure of measure preserving homeomorphisms in the vicinity of fixed points. It is shown that there must be arbitrarily small (but not degenerate) continua containing the fixed point and going into themselves under the transformation. This implies that there are points near the fixed point which remain near under positive iteration. (Received August 10, 1945.)

251. Deane Montgomery and Hans Samelson: *Fiberings with singularities.*

A mapping P of a polyhedron R onto a polyhedron B is called a singular fiber mapping if (1) P is open, (2) P is topological on a closed subset L of R, and (3) P is a fiber mapping (in the sense of Hurewicz-Steenrod) of the complement of L. The points of L are the singularities. It is proved that a (differentiable) singular fibering of an n-sphere cannot have exactly one singular point, and that, if the fiber has characteristic 0 and the singular points are finite in number, there are exactly 0 or 2, depending on the parity of n. (Received August 27, 1945.)

NEW PUBLICATIONS

GALLAND, J. S. An historical and analytical bibliography of the literature of cryptology. (Northwestern University Studies in the Humanities, no. 10.) Evanston, Northwestern University, 1945. 8+209 pp. $5.00.

HOOPER, A. The river mathematics. New York, Holt, 1945. 8+401 pp. $3.75.

TURNEY, T. H. Heaviside's operation calculus made easy. London, Chapman and Hall, 1944. 7+96 pp. 10s. 6d.

VALIRON, G. Théorie des fonctions. (Cours d'analyse mathématique, vol. 1.) Paris, Masson, 1942. 6+522 pp.

HERMITIAN QUADRATIC FORMS IN A QUASI-FIELD

GORDON PALL

1. Introduction. E. Witt[1] proved the following theorem concerning quadratic forms in a fairly general field:

THEOREM 1. *Let* $f_1 = ax_1^2 + \phi_1(x_2, \cdots, x_n)$ *and* $f_2 = ax_1^2 + \phi_2(x_2, \cdots, x_n)$ *be quadratic forms whose coefficients lie in a given field* F *in which* $2 \neq 0$. *Then the equivalence in* F *of* f_1 *and* f_2 *implies that of* ϕ_1 *and* ϕ_2.

It is our purpose here to generalize this theorem to any quasi-field (a field, except that multiplication may not be commutative) on which is defined a conjugate operation of period 2 with the usual properties

$$\overline{a+b} = \bar{a} + \bar{b}, \qquad \overline{ab} = \bar{b} \cdot \bar{a}.$$

Well known examples are any field with $\bar{a} = a$; the field of complex numbers with the usual complex conjugate; the system of quaternions with real coefficients and the usual conjugate. The analogue in a quasi-field of quadratic form in a field is the hermitian quadratic form

$$f = \bar{x}' A x = \sum_{i,j=2}^{n} \bar{x}_i a_{ij} x_j, \quad \text{where} \quad \overline{A}' = A, \quad \text{or} \quad \bar{a}_{ij} = a_{ji}.$$

The scalars of a quasi-field are the elements s such that $\bar{s} = s$. The diagonal elements of a hermitian matrix are therefore scalars. The process of completing squares is carried out in much the same way as in a field. Thus if, in f above, $a_{11} \neq 0$,

$$f = \left(\bar{x}_1 + \sum_{i=1}^{n} \bar{x}_i a_{i1} a_{11}^{-1}\right) a_{11} \left(x_1 + \sum_{i=2}^{n} a_{11}^{-1} a_{1i} x_i\right)$$
$$+ \sum_{j,k=2}^{n} \bar{x}_j (a_{jk} - a_{j1} a_{11}^{-1} a_{1k}) x_k.$$

Hence the analogue of a form like f_1 in Witt's theorem can be written

$$\bar{x}_1 a x_1 + \phi, \quad \text{where} \quad \phi = \sum_{i,j=2}^{n} \bar{x}_i b_{ij} x_j, \qquad \bar{b}_{ij} = b_{ji}.$$

Since determinants do not exist in a quasi-field (except for hermitian matrices), we cannot demonstrate that a matrix T is nonsingular

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[1] See bibliography.

by the nonvanishing of a determinant. Instead, we may construct explicitly the reciprocal matrix V, such that $VT=TV=I$.

We shall, in §3, consider automorphs of f, that is matrices T such that $\bar{T}'AT=A$. If z denotes the first column of T, then $\bar{z}'Az=a_{11}$, that is z is a representation of the leading coefficient a_{11} of f. If $a_{11}\neq 0$, we shall for any given representation z of a_{11} construct a corresponding automorph of f.

2. A generalization of Witt's theorem. The theorem we shall prove is the following.

THEOREM 2. *Let F be a quasi-field with a conjugate operation as described above, $2\neq 0$. Let a be a nonzero scalar, and B_1, B_2 nonsingular hermitian matrices of order $n-1$, with elements in F. Let*

$$A_1 = \begin{bmatrix} a & 0' \\ 0 & B_1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} a & 0' \\ 0 & B_2 \end{bmatrix}, \tag{1}$$

where 0 denotes a column, and 0′ a row, of $n-1$ zeros. Let T denote any transformation (with coefficients in F) of A_1 into A_2, that is let

$$A_2 = \bar{T}'A_1T. \tag{2}$$

Then we can construct a transformation of B_1 into B_2.

PROOF. We can write (2) in the form

$$\begin{bmatrix} a & 0' \\ 0 & B_2 \end{bmatrix} = \begin{bmatrix} \bar{x}_0 & \bar{x}' \\ \bar{y} & \bar{T}_1' \end{bmatrix} \begin{bmatrix} a & 0' \\ 0 & B_1 \end{bmatrix} \begin{bmatrix} x_0 & y' \\ x & T_1 \end{bmatrix}, \tag{3}$$

where x_0 is a constant, x and y are column vectors of $n-1$ components, T_1 a matrix of order $n-1$. Expanding (3) we get

$$\bar{x}_0ax_0 + \bar{x}'B_1x = a, \tag{4}$$

$$\bar{x}_0ay' + \bar{x}'B_1T_1 = 0', \qquad \bar{y}ax_0 + \bar{T}_1'B_1x = 0, \tag{5}$$

$$\bar{y}ay' + \bar{T}_1'B_1T_1 = B_2. \tag{6}$$

Our problem is to derive from (4)–(6) a transformation of B_1 into B_2.

Suppose we could secure $x_0=1$, $x=0$, to begin with. Then by (5), $y'=0'$, $\bar{y}=0$; and by (6), $\bar{T}_1'B_1T_1=B_2$. What we shall do is construct a nonsingular automorph U of A_1 whose first column is the same as that of T. Having done this, let

$$W = \begin{bmatrix} z_0 & w' \\ z & U_1 \end{bmatrix} \text{ be the reciprocal of } U = \begin{bmatrix} x_0 & \cdot \\ x & \cdot \end{bmatrix}.$$

Then $z_0x_0+w'x=1$, $zx_0+U_1x=0$, and so

$$WT = \begin{bmatrix} z_0 & w' \\ z & U_1 \end{bmatrix} \begin{bmatrix} x_0 & y' \\ x & T_1 \end{bmatrix} = \begin{bmatrix} 1 & u' \\ 0 & V \end{bmatrix},$$

say; and WT also replaces A_1 by A_2. By the preceding remark, $\bar{V}'B_1V = B_2$.

Hence Theorem 2 is a consequence of the following theorem; or see §4.

3. Automorphs with an assigned first column.

We now prove:

THEOREM 3. *Let $A = (a_{ij})$ be any nonsingular hermitian matrix with coefficients in a quasi-field F of characteristic not 2. Let z be a representation in F of a_{11}, that is $\bar{z}'Az = a_{11}$, and assume $a_{11} \neq 0$. Then there exists in F a nonsingular automorph of A with z as its first column.*

We first complete squares, which amounts to applying a transformation

$$\bar{P}'AP = \begin{bmatrix} a & 0' \\ 0 & B_1 \end{bmatrix} = A_1 \text{ say,} \quad \text{where } P = \begin{bmatrix} 1 & v' \\ 0 & I \end{bmatrix},$$
$$P^{-1} = \begin{bmatrix} 1 & -v' \\ 0 & I \end{bmatrix}. \tag{7}$$

Here $a = a_{11}$, and B_1 is hermitian with A. We have $\bar{z}'Az = \bar{u}'A_1u$, where $z = Pu$; and if we can construct an automorph U of A_1 with u as first column, then PUP^{-1} will be an automorph of A with z as first column. For, the first column of PU is Pu, and multiplication on the right by P^{-1} does not change the first column of PU.

We can therefore use the notations (4)–(6) with $B_2 = B_1$. Here x_0 and x are given as satisfying (4), and it is required to find y and T_1 to satisfy (5) and (6).

The cases $x_0 = 0$ and $x_0 \neq 0$ must be distinguished.

Let $x_0 = 0$. Then $\bar{x}'B_1x = a$, and we must choose y and T_1 to satisfy

$$\bar{x}'B_1T_1 = 0', \qquad \bar{y}ay' + \bar{T}_1'B_1T_1 = B_1. \tag{8}$$

The last equation can be replaced by $(\overline{xy'})'B_1(xy') + \bar{T}_1'B_1T_1 = B_1$, hence by

$$(\overline{xy' + T_1})'B_1(xy' + T_1) = B_1, \tag{9}$$

in view of (8_1). Then all of (8) will hold if we put

$$T_1 = I - xy', \qquad \bar{x}'B_1 - ay' = 0; \tag{10}$$

the last is satisfied if $y' = a^{-1}\bar{x}'B_1$. It will be found that $y'x = 1$, and

$$\begin{bmatrix} 0 & y' \\ x & T_1 \end{bmatrix} \begin{bmatrix} 0 & y' \\ x & T_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}. \tag{11}$$

Let $x_0 \neq 0$. Then (5) will hold if

$$\bar{y} = -\overline{T}_1' B_1 x x_0^{-1} a^{-1}, \qquad a y' = -\bar{x}_0^{-1} \bar{x}' B_1 T_1, \tag{12}$$

and then (6) becomes

$$\overline{T}_1'(B_1 + B_1 x (\bar{x}_0 a x_0)^{-1} \bar{x}' B_1) T_1 = B_1. \tag{13}$$

We therefore try to choose a number k in F to satisfy

$$(I + B_1 x k \bar{x}') B_1 (I + x \bar{k} \bar{x}' B_1) = B_1 + B_1 x (\bar{x}_0 a x_0)^{-1} \bar{x}' B_1. \tag{14}$$

In virtue of (4) this will be satisfied if

$$k + \bar{k} + k(a - \bar{x}_0 a x_0) \bar{k} = (\bar{x}_0 a x_0)^{-1}. \tag{15}$$

Here we try the substitution $k = t^{-1}$, and multiply left and right by t and $\bar{t}$ to obtain

$$\bar{t} + t + a - \bar{x}_0 a x_0 = t (\bar{x}_0 a x_0)^{-1} \bar{t}. \tag{16}$$

To satisfy this we put $t = h + \bar{x}_0 a x_0$, and find

$$h(\bar{x}_0 a x_0)^{-1} \bar{h} = a, \text{ which holds if } h = \pm a x_0, \quad t = (\bar{x}_0 \pm 1) a x_0. \tag{17}$$

Since $2 \neq 0$ in F we can choose the sign to make $t \neq 0$, so that k exists.

We can now solve for T_1 the equations

$$(I + x \bar{k} \bar{x}' B_1) T_1 = I = T_1 (I + x \bar{k} \bar{x}' B_1). \tag{18}$$

For if we put $T_1 = I + x m \bar{x}' B_1$, where m is a constant to be determined, then (18) will hold if

$$\bar{k} + m + \bar{k}(a - \bar{x}_0 a x_0) m = 0 = m + \bar{k} + m(a - \bar{x}_0 a x_0) \bar{k}. \tag{19}$$

Noting that $\bar{k}\bar{t} = \bar{t}\bar{k} = 1$ (since $k = t^{-1}$), we replace (19) by

$$m^{-1} + \bar{t} + a - \bar{x}_0 a x_0 = 0 = \bar{t} + m^{-1} + a - \bar{x}_0 a x_0, \tag{20}$$

which holds if $m^{-1} = -\bar{t} - a + \bar{x}_0 a x_0 = -(1 \pm \bar{x}_0) a$. Thus m exists, and

$$T_1 = I + x m \bar{x}' B_1. \tag{21}$$

Finally we verify that the automorph so constructed is nonsingular. By (3),

$$\begin{bmatrix} a^{-1} \bar{x}_0 a & a^{-1} \bar{x}' B_1 \\ B_1^{-1} \bar{y} a & B_1^{-1} \overline{T}_1' B_1 \end{bmatrix} \begin{bmatrix} x_0 & y' \\ x & T_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}. \tag{22}$$

Hence we have only to prove that[2]

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[2] The referee remarked that (23) need not be proved since it is known that if $(\alpha)(\beta) = 1$, where (α), (β) are matrices with elements in a division ring, then also $(\beta)(\alpha) = 1$.

$$\begin{bmatrix} x_0 & y' \\ x & T_1 \end{bmatrix} \begin{bmatrix} a^{-1}\bar{x}_0 a & a^{-1}\bar{x}' B_1 \\ B_1^{-1}\bar{y}a & B_1^{-1}\bar{T}_1' B_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}. \tag{23}$$

Put $r=\bar{x}_0 a x_0$. Using $\bar{x}'B_1 x=a-r$, and verifying that $m+\bar{m}+m(a-r)\bar{m}=-a^{-1}$, the proof of (23) is easy. For example, $x_0 a^{-1}\bar{x}_0 a+y'B_1^{-1}\bar{y}a=1$ if and only if $ra^{-1}r+\bar{x}'B_1(I+xm\bar{x}'B_1)B_1^{-1}(I+B_1 x\bar{m}\bar{x}')B_1 x=r$, or $ra^{-1}r+a-r+(a-r)(-a^{-1})(a-r)=r$, or $r=r$.

One additional remark is worth making for the case where F is a (commutative) field. If K is any square matrix of rank 1, it can of course be expressed as xy', where x and y are column vectors. The determinant of $I+K$ is readily found, since as is easily seen,

$$|I+xy'| = 1+x'y = 1+y'x = 1+\sum x_i y_i.$$

The reciprocal of a matrix of the type $I+hxy'$ (where h is a constant) can be found by noting that

$$(I+hxy')(I+kxy') = I+\{h+k+hky'x\}xy',$$

and choosing k to make $h+k+hky'x=0$.

4. **An alternative construction for Theorem 2.** For some purposes, it is more advantageous not to construct an automorph of A_1, but to continue the argument from (4)–(6) as follows. If $x_0=0$, $xy'+T_1$ replaces B_1 by B_2. Let $x_0\neq 0$. Then (5) is equivalent to (12), and (6) reduces to (13) with B_2 instead of B_1 on the right. We have (14)–(17) as before, and so $T_1+xk\bar{x}'B_1T_1$ is a transformation replacing B_1 by B_2. This transformation may in certain cases be integral (in a sense which we need not discuss here) even though no integral automorph of A_1 exists with x_0 and x as first column.

It should be mentioned that the preceding methods can be extended to the case where the element a is replaced by a nonsingular hermitian matrix of order greater than 1.

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NOTE ON THE EQUIVALENCE OF NONSINGULAR PENCILS OF HERMITIAN MATRICES

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In a previous paper[1] on the equivalence of nonsingular pencils of Hermitian matrices the restriction was made that the characteristic of the field in which the elements of the matrices lay be zero. It is the purpose of this note to show how this restriction can be removed and that all the results of this paper are true for fields of any characteristic different from two. Since many of the proofs involve division by the integer two, the case of characteristic two is not considered. To save unnecessary repetitions the notations used here conform with those of the previous paper.

If A, B and C, D are two pairs of Hermitian matrices, the pencil $A-\lambda B$ is said to be conjunctively equivalent to the pencil $C-\lambda D$ when there exists a nonsingular matrix P such that $P^*(A-\lambda B)P = C-\lambda D$, where P^* is the conjugate transposed of P. If B is nonsingular, the invariant factors of the pencil $A-\lambda B$ are the same as those of the pencil $AB^{-1}-\lambda E$. Let M be any matrix similar to AB^{-1} so that there exists a nonsingular matrix P such that $P^{-1}AB^{-1}P=M$. Then

$$P^*B^{-1}(A-\lambda B)B^{-1}P = P^*B^{-1}P(M-\lambda E) = RM-\lambda R.$$

Therefore the pencil $A-\lambda B$ is conjunctively equivalent to the pencil $RM-\lambda R$ and, since RM is Hermitian,

$$RM = M^*R. \tag{1}$$

Further, if Q is any nonsingular matrix satisfying

$$QM = M^*Q, \tag{2}$$

it follows from (1) that

$$R = QS, \tag{3}$$

where S is commutative with M. Moreover, if the pencil $RM-\lambda R$ is conjunctively equivalent to the pencil $GM-\lambda G$, so that there exists a nonsingular matrix W such that $W^*(RM-\lambda R)W=G(M-\lambda E)$, the matrix W is commutative with M.

In paper I the matrix M is taken in the Wedderburn canonical

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[1] John Williamson, *The equivalence of non-singular pencils of Hermitian matrices in an arbitrary field*, Amer. J. Math. vol. 57 (1935) pp. 475–490. This paper will be referred to as paper I.

form[2] and a canonical form for the matrix G is determined under a conjunctive transformation where the matrix W of the transformation is commutative with M. Since this canonical form is valid in fields of characteristic different from zero, provided that $AB^{-1}-\lambda E$ has no elementary factor $[p(\lambda)]^r$ where $p(\lambda)$ is an inseparable polynomial, the only case not covered in paper I is that in which $AB^{-1}-\lambda E$ has one or more elementary factors of the type $[p(\lambda)]^r$ where $p(\lambda)$ is inseparable.

If, however, M is taken to be the Jacobson canonical form[3] no exceptions need be made. In fact with a slightly different interpretation of the symbols involved each proof in paper I holds word for word. The reason for this is that the forms of the matrix Q, satisfying (2), and of the matrices S and W commutative with M are essentially the same as before.

Let $[p(\lambda)]^r$ be an elementary factor of the pencil $A-\lambda B$ where $p(\lambda)=\lambda^n-a_n\lambda^{n-1}-a_{n-1}\lambda^{n-2}-\cdots-a_1$, let

$$p=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$$

be the companion matrix of $p(\lambda)$ and let

$$\epsilon=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

If M is the Jacobson canonical form of AB^{-1}, M is a diagonal block matrix with one block the rn-rowed square matrix

$$\pi_r=\begin{pmatrix} p & \epsilon & 0 & \cdots & 0 \\ 0 & p & \epsilon & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & p \end{pmatrix}=pE_r+\epsilon U_r$$

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[2] J. H. M. Wedderburn, *Lectures on matrices*, Amer. Math. Soc. Colloquium Publications, vol. 17, New York, 1934, pp. 123–125.

[3] J. H. M. Wedderburn, *The canonical form of a matrix*, Ann. of Math. vol. 39 (1938) pp. 178–180. C. C. MacDuffee, *Vectors and matrices*, Carus Mathematical Monographs, 1943, p. 136.

corresponding to the elementary factor $[p(\lambda)]^r$. Any matrix commutative with π_r is a polynomial $f(\pi_r)$ in π_r. Since the minimal polynomial of π_r is $[p(x)]^r$ of degree nr,

$$(4) \qquad f(\pi_r) = f_0(\pi_r) + f_1(\pi_r)p(\pi_r) + \cdots + f_{r-1}(\pi_r)[p(\pi_r)]^{r-1}$$

where $f_i(\pi_r)$ is a polynomial in π_r of degree not greater than $n-1$, $i=0, 1, \cdots, r-1$.

A routine but long calculation shows that $p(\pi_r)=eU_r$ where e is the unit matrix of order n. It is however not really necessary to make this calculation. The coefficient of U_r in π_r^{t+1} is $w_{t+1}=\sum_{j=0}^{t} p^j \epsilon p^{t-j}$ and therefore $pw_{t+1}-w_{t+1}p=p^{t+1}\epsilon-\epsilon p^{t+1}$. Consequently, if h is the coefficient of U_r in $p(\pi_r)$, $ph-hp=p(p)\epsilon-\epsilon p(p)=0$ so that h is commutative with p. Since h is not zero, h is nonsingular and, if, in π_r, ϵ is replaced by $h^{-1}\epsilon$, $p(\pi_r)=h^{-1}hU_r=eU_r$.

Accordingly (4) assumes the form

$$(5) \qquad f(\pi_r) = f_0(\pi_r) + f_1(\pi_r)U_r + \cdots + f_{r-1}(\pi_r)U_r^{r-1}.$$

If $p(\lambda)$ is separable and M is the Wedderburn canonical form, in paper I $N_r=pE_r+eU_r$ takes the place of π_r. Any matrix commutative with N_r is of the form

$$(6) \qquad f(N_r) = f_0(p)E_r + f_1(p)U_r + \cdots + f_{r-1}(p)U_r^{r-1}.$$

It is obvious that (5) can be obtained from (6) by replacing p by π_r. Further, if

$$(7) \qquad q = \begin{Vmatrix} -a_2 & -a_3 \cdots & -a_n & 1 \\ -a_3 & -a_4 \cdots & 1 & 0 \\ \cdot & \cdot\cdot\cdot\cdot & \cdot & \cdot \\ -a_n & 1 \cdots & 0 & 0 \\ 1 & 0 \cdots & 0 & 0 \end{Vmatrix},$$

$q=q'$, $qp=p'q$ and $q\epsilon=\epsilon'q$. Accordingly the matrix

$$Q_r = qT_r = \begin{Vmatrix} 0 & 0 \cdots 0 & q \\ 0 & 0 \cdots q & 0 \\ \cdot & \cdot\cdot\cdot\cdot\cdot & \cdot \\ 0 & q \cdots 0 & 0 \\ q & 0 \cdots 0 & 0 \end{Vmatrix}$$

satisfies the equation $Q_r\pi_r=\pi_r' Q_r$. Apart from the definition of q, the matrix Q_r is the same as the corresponding matrix in paper I (see formula (11)).

In the proofs of paper I the only use made of the fact that the elements of the matrices under consideration are polynomials in p is the following. If $g(x)$ is a polynomial of degree not greater than $n-1$ and if the matrix $g(p)$ is singular, then $g(p)$ is zero. However the same is true if p is replaced by π_r, for $|g(\pi_r)| = |g(p)|^r$. Accordingly, if p is replaced by π_r and q by the matrix defined in (7), the proofs and results of paper I are valid for fields of any characteristic different from two.

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ON A LEMMA OF LITTLEWOOD AND OFFORD

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Recently Littlewood and Offord[1] proved the following lemma: Let $x_1, x_2, \cdots, x_n$ be complex numbers with $|x_i| \geqq 1$. Consider the sums $\sum_{k=1}^n \epsilon_k x_k$, where the ϵ_k are ± 1. Then the number of the sums $\sum_{k=1}^n \epsilon_k x_k$ which fall into a circle of radius r is not greater than

$$cr2^n(\log n)n^{-1/2}.$$

In the present paper we are going to improve this to

$$cr2^n n^{-1/2}.$$

The case $x_i = 1$ shows that the result is best possible as far as the order is concerned.

First we prove the following theorem.

THEOREM 1. *Let $x_1, x_2, \cdots, x_n$ be n real numbers, $|x_i| \geqq 1$. Then the number of sums $\sum_{k=1}^n \epsilon_k x_k$ which fall in the interior of an arbitrary interval I of length 2 does not exceed $C_{n,m}$ where $m = [n/2]$. ($[x]$ denotes the integral part of x.)*

Remark. Choose $x_i = 1$, n even. Then the interval $(-1, +1)$ contains $C_{n,m}$ sums $\sum_{k=1}^n \epsilon_k x_k$, which shows that our theorem is best possible.

We clearly can assume that all the x_i are not less than 1. To every sum $\sum_{k=1}^n \epsilon_k x_k$ we associate a subset of the integers from 1 to n as follows: k belongs to the subset if and only if $\epsilon_k = +1$. If two sums $\sum_{k=1}^n \epsilon_k x_k$ and $\sum_{k=1}^n \epsilon_k' x_k$ are both in I, neither of the corresponding subsets can contain the other, for otherwise their difference would clearly be not less than 2. Now a theorem of Sperner[2] states that in any collection of subsets of n elements such that of every pair of subsets neither contains the other, the number of sets is not greater than $C_{n,m}$, and this completes the proof.

An analogous theorem probably holds if the x_i are complex numbers, or perhaps even vectors in Hilbert space (possibly even in a Banach space). Thus we can formulate the following conjecture.

CONJECTURE. *Let $x_1, x_2, \cdots, x_n$ be n vectors in Hilbert space, $\|x_i\| \geqq 1$. Then the number of sums $\sum_{k=1}^n \epsilon_k x_k$ which fall in the interior of an arbitrary sphere of radius 1 does not exceed $C_{n,m}$.*

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[1] Rec. Math. (Mat. Sbornik) N.S. vol. 12 (1943) pp 277–285.

[2] Math. Zeit. vol. 27 (1928) pp. 544–548.

At present we can not prove this, in fact we can not even prove that the number of sums falling in the interior of any sphere of radius 1 is $o(2^n)$.

From Theorem 1 we immediately obtain the following corollary.

COROLLARY. *Let r be any integer. Then the number of sums $\sum_{k=1}^{n}\epsilon_k x_k$ which fall in the interior of any interval of length $2r$ is less than $rC_{n,m}$.*

THEOREM 2. *Let the x_i be complex numbers, $|x_i| \geqq 1$. Then the number of sums $\sum_{k=1}^{n}\epsilon_k x_k$ which fall in the interior of an arbitrary circle of radius r (r integer) is less than*

$$crC_{n,m} < c_1 r 2^n n^{-1/2}.$$

We can clearly assume that at least half of the x_i have real parts not less than 1/2. Let us denote them by $x_1, x_2, \cdots, x_t$, $t \geqq n/2$. In the sums $\sum_{k=1}^{n}\epsilon_k x_k$ we fix $\epsilon_{t+1}, \cdots, \epsilon_n$. Thus we get 2^t sums. Since we fixed $\epsilon_{t+1}, \cdots, \epsilon_n$, $\sum_{k=1}^{t}\epsilon_k x_k$ has to fall in the interior of a circle of radius r. But then $\sum_{k=1}^{t}\epsilon_k R(x_k)$ has to fall in the interior of an interval of length $2r$ ($R(x)$ denotes the real part of x). But by the corollary the number of these sums is less than

$$crC_{t,[t/2]} < c_1 r 2^t / t^{1/2}.$$

Thus the total number of sums which fall in the interior of a circle of radius r is less than

$$c_2 r 2^n / n^{1/2},$$

which completes the proof.

Our corollary to Theorem 1 is not best possible. We prove:

THEOREM 3. *Let r be any integer, the x_i real, $|x_i| \geqq 1$. Then the number of sums $\sum_{k=1}^{n}\epsilon_k x_k$ which fall into the interior of any interval of length $2r$ is not greater than the sum of the r greatest binomial coefficients (belonging to n).*

Clearly by choosing $x_i = 1$ we see that this theorem is best possible. The same argument as used in Theorem 1 shows that Theorem 3 will be an immediate consequence of the following theorem.

THEOREM 4. *Let $A_1, A_2, \cdots, A_u$ be subsets of n elements such that no two subsets A_i and A_j satisfy $A_i \supset A_j$ and $A_i - A_j$ contains more than $r-1$ elements. Then u is not greater than the sum of the r largest binomial coefficients.*

Let us assume for sake of simplicity that $n = 2m$ is even and $r = 2j+1$ is odd. Then we have to prove that

$$u \leqq \sum_{i=-j}^{+j} C_{2m,n+i}.$$

Our proof will be very similar to that of Sperner.[2] Let $A_1, A_2, \cdots, A_u$ be a set of subsets which have the required property and for which u is maximal. It will suffice to show that in every A the number of elements is between $n-j$ and $n+j$. Suppose this were not so, then by replacing if need be each A by its complement we can assume that there exist A's having less than $n-j$ elements. Consider the A's with fewest elements; let the number of their elements be $n-j-y$ and let there be x A's with this property. Denote these A's by $A_1, A_2, \cdots A_x$. To each A_i, $i=1, 2, \cdots, x$, add in all possible ways r elements from the $n+j+y$ elements not contained in A. We clearly can do this in $C_{n+j+y,r}$ ways. Thus we obtain the sets $B_1, B_2, \cdots$, each having $n+j-y+1$ elements. Clearly each set can occur at most $C_{n+j-y+1,r}$ times among the B's. Thus the number of different B's is not less than

$$xC_{n+j+y,r}(C_{n+j-y+1,r})^{-1} > x.$$

Hence if we replace $A_1, A_2, \cdots, A_x$ by the B's and leave the other A's unchanged we get a system of sets which clearly satisfies our conditions (the B's contain $n+j-y+1$ elements and all the A's now contain more than $n-j-y$ elements, thus $B-A$ can not contain more than $r-1$ elements and also $B(\not\subset A)$ and has more than u elements, this contradiction completes our proof.

By more complicated arguments we can prove the following theorem.

THEOREM 5. *Let $A_1, A_2, \cdots, A_u$ be subsets of n elements such that there does not exist a sequence of $r+1$ A's each containing the previous one. Then u is not greater than the sum of the r largest binomial coefficients.*

As in Theorem 4 assume that $n=2m$, $r=2j+1$, and that there are x A's with fewest elements, and the number of their elements is $n-j-y$. We now define a graph as follows: The vertices of our graph are the subsets containing z elements, $n-j-y \leqq z \leqq n+j+y$. Two vertices are connected if and only if one vertex represents a set containing z elements, the other a set containing $z+1$ elements, and the latter set contains the former. Next we prove the following lemma.

LEMMA. *There exist $C_{2n,n-j-y}$ disjoint paths connecting the vertices containing $n-j-y$ elements to the vertices containing $n+j+y$ elements.*

Our lemma will be an easy consequence of the following theorem of Menger:[3] *Let G be any graph, V_1 and V_2 two disjoint sets of its vertices. Assume that the minimum number of points needed for the separation of V_1 and V_2 is w. Then there exist w disjoint paths connecting V_1 and V_2. (A set of points w is said to separate V_1 and V_2, if any path connecting V_1 with V_2 passes through a point of w.)*

Hence the proof of our lemma will be completed if we can show that the vertices V_1 containing $n-j-y$ elements can not be separated from the vertices V_2 containing $n+j+y$ elements by less than $C_{2n,n-j-y}$ vertices. A simple computation shows that V_1 and V_2 are connected by

$$C_{2n,n-j-y}(n+j+y)(n+j+y-1)\cdots(n-j-y+1)$$

paths. Let z be any vertex containing $n+i$ elements, $-j-y\leqq i\leqq j+y$. A simple calculation shows the the number of paths connecting V_1 and V_2 which go through z equals

$$(n+i)(n+i-1)\cdots(n-j-y+1)(n-i)(n-i-1)\cdots(n-j-y+1)$$
$$\leqq(n+j+y)(n+j+y-1)\cdots(n-j-y+1).$$

Thus we immediately obtain that V_1 and V_2 can not be separated by less than $C_{2n,n-j-y}$ vertices, and this completes the proof of our lemma.

Let now $A_1^{(1)}, A_2^{(1)}, \cdots, A_x^{(1)}$ be the A's containing $n-j-y$ elements. By our lemma there exist sets $A_i^{(l)}$, $i=1, 2, \cdots, x$; $l=1, 2, \cdots, 2j+2y+1$, such that $A_i^{(2j+2y+1)}$ has $n+j+y$ elements and $A_i^{(l)}\subset A_i^{(l+1)}$ and all the A's are different. Clearly not all the sets $A_i^{(l)}$, $l=1, 2, \cdots, 2j+2y+1$, can occur among the $A_1, A_2, \cdots, A_u$. Let $A_i^{(s)}$ be the first A which does not occur there. Evidently $s\leqq r$. Omit $A_i^{(1)}$ and replace it by $A_i^{(s)}$. Then we get a new system of sets having also u elements which clearly satisfies our conditions, and where the sets containing fewest elements have more than $n-j-y$ elements and the sets containing most elements have not more than $n+j+y$ elements. By repeating the same process we eventually get a system of A's for which the number of elements is between $n-j$ and $n+j$. This shows that

$$u\leqq\sum_{i=-j}^{+j}C_{2n,n+i},$$

which completes the proof.

One more remark about our conjecture: Perhaps it would be easier to prove it in the following stronger form: Let $|x_i|\geqq 1$, then the number of sums $\sum_{k=1}^{n}\epsilon_k x_k$ which fall in the interior of a circle of radius 1 plus one half the number of sums falling on the circumference of the circle is not greater than $C_{n,m}$. If the x_i are real it is quite easy to prove this.

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[3] See, for example, D. König, *Theorie der endlichen und unendlichen Graphen*, p. 244.

We state one more conjecture.

(1). Let $|x_i|=1$. Then the number of sums $\sum_{k=1}^{n}\epsilon_k x_k$ with $\left|\sum_{k=1}^{n}\epsilon_k x_k\right|\leqq 1$ is greater than $c2^n n^{-1}$, c an absolute constant.

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NULL SYSTEMS IN PROJECTIVE SPACE

REINHOLD BAER

If P is an (abstract) n-dimensional projective space, then we define a *polarity* in P as a correspondence p associating with every point Q in P a hyperplane Q^p and with every hyperplane h in P a point h^p in such a way that:

(i) $Q=Q^{p^2}$ for every point Q and $h=h^{p^2}$ for every hyperplane h.

(ii) The point Q is on the hyperplane h if, and only if, the hyperplane Q^p passes through the point h^p.

It is an immediate consequence of (i) that polarities are 1:1 correspondences.

We shall term p a *null-polarity* if the polarity p has the additional property that:

(iii) Every point Q is on the corresponding hyperplane Q^p, and consequently every hyperplane h passes through the corresponding point h^p.

Extending a result of Veblen and Young, R. Brauer[1] has shown that the existence of a null-polarity in P implies that the number n of dimensions of P is odd, and he has connected the null-polarities with the so-called null-systems, provided P is the n-dimensional projective space over a commutative field of coordinates. It is the object of the present note to show that this last hypothesis may be omitted; more precisely we are going to show that if the dimension of P is greater than 1, then the existence of a null polarity is equivalent to the fact that P is of odd dimension and is a projective space over a commutative field of coordinates.

If P is a projective space of dimension 1, then the hyperplanes are points too. The identity transformation on the points of the line P is therefore the null-polarity of P. For this reason we shall assume throughout the remainder of this note that P be of dimension greater than 1.

The case of a projective plane P has to be treated separately from the others, since the Theorem of Desargues need not hold true in a projective plane, though it is true for all the higher-dimensional projective spaces.

A projective plane is a system of points and lines such that any two different lines meet in one and only one point, any two different

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[1] R. Brauer, *A characterization of null systems in projective space*, Bull. Amer. Math. Soc. vol. 42 (1936) pp. 247–254.

points determine one and only one line (on which they are both situated) and such that every line carries at least three points (every point is on at least three lines).

LEMMA. *If p is a polarity of the projective plane P, if Q and R are different points in P such that Q is on the line Q^p and R is on the line R^p, then Q is not on R^p (nor is R on Q^p).*

PROOF. If Q were on R^p, then R^p would be the uniquely determined line through the two different points R and Q, since R is on R^p. Furthermore R would be on Q^p; and it would follow likewise that Q^p is the uniquely determined line through R and Q. Thus $R^p = Q^p$ and hence $R = Q$, a contradiction proving our contention.

COROLLARY. *There does not exist a null-polarity in a projective plane.*

This is an immediate consequence of the lemma.

Because of the corollary we shall assume throughout the remainder of this note that the dimension of the projective space P is at least 3. In this case P is the n-dimensional projective space over an essentially uniquely determined, not necessarily commutative, field F.

If F is any field (commutative or not), and if n is an integer not less than 3, then we denote by (F, n) an additively written abelian group, admitting the elements in F as left-multipliers, and having the rank $n+1$ over F. The n-dimensional projective space over F is then essentially the same as the partially ordered set $P(F, n)$ of all the F-admissible subgroups of (F, n), the points being of the form Fx with $x \neq 0$, and the hyperplanes being of rank n.

Every polarity p of $P(F, n)$ may, as is well known,[2] be represented in the following form: There exist an anti-automorphism a of F (satisfying $a^2 = 1$) and an F-valued function $f(x, y)$, for x, y in (F, n), satisfying:[3]

$f(x, y) = 0$ if, and only if, $f(y, x) = 0$;

$f(ux+vy, z) = uf(x, z) + vf(y, z)$, $f(z, ux+vy) = f(z, x)u^a + f(z, y)v^a$.

The point Fx is on the hyperplane $(Fy)^p$ if, and only if, $f(x, y) = 0$.

Assume now that the polarity p be a null-polarity. This is equivalent to saying $f(x, x) = 0$ for every x in (F, n). If $x \neq 0$, then there exists at least one y such that $f(x, y) \neq 0$, since otherwise $(Fx)^p$ would be the whole space instead of only a hyperplane. Let x, y be any pair of elements such that $f(x, y) \neq 0$. If t is some element in F, then we find

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[2] A proof of this fact may be effected in essentially the same fashion as done by R. Brauer, op. cit. pp. 251, 252, in the case of commutative F; for a detailed proof of a more comprehensive fact see R. Baer, *A unified theory of projective spaces and finite abelian groups*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 315–317.

[3] We state here only such properties of the function $f(x, y)$ as will be needed later. Further properties have to be imposed to assure that, conversely, every such $f(x, y)$ defines a polarity. Note in particular that no use has been made of the involutorial character of a.

$$\begin{aligned} 0 = f(x + ty, x + ty) &= f(x, x) + f(x, y)t^a + tf(y, x) + tf(y, y)t^a \\ &= f(x, y)t^a + tf(y, x). \end{aligned}$$

Substituting $t=1$, we obtain $f(x, y)+f(y, x)=0$; and thus the above equation reduces to $0=f(x, y)t^a-tf(x, y)$. Since $f(x, y) \neq 0$, this implies $t^a=f(x, y)^{-1}tf(x, y)$ for every t in F, proving that the anti-automorphism a of F is an inner automorphism of F. Hence F is commutative and $a=1$. Combining this with the fact that $f(x, y)=0$ if, and only if, $f(y, x)=0$, we see that $f(x, y)$ is actually a skew-symmetric bilinear form.

It is well known[4] that there exists to every skew-symmetric, F-valued bilinear form $f(x, y)$ over (F, n) a basis $x(1), \cdots, x(m)$, $y(1), \cdots, y(m)$, $z(1), \cdots, z(k)$ of (F, n), meeting the following requirements:

$$\text{(a)} \qquad 0 \leqq m, \qquad 0 \leqq k, \qquad 2m + k = n + 1;$$

$$\begin{aligned} \text{(b)} \quad & f(x(i), y(i)) = 1 \quad \text{for} \quad 1 \leqq i \leqq m; \\ & f(x(i), x(j)) = f(y(i), y(j)) = 0 \quad \text{for every } i \text{ and } j; \\ \text{(c)} \quad & f(x(i), z(j)) = f(y(i), z(j)) = f(z(i), z(j)) = 0 \text{ for every } i \text{ and } j; \\ & f(x(i), y(j)) = 0 \quad \text{for every } i \neq j. \end{aligned}$$

This implies in particular $f(x, z(i))=0$ for every x in (F, n) so that the hyperplane $(Fz(i))^p$ would contain every point of the space, an impossibility proving $k=0$ and $n=2m-1$. Summarizing our results we obtain:

If P is a projective space of dimension n, greater than 1, *and if p is a null-polarity in P, then $n=2m-1$ for m a positive integer, P is the n-dimensional projective space over a commutative field F, and there exists a system of homogeneous coordinates in P such that the point $F(x_0, \cdots, x_{2m-1})$ is on the hyperplane $[F(y_0, \cdots, y_{2m-1})]^p$* if, and only if,

$$0 = \sum_{i=0}^{m-1} (x_{2i+1}y_{2i} - y_{2i+1}x_{2i}).$$

Combining all our results one deduces without difficulty the following facts.

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[4] See for instance C. C. MacDuffee, *The theory of matrices*, Ergebnisse der Mathematik under ihrer Grenzgebiete, vol. 2, part 5, 1933, pp. 52, 53, Theorem 32.2.

THEOREM A. *In an n-dimensional projective space there exists essentially at most one null-polarity.*

THEOREM B. *In the n-dimensional projective space P with* $1 < n$ *there exists a null-polarity if, and only if, n is odd and P is the projective space of dimension n over a commutative field.*

UNIVERSITY OF ILLINOIS

QUADRATIC FORMS

D. G. BOURGIN

This note concerns itself with the concept of the signature of a real quadratic form and with Sylvester's classical theorem of "inertia." The standard treatment is elementary enough and involves merely simple algebraic algorithms. The usual extensions have been in the direction of replacing the real number field by other fields. Roughly speaking, the viewpoint has been algebraic. The viewpoint developed below is different. It leads to an interesting identification of the signature with a certain topological invariant. This suggests that the corresponding invariant for other forms may be taken as a natural generalization of the signature. Moreover, although the signature and rank are sufficient to characterize quadratic forms, it would seem that for more general forms the topological aspects provide a basis for defining other numerical invariants.

Write

$$Q = \sum_{0}^{k} x_i^2 - \sum_{k+1}^{n+1} x_j^2, \qquad k \geqq 0.$$

Since we may equally well use $-Q$, there is no restriction in assuming $k \leqq n-k$. If $k \geqq 0$, the equation $Q=0$ defines a hypersurface in an $(n+1)$-dimensional real projective space, with coordinates $x_0:x_1: \cdots :x_{n+1}$. Let R_j, $j=0, \cdots, n$, be the mod 2 Betti numbers.

THEOREM. *The signature of Q is $n+2-\sum_0^n R_j$. The rank and the signature are invariant under real (nonsingular) projective transformations.*

It is worth noting the appearance of the sum of the Betti numbers and not the alternating sum. Indeed so far as the writer is aware, this is the first time that $\sum R_j$ has entered in a natural way.

The second assertion of the theorem is, of course, a trivial reflection of the invariance under homeomorphism of the Betti groups and the dimension (n) which is 2 less than the rank. To establish the first part we need the explicit values of the Betti numbers R_j. We can calculate these for $k>0$ by making use of results of Steenrod and Tucker,[1] for instance. An obvious conclusion from their work is that the mod 2 homology groups for $Q=0$ are isomorphic to those of $P^k \times S^{n-k}$ where

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[1] N. E. Steenrod and A. W. Tucker, *Real n quadrics as sphere bundles*, Bull. Amer. Math. Soc. vol. 47 (1941) p. 399.

P^k is a k-dimensional projective space and S^{n-k} is an $(n-k)$-dimensional sphere.

Write ${}_1R_j$ and ${}_2R_j$ for the j-dimensional mod 2 Betti numbers of P^k and S^{n-k} respectively. It is well known that

$$ {}_1R_j = 1, \qquad j = 0, \cdots, k, \tag{1a}$$

$$ {}_2R_i = \delta_{0i} + \delta_{n-k\, i}, \qquad i = 0, \cdots, n - k, \tag{1b}$$

where, of course, δ_{ij} is the Kronecker symbol. The Kunneth relations[2] lead to

$$ \sum_0^n R_j = \sum_0^k {}_1R_j \sum_0^{n-k} {}_2R_j = 2(k+1). \tag{2}$$

For $k=0$ the observation that $Q=0$ is a sphere leads immediately to a result formally consistent with equation 2, namely

$$ \sum_0^n R_j = 2. \tag{2a}$$

The signature of Q is evidently $n-2k$. In view of equations 2 and 2a, this is the value of $n+2-\sum_0^n R_j$. The theorem is therefore established.

The case of definite forms is not compassed by the theorem, for then the configuration $Q=0$ has no meaning in projective geometry. Of course, in this case we need not differentiate between the signature and the rank. Alternatively we may agree to identify the signature of Q with that of $Q+x_{-1}^2-x_{-2}^2$ with transformations restricted to the variables entering Q alone.

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[2] S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications, vol. 27, p. 141.

ON GENERA OF BINARY QUADRATIC FORMS

IRVING REINER

Let $\beta = ax^2+2bxy+cy^2$ be a properly primitive form with integral coefficients, and let the determinant $D = ac-b^2$ be written as $D = \pm 2^s\Delta$, where Δ is odd and positive, and the factorization of Δ into distinct primes is $\Delta = q_1^{\alpha_1} \cdots q_r^{\alpha_r}$. Let us suppose that a is positive and prime to $2D$. The genus of β is then completely determined by the Legendre symbols $(a|q_1), \cdots, (a|q_r)$, and $(-1|a)$ if $D \equiv 0$ or 1 (mod 4), $(2|a)$ if $D \equiv 0$ or 6 (mod 8), and $(-2|a)$ if $D \equiv 0$ or 2 (mod 8).[1] These characters are not independent, however, since $-D = b^2 - ac$ and $(a, b) = 1$ imply that $(-D|a) = 1$; from this, using the law of quadratic reciprocity, we get

$$\begin{aligned} 1 = (-D \mid a) &= (2 \mid a)^s (a \mid \Delta)(-1)^{(a-1)(\mp\Delta-1)/4} \\ &= \epsilon \cdot (a \mid q_1)^{\alpha_1} \cdots (a \mid q_r)^{\alpha_r}, \end{aligned}$$

where

$$\epsilon = (2 \mid a)^s (-1 \mid a)^{(\mp\Delta-1)/2}$$

is a character or is trivially $+1$. Thus, the characters of any existing genus must satisfy

$$\epsilon \cdot (a \mid q_1)^{\alpha_1} \cdots (a \mid q_r)^{\alpha_r} = +1. \tag{1}$$

Conversely, given any set of characters satisfying (1), a long but elementary proof showing the existence of the corresponding genus was given by Gauss,[2] who used the method of composition of forms; he also demonstrated by this means that all existing primitive genera of the same determinant contain the same number of classes.[2] Elementary proofs were also given by Hilbert for the analogous case of ideal classes in quadratic fields.[3] The purpose of this paper is to furnish, by use of Dirichlet's theorem on the infinitude of primes in an arithmetic progression, simple proofs of the results mentioned above for forms with integral coefficients. These may be stated precisely as follows:

THEOREM 1. *For any preassigned set of characters satisfying* (1), *there exists a genus with the given characters.*

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[1] Mathews, *Theory of numbers*, Part I, 1927 reprint, p. 134.

[2] Gauss, *Disquisitiones arithmeticae*, arts. 234–265.

[3] Hilbert, Jber. Deutschen Math. Verein. vol. 4 (1894–1895) pp. 286–316.

THEOREM 2. *All existing properly primitive genera of binary quadratic forms of a given determinant contain the same number of classes.*

Remark. The proofs given in this paper may easily be extended to the case of primitive forms with odd middle coefficient.

For the proof of Theorem 1, we observe that by Dirichlet's theorem, a prime a may be found such that $(a, 2D)=1$ and the set of Legendre symbols $(a|q_1), \cdots, (a|q_r)$ and $(-1|a)$ or $(2|a)$ or $(-2|a)$ (or combinations of the last three symbols, depending on the value of D) coincide with the preassigned set of values of the characters. (1) then implies that $(-D|a)=+1$, whence there exists an integer b for which $b^2\equiv -D \pmod{a}$, that is, $-D=b^2-ac$ with integral a, b and c; the form $ax^2+2bxy+cy^2$ then has the preassigned characters.

The proof of Theorem 2 will proceed as follows: Let $\beta_1, \cdots, \beta_k$ be nonequivalent representative forms of the classes in a given genus β, and let $\phi_1, \cdots, \phi_{k'}$ be the forms of another given genus ϕ, both genera having the same determinant. It is sufficient to show that we can find a transformation taking the β_i into the ϕ_i, and that no two of the ϕ_i thus obtained are equivalent. In that case $k'\geqq k$. By reversing the process, we shall have $k\geqq k'$, from which it will follow that $k=k'$, which gives the theorem.

The following will be shown:

A. Starting with a form β of a given genus, it can be transformed into a form $p\phi$, where ϕ is a form in another given genus of the same determinant as β, and where p is a prime determined by the genera β and ϕ.

B. If from β_1 and β_2 we get forms ϕ_1 and ϕ_2, then $\phi_1\sim\phi_2$ implies $\beta_1\sim\beta_2$.[4]

Let the given form be $\beta=ax^2+2bxy+cy^2$, and set $ac-b^2=D=\pm 2^s\Delta$ as before. Let p be a prime such that $(p, 2aD)=1$ and $(-D|p)=1$. Then there exist integers x and y, with $y\not\equiv 0 \pmod{p}$, such that

$$(ax+by)^2 \equiv -Dy^2 \pmod{p}.$$

Hence there exists an integer r for which

$$ar^2+2br+c\equiv 0 \pmod{p}.$$

The transformation J:

$$x\rightarrow px+ry, \qquad y\rightarrow y$$

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[4] The notation $\phi_1\sim\phi_2$ will mean that ϕ_1 is equivalent to ϕ_2, that is, there is a transformation with integral elements and of determinant $+1$ taking ϕ_1 into ϕ_2.

of determinant p takes β into $p\psi$, where

$$\psi = apx^2 + 2(ar + b)xy + dy^2,$$

and d is an integer; it is clear that β and ψ have the same determinant.

Let $\phi = a'x^2 + 2b'xy + c'y^2$ be a form of another given genus of the same determinant as β. If we show that p can be chosen so that $\psi \text{ v } \phi$,[5] we shall have proved A. Since β and ϕ are representatives of classes of properly primitive forms, and in every such class there is a form with leading coefficient odd and prime to the determinant of the form,[6] we may take $(a, 2D) = (a', 2D) = 1$. If the following relations are satisfied it will follow that $\psi \text{ v } \phi$, for their generic characters will be the same.

$$(2) \qquad \begin{gathered} (ap \mid q_1) = (a' \mid q_1), \cdots, (ap \mid q_r) = (a' \mid q_r), \\ (-1 \mid ap) = (-1 \mid a'), \qquad (2 \mid ap) = (2 \mid a'). \end{gathered}$$

From these we may conclude that $(-D \mid ap) = (-D \mid a')$. Since $(-D \mid a) = (-D \mid a') = 1$, this means that $(-D \mid p) = 1$. Hence, if we choose a prime p which satisfies (2) and is prime to $2aD$ (this is possible by Dirichlet's theorem), then there exists a transformation J taking β into ϕ. This completes the proof of A.

To prove B, let $\beta_1 \text{ v } \beta_2$; we may choose $\beta_1 \equiv \beta_2 \pmod{p}$, for we shall show that there exists a form in the class of β_2 which is congruent $\pmod{p}$ to β_1. To show this, it is sufficient to show that by a transformation of determinant $+1$, β_2 may be taken into a form congruent to β_1. Let

$$(3) \qquad \beta_1 = a_1x^2 + 2b_1xy + c_1y^2, \qquad \beta_2 = a_2x^2 + 2b_2xy + c_2y^2,$$

where $(a_1, 2D) = (a_2, 2D) = 1$. If all congruences are modulo p, the congruence

$$a_2\beta_2 \equiv (a_2x + b_2y)^2 + Dy^2 \equiv a_1a_2$$

certainly has a solution with not both x and y congruent to 0, provided that we impose the restriction $(p, 2a_1a_2D) = 1$. Let $x = x_0$, $y = y_0$ be such a solution, and choose integers x_1 and y_1 such that $x_0y_1 - x_1y_0 = 1$. The transformation of determinant $+1$:

$$x \to x_0x + (x_0t + x_1)y, \qquad y \to y_0x + (y_0t + y_1)y$$

takes β_2 into a form

$$\beta_3 = a_3x^2 + 2(a_3t + b_3)xy + c_3y^2,$$

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[5] $\phi_1 \text{ v } \phi_2$ means that ϕ_1 and ϕ_2 are of the same genus.

[6] Mathews, loc. cit. p. 133.

where $a_3 \equiv a_1 \not\equiv 0$. If we choose t so that $a_3t+b_3 \equiv b_1$, it is clear from the fact that β_1 and β_3 have the same determinant that $\beta_3 \equiv \beta_1 \pmod{p}$.

Thus, let $\beta_2 \equiv \beta_1 \pmod{p}$, where β_1 and β_2 are defined by equations (3). The transformation J obtained for β_1 may also be used for β_2. If now ϕ_1 and ϕ_2 are equivalent forms obtained from β_1 and β_2 respectively by use of J, and if A takes ϕ_1 into ϕ_2, then JAJ^{-1} takes β_1 into β_2. Let

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

Then

$$\begin{aligned} JAJ^{-1} &= \begin{bmatrix} p & r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} 1/p & -r/p \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha + r\gamma/p & -r\alpha - r^2\gamma/p + p\beta + r\delta \\ \gamma/p & -r\gamma/p + \delta \end{bmatrix}. \end{aligned}$$

If we show that p divides γ we are through, since $|A| = 1$ implies $|JAJ^{-1}| = 1$, whence $\beta_1 \sim \beta_2$. We have

$$\phi_1 = a_1px^2 + 2(a_1r + b_1)xy + d_1y^2,$$
$$\phi_2 = a_2px^2 + 2(a_2r + b_2)xy + d_2y^2.$$

Since A takes ϕ_1 into ϕ_2, we obtain

$$a_1p\alpha^2 + 2(a_1r + b_1)\alpha\gamma + d_1\gamma^2 = a_2p,$$
$$a_1p\alpha\beta + (a_1r + b_1)(\alpha\delta + \beta\gamma) + d_1\gamma\delta = a_2r + b_2.$$

If all congruences are modulo p, we have

$$2(a_1r + b_1)\alpha\gamma + d_1\gamma^2 \equiv 0, \quad (a_1r + b_1)(\alpha\delta + \beta\gamma) + d_1\gamma\delta \equiv a_2r + b_2.$$

If d is eliminated between these two congruences and the fact that $\alpha\delta - \beta\gamma = 1$ is used, it follows that

$$\gamma(a_1r + b_1) \equiv -\gamma(a_2r + b_2).$$

If p does not divide γ, then this shows that $a_1r + b_1 \equiv 0$, since $\beta_1 \equiv \beta_2$. But in this case it follows that $b_1r + c_1 \equiv 0$, by virtue of $a_1r^2 + 2b_1r + c_1 \equiv 0$. Eliminating r between the two congruences, we get $D = a_1c_1 - b_1^2 \equiv 0$, or $(p, D) \neq 1$. Since this is impossible, p must divide γ, and Theorem 2 is proved.

CORNELL UNIVERSITY

ON THREE PROBLEMS CONCERNING NIL-RINGS

JAKOB LEVITZKI

1. **Introduction.** In the present note three problems concerning nil-rings are proposed and certain relations linking these problems to one another are discussed.

First problem. The sum of all two-sided[1] nil-ideals of a ring S has been defined by G. Koethe [2, §3][2] as the radical of S, provided that this sum contains also all one-sided nil-ideals of S. We shall henceforth refer to this radical as the K-radical[3] of S. It is an open question whether or not there are rings in which the K-radical does not exist.

Second problem. A ring T is called semi-nilpotent (see [3, §2]) if each finite set of elements in T generates a nilpotent ring. A ring which is not semi-nilpotent is called semi-regular. Each semi-nilpotent ring is evidently a nil-ring. It is an open question whether or not there exist semi-regular nil-rings. As may easily be seen, this problem is equivalent to the question whether or not there exist semi-regular nil-rings which are generated by a finite set of elements.

Third problem. A nil-ideal P of a ring S has been termed by R. Baer [1, §1] a radical ideal if the quotient-ring S/P does not contain nilpotent ideals other than zero. The sum $U(S)$ and the crosscut $L(S)$ of all radical ideals of a ring S are again radical ideals which are called the upper radical and the lower radical respectively (see Baer [1, §1]). As indicated by Baer ideals may exist between $U(S)$ and $L(S)$ which are not radical ideals. R. Baer has also constructed an interesting example which illustrates this possibility. Our results in the present note show that this phenomenon can not be considered as an exception to the rule but on the contrary rather as the rule itself, and thus the following problem presents itself: Are there or are there not rings S in which $U(S)\supset L(S)$ and in which furthermore each ideal which lies between $U(S)$ and $L(S)$ is also a radical ideal?

In the present note the following results are obtained: Suppose that S is a ring in which the K-radical does not exist, then S contains an infinite number of right ideals as well as of left ideals which are semi-regular nil-rings (see Theorem 4 in §3). Suppose that S is a ring with a semi-regular upper radical $U(S)$, then S contains a subring S' so that $U(S)\supseteq S'=U(S')\supset L(S')$ and so that S' contains an infinite

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[1] We shall write henceforth in short *ideals* instead of *two-sided ideals.*

[2] Numbers in brackets refer to the Bibliography at the end of the paper.

[3] For the sake of convenience we shall reserve in this paper the term radical for the sum of all semi-nilpotent ideals of the ring (see [3, §2]).

number of ideals between $U(S')$ and $L(S')$ which are not radical ideals (see Theorem 5 in §3). If S is a ring with a semi-nilpotent upper radical $U(S)$, and if $U(S)\supset L(S)$, then S contains an infinite number of ideals between $U(S)$ and $L(S)$ which are not radical ideals (see Theorem 6 in §3).

Remarks. From Theorem 4 it follows that if the first problem is answered in the affirmative then this will apply also to the second problem. From Theorem 5 it follows that if the second problem is answered in the negative then this will also apply to the third problem. The above-mentioned example of R. Baer is of a nil-ring U of which it can be easily proved that it is semi-nilpotent. In fact, the proof for the semi-nilpotency of U is implicitly contained in Baer's proof for the nillity of U (see [1, §2]). Thus our Theorem 6 explains the phenomenon which was described by Baer in his example. This theorem in conjunction with Theorem 5 seems to justify the conjecture that the answer to the third problem is in the negative.

Notations. If the ring T is generated by the finite set of elements $t_1, t_2, \cdots, t_n$, then we write $T=\{t_1, t_2, \cdots, t_n\}$ or $T=\{\cdots, t_i, \cdots\}$. The sum[4] of a finite number of right ideals (left ideals) $A_1, A_2, \cdots, A_n$ will be denoted by $A_1+A_2+\cdots+A_n=\sum_{i=1}^{n}A_i$.

The ideal, the right ideal and the left ideal in the ring S which are generated by a finite set of elements $a_1, a_2, \cdots, a_n$ will be denoted by $(a_1, a_2, \cdots, a_n)$, $(a_1, a_2, \cdots, a_n)_r$ and $(a_1, a_2, \cdots, a_n)_l$ respectively.

If $A=(a_1, a_2, \cdots, a_n)_r$, then $A\supseteq\sum_{i=1}^{n}a_iS$. In case $A=\sum_{i=1}^{n}a_iS$ we say that A possesses a proper right basis. Similarly a proper left basis is defined.

If $A=(a_1, a_2, \cdots, a_n)$, then $A\supseteq\sum_{i=1}^{n}S\cdot a_iS$. In case $A=\sum_{i=1}^{n}Sa_iS$ we say that A possesses the proper basis $a_1, a_2, \cdots, a_n$.

If the ring S has an identity element, then all bases are of course proper bases.

2. On nil-ideals with proper bases.

In this section we shall derive certain properties of nil-ideals with proper bases which will be used in §3 for the proof of the assertions made in the introduction.

THEOREM 1. *If R is a nonzero nil-ideal with a proper right basis in a ring S, then $R^2\subset R$.*

PROOF. By assumption we may put $R=\sum_{i=1}^{n}a_iS$, where $a_i\in R$, $i=1, \cdots, n$. Since further $Sa_iS\subseteq R$, we have $R^2=\sum_{i,k=1}^{n}a_iSa_kS \subseteq\sum_{i=1}^{n}a_iR\subseteq R^2$, and hence $R^2=\sum_{i=1}^{n}a_iR$. Now suppose that $R^2\not\subset R$, then by $R^2\subseteq R$ we would have $R^2=R$, that is,

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[4] We shall not use direct sums in this paper.

$$R = \sum_{i=1}^{n} a_iR, \qquad a_i \in R,\ i = 1, \cdots, n. \tag{1}$$

In case $n=1$ we obtain $R=a_1R$, which by successive left multiplication yields $R=a_1^kR$ for each k, that is, $a_1^k \neq 0$ for each k, which is a contradiction to the nillity of R. Now suppose that $n \geqq 2$ and define m so that

$$R = \sum_{i=1}^{m} a_iR, \qquad R \supset \sum_{i=1}^{m-1} a_iR, \qquad 2 \leqq m \leqq n. \tag{2}$$

By putting $T=\sum_{i=1}^{m-1} a_iR$ we then have

$$R = T + a_mR, \qquad a_m \neq 0. \tag{3}$$

By (3) it follows that a_m can be represented in the form

$$a_m = b_1 + a_mc, \qquad b_1 \in T,\ c \in R. \tag{4}$$

Now suppose that for a certain r it has been proved that $a_m=b_r+a_mc^r$, $b_r \in T$, then by (4) this would yield $a_m=b_r+(b_1+a_mc)c^r=b_{r+1}+a_mc^{r+1}$, where $b_{r+1}=b_r+b_1c^r$. In view of the fact that T is a right ideal, it follows that $b_{r+1} \in T$, and thus by induction we have proved that for each s the element a_m has a representation of the form

$$a_m = b_s + a_mc^s, \qquad b_s \in T,\ c \in R. \tag{5}$$

Now take for s the index of the nilpotent element c, then $c^s=0$, and we have $a_m=b_s \in T$, and hence $a_mR \subseteq T$, or $R=\sum_{i=1}^{m-1} a_iR$, which is a contradiction to (2). Consequently equation (1) is impossible, which implies that $R^2 \subset R$.

THEOREM 2. *If R is a nonzero semi-nilpotent right ideal with a proper right basis in a ring S, then $R^2 \subset R$.*

PROOF. For a certain integer n we have by assumption $R=\sum_{i=1}^{n} a_iS$ with $a_i \in R$, $i=1, \cdots, n$. Now denote by N the radical[8] of S, then $R \subseteq N$, and $Sa_iS \subseteq N$ (see [3, §2]). Hence we have $R^2=\sum_{i,k=1}^{n} a_iSa_kS \subseteq \sum_{i=1}^{n} a_iN$. Now suppose that $R^2 \not\subset R$, then we would have $R^2=R$, that is, $R \subseteq \sum_{i=1}^{n} a_iN \subseteq R$, or

$$R = \sum_{i=1}^{n} a_iN, \qquad a_i \in R,\ i = 1, \cdots, n. \tag{6}$$

From (6) follows

$$a_i = \sum_{k=1}^{n} a_k b_{ik}, \quad b_{ik} \in N,\ i,\ k = 1, \cdots, n. \tag{7}$$

By considering the ring $B = \{ \cdots, b_{ik}, \cdots \}$, we have in view of (7) the relation $a_i \in \sum_{k=1}^{n} a_k B$, $i = 1, \cdots, n$, which by right multiplication yields $a_i B \subseteq \sum_{k=1}^{n} a_k B^2$, $i = 1, \cdots, n$, and hence $a_i \in \sum_{k=1}^{n} a_k B^2$, $i = 1, \cdots, n$. Thus by successive right multiplication we obtain for each m

$$a_i \in \sum_{k=1}^{n} a_k B^m, \qquad i = 1, \cdots, n, \tag{8}$$

which implies that $B^m \neq 0$ for each m. But this is a contradiction, since N is semi-nilpotent, $B \subseteq N$, and B is generated by a finite set of elements.

Remark. Each nil-ideal of a ring S is contained in the upper radical $U(S)$ of S. Only a slight modification of the proof of Theorem 1 is necessary in order to extend the validity of that theorem to one-sided ideals of S which are contained in $U(S)$. In this generalized form, Theorem 1 would include Theorem 2 as a special case, since each semi-nilpotent one-sided ideal lies in the radical[2] N of S, which in turn is a subset of the upper radical.

THEOREM 3. *If R is a nonzero semi-nilpotent ideal with a proper basis in S, then $R^2 \subset R$.*

PROOF. For a certain integer n we have by assumption $R = \sum_{i=1}^{n} S a_i S$, $a_i \in R$, $i = 1, \cdots, n$. If, again, N denotes the radical[3] of S, then $S a_i S \subseteq N$ for each i, and hence we have $S a_i S S a_k S S a_j S \subseteq N a_k N$ for any triple of indices i, j, k. Thus it follows that $R^3 = \sum_{i,k,j=1}^{n} S a_i S S a_k S S a_j S \subseteq \sum_{k=1}^{n} N a_k N$. Now suppose that $R^2 \not\subset R$, then we would have $R^2 = R$, and hence also $R^3 = R$, which by $R = R^3 \subseteq \sum_{k=1}^{n} N c_k N \subseteq R$ implies

$$R = \sum_{k=1}^{n} N a_k N, \qquad a_k \in R,\ k = 1, \cdots, n. \tag{9}$$

By (9) it follows that each a_k has the form

$$a_k = \sum_{i=1}^{n} b_{ki} a_i c_{ki}, \qquad b_{ki} \in N,\ c_{ki} \in N,\ i,\ k = 1, \cdots, n. \tag{10}$$

Putting $B = \{ \cdots, b_{ik}, \cdots \}$ and $C = \{ \cdots, c_{ik}, \cdots \}$ we can write (10) in the form $a_k \in \sum_{i=1}^{n} B a_i C$, $k = 1, \cdots, n$, which by successive right and left multiplication yields

$$a_k \in \sum_{i=1}^{n} B^m a_i C^m, \qquad k = 1, \cdots, n. \tag{11}$$

By (11) it follows that $B^m \neq 0$, $C^m \neq 0$ for each m, which is a contradiction to the semi-nilpotency of N.

3. **Proof of statements made in the introduction.** The proof of the following Theorem 4 follows directly from the author's results in [3, §2], while Theorems 5 and 6 are based on our results in §2.

THEOREM 4. *If S is a ring in which the K-radical does not exist, then S contains an infinite number of right ideals as well as of left ideals which are semi-regular nil-rings.*

PROOF. First note that S contains right nil-ideals as well as left nil-ideals which are not in the upper radical $U(S)$ of S. Indeed, by assumption S contains a one-sided nil-ideal A so that $A \not\subset U(S)$. Now define the element a so that $a \in A$ but $a \notin U(S)$, then the right ideal $(a)_r$ and the left ideal $(a)_l$ are nil-ideals which are not in $U(S)$. Now denote by R any right nil-ideal so that $R \not\subset U(S)$, then also $R^2 \not\subset U(S)$, since $U(S)$ is a radical ideal. Consequently R contains an element a_1 so that $a_1 R \not\subset U(S)$. By $a_1 R \subseteq R$ and by the nillity of R follows easily $R \supset a_1 R$. Now put $R_1 = a_1 R$; then in view of $R_1 \not\subset U(S)$ we may repeat with R_1 the same procedure, and thus by induction obtain an infinite sequence of right nil-ideals $R_1, R_2, R_3, \cdots$, each satisfying the relation $R_\nu \not\subset U(S)$. Now each semi-nilpotent right ideal of S is contained in the radical[3] N of S (see [3, §2]) which is a subset of $U(S)$. This implies that the right ideals of the sequence $R_1, R_2, \cdots$ are semi-regular; since a similar result holds for left ideals, the proof of our theorem is thus completed.

LEMMA 1. *If T is a ring with a finite set of generators, then for each positive integer n also T^n is a ring with a finite number of generators.*

PROOF. Write $T = \{a_1, a_2, \cdots, a_m\}$, and put $b_{i_1, i_2, \cdots, i_{n+k}} = a_{i_1} a_{i_2} \cdots a_{i_{n+k}}$, where $0 \leqq k < n$ and $1 \leqq i_j \leqq m$ for $j = 1, \cdots, n+k$; then, as may easily be seen, $T^n = \{\cdots, b_{i_1, i_2, \cdots, i_{n+k}}, \cdots\}$, q.e.d.

LEMMA 2. *If T is a semi-regular nil-ring with a finite set of generators, then $T^n \supset T^{n+1}$ for each positive integer n.*

PROOF. By Lemma 1 we may put $T^n = \{b_1, b_2, \cdots, b_r\}$. Now suppose that $T^n = T^{n+1}$, then by successive multiplication follows $T^n = T^{n+m}$ for each m, and hence by putting $T^n = W$ we have $W = W^k$ for each k. In view of $W = \{b_1, \cdots, b_r\}$ it follows easily that

$W^2=\sum_{i=1}^{r} b_i W=W$, which is a contradiction to Theorem 1.

LEMMA 3. *If P and Q are nil-ideals of a ring S and if $P\supseteq Q$, then P is a radical ideal if and only if P/Q is a radical ideal in the ring S/Q. In particular, Q is a radical ideal if and only if the zero ideal is a radical ideal in S/Q.*

PROOF. Denote by A a two-sided ideal of S so that $A\supseteq P$, then by the so-called "second law of isomorphisms" (see [4, p. 149]) we have $(A/Q)/(P/Q)\cong A/P$. Our lemma follows from the fact that A/P is nilpotent if and only if $(A/Q)/(P/Q)$ is nilpotent.

LEMMA 4. *If the lower radical of a ring S is zero and if the upper radical $U(S)$ is semi-regular, then S contains a subring T so that $U(S)\supseteq T=U(T)\supset L(T)$ and T contains an infinite number of ideals between $U(T)$ and $L(T)$ which are not radical ideals.*

PROOF. By assumption $U(S)$ contains a finite set of elements $a_1, a_2, \cdots, a_n$ so that the ring $T=\{a_1, a_2, \cdots, a_n\}$ is a semi-regular nil-ring. In view of the semi-regularity of $U(T)$ and the semi-nilpotency of the lower radical[5] it follows that $T=U(T)\supset L(T)$, and that also the quotient-ring $W=T/L(T)$ is a semi-regular nil-ring. If now b_i denotes the image of a_i in the homomorphism $T\sim W$, we may evidently write $W=\{b_1, b_2, \cdots, b_n\}$. By Lemma 2 we have $W^{m+r}\subset W^m$ for any pair of positive integers m, r, that is, the nonzero ideal W^m/W^{m+r} of the ring T/W^{m+r} is nilpotent. Consequently, W^{m+r} is not a radical ideal of W. Now define for each positive integer r the ideal A_r of T by the relations $A_r\supseteq L(T)$, $A_r/L(T)\cong W^r$, then A_r is uniquely determined, and in view of Lemma 3 none of the ideals of the infinite sequence $A_2, A_3, \cdots$ is a radical ideal, q.e.d.

LEMMA 5. *If the lower radical of a ring S is zero and if A is a nonzero semi-nilpotent ideal of S, then S contains an infinite number of ideals which are not radical ideals and which are subsets of A.*

PROOF. Denote by $a_1, a_2, \cdots, a_n$ an arbitrary finite set of nonzero elements of A and consider the nonzero ideal $A_1=(a_1, a_2, \cdots, a_n)$ of S which lies in A. As may easily be verified, we then have $A_1^3=\sum_{i=1}^{n} A_1 a_i A_1$. Now suppose that $A_1^3=A_1$, then we would have $A_1=\sum_{i=1}^{n} A_1 a_i A_1\subseteq\sum_{i=1}^{n} S a_i S\subseteq A_1$, that is, $A_1=\sum_{i=1}^{n} S a_i S$, which is a contradiction to Theorem 3 in §2. Hence we have $A_1^3\subset A_1$, which implies that the ideal A_1/A_1^3 of the ring S/A_1^3 is nilpotent, that is,

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[5] The lower radical is semi-nilpotent since it is a subset of the radical N (see footnote 2).

$B_1 = A_1^3$ is not a radical ideal. By $L(S)=0$ it follows that $B_1 \supset 0$. Since $0 \subset B_1 \subset A$, we may repeat with B_1 the same procedure, and thus (by induction) obtain an infinite sequence of semi-nilpotent ideals $B_1 \supset B_2 \supset B_3 \supset \cdots$ none of which is a radical ideal, q.e.d.

THEOREM 5. *If S is a ring with a semi-regular upper radical $U(S)$, then S contains a subring S' so that $U(S) \supseteq S' = U(S') \supset L(S')$ and so that S' contains an infinite number of ideals between $U(S')$ and $L(S')$ which are not radical ideals.*

PROOF. By Lemma 3 (see also Baer [1, p. 539]) the lower radical of $S^* = S/L(S)$ is zero, while the upper radical $U(S)/L(S) = U(S^*)$ of S^* in virtue of the semi-regularity of $U(S)$ and of the semi-nilpotency[5] of $L(S)$ is also semi-regular. Hence by Lemma 4 the ring S^* contains a subring T so that $U(S^*) \supseteq T = U(T) \supset L(T)$ and so that S^* contains an infinite number of ideals $A_1^*, A_2^*, \cdots$ between $U(T)$ and $L(T)$ which are not radical ideals. Now define a subring S' of S so that $S' \supseteq L(S)$ and $S'/L(S) \cong T$. By Lemma 3 it follows then that S' has the required properties.

THEOREM 6. *If S is a ring with a semi-nilpotent upper radical $U(S)$, and if $U(S) \supset L(S)$, then S contains an infinite number of ideals between $U(S)$ and $L(S)$ which are not radical ideals.*

PROOF. Since by Lemma 3 the lower radical of the ring $S^* = S/L(S)$ is zero, it follows by Lemma 5 that the nonzero semi-nilpotent ideal $U(S)/L(S)$ of S^* contains an infinite number of ideals which are not radical ideals. Our theorem follows now as a consequence of Lemma 3.

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SOLUTION OF A CLASS OF SINGULAR INTEGRAL EQUATIONS

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The following class of integral equations may be of some importance in the applications:

$$(1) \qquad g(x) = \frac{1}{\pi} \oint_{-1}^{1} f(\xi) \left\{ \frac{1}{\xi - x} + \sum_{n=0}^{N} c_n (\xi - x)^{2n+1} \right\} d\xi.$$

The symbol $\oint$ indicates that the principal value of the integral is to be taken and the coefficients c_n are given constants. The special case of all $c_n = 0$ has been dealt with extensively, for instance by Glauert [1], Fuchs [2], Hamel [3], Schroeder [4] and Söhngen [5].[1] The values of the coefficients c_n might be determined by the condition that a given kernel $K(\xi - x)$, for instance $K = 1/\sinh(\xi - x)$, is approximated as nearly as possible by the kernel of equation (1).

The purpose of the present note is to derive the solution of (1) for a finite number of nonvanishing c_n. The method of solution is an extension of the method applicable when all $c_n = 0$.

Equation (1) is first transformed by the substitutions

$$(2) \qquad x = \cos\phi, \qquad \xi = \cos\theta,$$

$$(3) \qquad g(x) = G(\phi), \qquad f(\xi) = F(\theta)$$

into

$$(4) \quad G(\phi) = \frac{1}{\pi} \oint_{0}^{\pi} F(\theta) \left\{ \frac{1}{\cos\theta - \cos\phi} + \sum_{n=0}^{N} c_n (\cos\theta - \cos\phi)^{2n+1} \right\} \sin\theta d\theta.$$

The function $G(\phi)$ is thought to be developed in the interval $(0, \pi)$ in the following form:

$$(5) \qquad \sin\phi\, G(\phi) = \sum_{m=1}^{\infty} B_m \sin m\phi.$$

It is then to be shown that the following representation of $F(\theta)$

$$(6) \qquad \sin\theta\, F(\theta) = \sum_{m=0}^{\infty} A_m \cos m\theta$$

permits the explicit determination of the unknown coefficients A_m in

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[1] Numbers in brackets refer to the references cited at the end of the paper.

terms of the known coefficients B_m and c_n.

Substituting equations (5) and (6) in equation (4) there occur these integrals

$$\text{(7)} \qquad \frac{1}{\pi}\fint_0^{\pi} \frac{\cos m\theta \, d\theta}{\cos\theta - \cos\phi} = \frac{\sin m\phi}{\sin\phi},$$

$$\text{(8)} \qquad \frac{1}{\pi}\int_0^{\pi} \cos m\theta(\cos\theta - \cos\phi)^{2n+1} d\theta = \sum_{k=1}^{2n+2} D_k(m, n) \frac{\sin k\phi}{\sin\phi}.$$

Equation (7) may be found in reference [1]. The validity of equation (8) with suitable coefficients D_k follows from the fact that its left side may be written as a polynomial of degree $2n+1$ in $\cos\phi$ and therefore also as a series of the form $\sum_{j=0}^{2n+1} a_j \cos j\theta$. It is important to note that the coefficients D_k satisfy the following conditions,

$$\text{(9)} \qquad D_k(m, n) = 0, \qquad m = 2n+2, 2n+3, \cdots; \ k = 2n+3, 2n+4, \cdots.$$

On the basis of equations (5) to (8) equation (4) takes on the form

$$\text{(10)} \qquad \sum_{m=1}^{\infty} B_m \sin m\phi = \sum_{m=1}^{\infty} A_m \left\{ \sin m\phi + \sum_{n=0}^{N} c_n \left[\sum_{k=1}^{2n+2} D_k(m, n) \sin k\phi \right] \right\}.$$

This is equivalent to the following set of simultaneous equations for the quantities A_m,

$$\text{(11)} \qquad B_j = A_j + \sum_{m=1}^{\infty} A_m \left\{ \sum_{n=0}^{N} c_n D_j(m, n) \right\}, \qquad j = 1, 2, \cdots.$$

But in view of equations (9) the system (11) may be written as

$$\text{(12)} \qquad \begin{aligned} &B_j = A_j, && j = 2N+3, 2N+4, \cdots, \\ &B_j = A_j + \sum_{m=1}^{2N+1} A_m \left\{ \sum_{n=0}^{N} c_n D_j(m, n) \right\}, && j = 1, 2, \cdots, 2N+2. \end{aligned}$$

Thus, it remains to solve a simultaneous system of $2N+2$ equations for the $2N+3$ unknowns $A_0, A_1, \cdots, A_{2N+2}$. In analogy to the procedure when all $c_n = 0$ we may express $A_1, \cdots, A_{2N+2}$ in terms of A_0 and leave A_0 arbitrary or determine it by an extraneous condition such as for instance $F(0) = \sum_0^{\infty} A_m = 0$.

It is to be underlined that the above reduction of the problem to a system of simultaneous equations for a finite number of unknowns depends on the fact that the regular part of the kernel in equation (1) consists of a polynomial and not of an infinite series. It is further to be noted that there may be critical values of the coefficients c_n for which equations (12) have a solution only if certain relations between the quantities B_j hold.

The following example may illustrate the foregoing results. Taking $N=1$ one finds, in this special case most easily directly from equations (4) to (7), that equations (12) become

$$\begin{aligned} B_1 &= \left(1 + \frac{c_0}{2} + \frac{3c_1}{4}\right)A_1 + \frac{c_1}{8}A_3, \\ B_2 &= \left(1 - \frac{3c_1}{8}\right)A_2 - \left(\frac{c_0}{2} + c_1\right)A_0, \qquad (13) \\ B_3 &= A_3 + \frac{3c_1}{8}A_1, \quad B_4 = A_4 - \frac{c_1}{8}A_0, \quad B_j = A_j, \quad j = 5, 6, \cdots. \end{aligned}$$

Solving for A_j, we obtain

$$\begin{aligned} A_1 &= \frac{B_1 - c_1B_3/8}{1 + c_0/2 + 3c_1/4 - 3c_1^2/64}, \qquad A_2 = \frac{B_2 + (c_0/2 + c_1)A_0}{1 - 3c_1/8}, \\ A_3 &= \frac{(1 + c_0/2 + 3c_1/4)B_3 - 3c_1B_1/8}{1 + c_0/2 + 3c_1/4 - 3c_1^2/64}, \qquad (14) \\ A_4 &= B_4 + \frac{c_1}{8}A_0, \qquad A_5 = B_5, \end{aligned}$$

and so on.

Evidently, exceptional conditions exist when one or both denominators in equation (14) have the value zero. The meaning of this occurrence is that under those circumstances equation (4) has solutions of one or both of the forms $F=\cos 2\theta/\sin\theta$, $F=(\cos\theta-\alpha\cos 3\theta)/\sin\theta$ when $G=0$.

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ON A CHARACTERISTIC PROPERTY OF LINEAR FUNCTIONS

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1. **Introduction.** If the real function $y=g(x)$, defined and continuous in the closed and bounded interval $a \leqq x \leqq b$, or in the open interval $a<x<b$, is linear there,

$$y = px + q,$$

then for all x_0, h, with $h>0$, such that x_0-h and x_0+h lie in the interval of definition, we have

$$(1) \qquad g(x_0) = [g(x_0 - h) + g(x_0 + h)]/2.$$

Conversely, if the real function $y=g(x)$, defined and continuous in $a \leqq x \leqq b$ or in $a<x<b$, satisfies (1) for all x_0, h, with $h>0$, such that x_0-h and x_0+h lie in the interval of definition, then [4, p. 189][1] $y=g(x)$ is a linear function of x.

If, however, in the converse it is given only that for each x_0, $a<x_0<b$, there exists a positive $h_0=h_0(x_0)$, such that x_0-h_0 and x_0+h_0 lie in the interval of definition, and for which we have

$$(2) \qquad g(x_0) = [g(x_0 - h_0) + g(x_0 + h_0)]/2,$$

then the implications are different in the case that $g(x)$ is defined and continuous in the closed and bounded interval and in the case that $g(x)$ is defined and continuous only in the open interval; for in the former case it still follows [3, p. 253] that $g(x)$ must be linear, while in the latter case $g(x)$ is not necessarily linear [3, pp. 253–255].

A proof of the above result, that if $g(x)$ is defined and continuous in the closed and bounded interval and satisfies (2) then $g(x)$ necessarily is linear, can be given, as we shall show, which applies equally well to characterize, in terms of equalities analogous to (2), classes of functions [1] differing, and even topologically distinct [2], from the class of linear functions.

2. **Theorem.** We shall establish the following result.

THEOREM. *Let $\{f(x)\}$ be a class of functions defined and continuous in the closed and bounded interval $a \leqq x \leqq b$, and such that for all real (x_1, y_1), (x_2, y_2) with $a \leqq x_1 < x_2 \leqq b$ there is a unique member*

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[1] Numbers in brackets refer to the references cited at the end of the paper.

$$f(x) \equiv f(x; x_1, y_1; x_2, y_2)$$

of the family $\{f(x)\}$ *satisfying*

$$f(x_1) = y_1, \qquad f(x_2) = y_2.$$

If the real function $g(x)$ *is defined and continuous in the interval* $a \leqq x \leqq b$ *and if for each* x_0 *satisfying* $a < x_0 < b$ *there is a positive* $h_0 = h_0(x_0)$, $a \leqq x_0 - h_0 < x_0 + h_0 \leqq b$, *such that*

$$(3) \qquad g(x_0) = f[x_0; x_0 - h_0, g(x_0 - h_0); x_0 + h_0, g(x_0 + h_0)],$$

then $g(x)$ *coincides with a member of the family* $\{f(x)\}$.

The above theorem contains as special case the result concerning linear functions mentioned in the last paragraph of §1.

The converse of the theorem clearly holds, so that condition (3) is both necessary and sufficient in order that a function defined and continuous in the closed and bounded interval be a member of $\{f(x)\}$.

3. **Lemmas.** We shall use the following result [1].

LEMMA 1. *For a given* x_0 *with* $a \leqq x_0 \leqq b$, *let* $f_r(x)$ *and* $f_s(x)$ *be two members of the family* $\{f(x)\}$, *such that*

$$(4) \qquad f_r(x_0) = f_s(x_0)$$

and

$$(5) \qquad f_r(x) \not\equiv f_s(x), \qquad a \leqq x \leqq b;$$

then $f_r(x) < f_s(x)$ *for all* x *in* $a \leqq x \leqq b$ *on one side of* x_0, *while* $f_r(x) > f_s(x)$ *for all* x *in* $a \leqq x \leqq b$ *on the other side of* x_0.

PROOF. By (4), (5) and the uniqueness property of the family $\{f(x)\}$, we have $f_r(x) \neq f_s(x)$ in $a \leqq x \leqq b$ except at x_0. Consequently, by the continuity of the members of $\{f(x)\}$, on either side of x_0 in $a \leqq x \leqq b$, one of $f_r(x)$ and $f_s(x)$ is greater than the other.

If $x_0 = a$ or $x_0 = b$, the theorem now follows.

If $a < x_0 < b$, we shall obtain a contradiction from the assumption that it could be the same one of $f_r(x)$ and $f_s(x)$, say $f_s(x)$, which is greater on each side of x_0.

Let x_1, x_2 satisfy $a < x_1 < x_0 < x_2 < b$, and consider the member $f_t(x)$ of $\{f(x)\}$, determined by

$$f_t(x) \equiv f[x; x_1, f_s(x_1); x_2, f_r(x_2)].$$

We have $f_t(x_2) < f_s(x_2)$, so that $f_t(x) < f_s(x)$, $x_1 < x \leqq b$; in particular,

$$(6) \qquad f_t(x_0) < f_s(x_0).$$

Similarly,

$$f_l(x_0) > f_r(x_0). \tag{7}$$

Now (6) and (7) contradict (4).

We shall use also the following lemma.

LEMMA 2. *For any positive M there is a member $f'(x)$ of $\{f(x)\}$ such that*

$$f'(x) > M, \qquad a \leqq x \leqq b. \tag{8}$$

PROOF. By the continuity of the members of $\{f(x)\}$, there is a positive h such that if $I(a)$ and $I(b)$ denote the intervals $a \leqq x < a+h$ and $b-h < x \leqq b$ respectively, then we have

$$f_0'(x) \equiv f[x;\, a,\, 2M;\, b,\, 2M] > M \tag{9}$$

for all x in $I(a)$ and for all x in $I(b)$. Similarly, for each x' in $a < x < b$ there is a positive $h = h(x')$ such that if $I(x')$ denotes the interval $x'-h < x < x'+h$, then we have

$$f[x;\, a,\, 2M;\, x',\, 2M] > M \tag{10}$$

for all x in $I(x')$.

By the Heine-Borel theorem there is a finite set $a, b, x_1', \cdots, x_n'$ of numbers in $a \leqq x \leqq b$ such that each number in $a \leqq x \leqq b$ is contained in at least one of $I(a), I(b), I(x_1'), \cdots, I(x_n')$. Hence there is a finite set

$$f_0'(x), f_1'(x), \cdots, f_n'(x) \tag{11}$$

of functions in the left-hand members of (9) and (10) such that for each x in $a \leqq x \leqq b$ we have

$$\max\,[f_0'(x), f_1'(x), \cdots, f_n'(x)] > M. \tag{12}$$

If now $f'(x)$ is one of the functions (11), such that

$$f'(b) = \max\,[f_0'(b), f_1'(b), \cdots, f_n'(b)],$$

then since $f_j'(a) = 2M$, $j = 0, 1, \cdots, m$, it follows from (12) and Lemma 1 that $f'(x)$ satisfies (8).

4. A Dedekind section. Let $f_0(x)$ be the member of $\{f(x)\}$ coinciding with $g(x)$ at $x=a$ and at $x=b$; that is,

$$f_0(x) \equiv f[x;\, a,\, g(a);\, b,\, g(b)], \tag{13}$$

and consider the members $f_\alpha(x)$ of $\{f(x)\}$ defined, for $-\infty < \alpha < +\infty$, by

$$f_\alpha(a) = g(a) + \alpha = f_0(a) + \alpha, \quad f_\alpha(b) = g(b) + \alpha = f_0(b) + \alpha.$$

Let a Dedekind section (A, B) of the real numbers be defined as follows: B contains all real numbers β such that

$$f_\beta(x) > g(x), \qquad a \leqq x \leqq b,$$

and A contains all other real numbers α.

To see that we actually have defined a Dedekind section of the real numbers, we note first that by definition all real numbers are included in A and B together. Further, A is not vacuous; for $f_0(x)$ coincides with $g(x)$ at $x=a$, so that 0 is contained in A. And by Lemma 2, B is not vacuous, since the function $g(x)$, continuous in the closed and bounded interval $a \leqq x \leqq b$, is bounded there. Finally, each α of A is less than each β of B; for if we should have $\beta_0 < \alpha_0$, then by Lemma 1 we would have

$$f_{\beta_0}(x) < f_{\alpha_0}(x), \qquad a \leqq x \leqq b,$$

and therefore, by the definition of B,

$$g(x) < f_{\beta_0}(x) < f_{\alpha_0}(x), \qquad a \leqq x \leqq b,$$

and α_0 would be a member of B.

Hence (A, B) is a Dedekind section of the real numbers and determines a real number γ. Since 0 is contained in A, it follows that

$$(14) \qquad \gamma \geqq 0.$$

In §6 we shall show that $\gamma = 0$.

5. **Properties of a function determined by the section.** The function $f_\gamma(x)$ has the following properties:

$$(15) \qquad f_\gamma(x) \geqq g(x), \qquad a \leqq x \leqq b;$$

$$(16) \qquad f_\gamma(x_0) = g(x_0)$$

for some x_0, $a \leqq x_0 \leqq b$.

If (15) did not hold, by (13) and (14) there would be an x_1, $a < x_1 < b$, such that

$$(17) \qquad f_\gamma(x_1) < g(x_1).$$

Consider the functions

$$(18) \qquad f^{(1)}(x) \equiv f[x;\, a, f_\gamma(a);\, x_1, 2f_\gamma(x_1)/3 + g(x_1)/3]$$

and

$$(19) \qquad f^{(2)}(x) \equiv f[x;\, x_1, f_\gamma(x_1)/3 + 2g(x_1)/3;\, b, f^{(1)}(b)],$$

and let δ be defined by

$$\delta = \min\, [f^{(2)}(a) - g(a), f^{(2)}(b) - g(b)]. \tag{20}$$

By (17), (18), (19) and Lemma 1, we have $\delta > \gamma$, so that δ is contained in the set B. But by (17), (19), (20) and Lemma 1, we have

$$f_\delta(x_1) < g(x_1),$$

whence, by the definition of A, δ is contained in the set A. But δ can not be contained both in A and in B; this contradiction establishes (15).

If (16) did not hold, by (15) we would have

$$f_\gamma(x) > g(x), \qquad a \leqq x \leqq b. \tag{21}$$

Since the members of $\{f(x)\}$ are continuous, by (21) there is a positive k such that if $J(a)$ and $J(b)$ denote the intervals $a \leqq x < a+k$ and $b-k < x \leqq b$ respectively, then

$$f_0^*(x) \equiv f[x;\, a, f_\gamma(a);\, b, g(b)/2 + f_\gamma(b)/2] > g(x) \tag{22}$$

for all x in $J(a)$ and for all x in $J(b)$. Similarly, for each x^* in $a < x < b$ there is a positive $k = k(x^*)$ such that if $J(x^*)$ denotes the interval $x^* - k < x < x^* + k$, then

$$f[x;\, a, f_\gamma(a);\, x^*, g(x^*)/2 + f_\gamma(x^*)/2] > g(x) \tag{23}$$

for all x in $J(x^*)$.

By the Heine-Borel theorem there is a finite set $a, b, x_1^*, \cdots, x_m^*$ of numbers in $a \leqq x \leqq b$ such that each number in $a \leqq x \leqq b$ is contained in at least one of $J(a)$, $J(b)$, $J(x_1^*)$, $\cdots$, $J(x_m^*)$. Hence there is a finite set

$$f_0^*(x), f_1^*(x), \cdots, f_m^*(x) \tag{24}$$

of functions in the left-hand members of (22) and (23) such that for each x in $a \leqq x \leqq b$ we have

$$\max\, [f_0^*(x), f_1^*(x), \cdots, f_m^*(x)] > g(x). \tag{25}$$

If now $f^*(x)$ is one of the functions (24) such that

$$f^*(b) = \max\, [f_0^*(b), f_1^*(b), \cdots, f_m^*(b)],$$

then since $f_j^*(a) = f_\gamma(a)$, $j = 0, 1, \cdots, m$, it follows from (25) and Lemma 1 that $f^*(x)$ satisfies

$$f^*(x) > g(x), \qquad a \leqq x \leqq b. \tag{26}$$

By (21), (22), (23) and Lemma 1, we have

$$f_j^*(x) < f_\gamma(x), \qquad j = 0, 1, \cdots, m;\ a < x \leqq b;$$

in particular,

$$(27) \qquad f^*(x) < f_\gamma(x), \qquad a < x \leqq b,$$

with

$$(28) \qquad f^*(a) = f_\gamma(a).$$

Starting with (26) in place of (21), and interchanging the rôles of a and b, we can determine analogously the existence of a member $f''(x)$ of $\{f(x)\}$, such that

$$(29) \qquad f''(x) > g(x), \qquad a \leqq x \leqq b,$$

and

$$(30) \qquad f''(x) < f^*(x), \qquad a \leqq x < b,$$

with

$$(31) \qquad f''(b) = f^*(b).$$

From (27), (28), (30) and (31) we obtain

$$(32) \qquad f''(x) < f_\gamma(x), \qquad a \leqq x \leqq b.$$

Let ϵ be defined by

$$(33) \qquad \epsilon = \max\,[f''(a) - g(a), f''(b) - g(b)].$$

By (29) and (32) we have

$$(34) \qquad 0 < \epsilon < \gamma.$$

Hence from (29), (33) and Lemma 1 we obtain

$$f_\epsilon(x) > g(x), \qquad a \leqq x \leqq b,$$

so that ϵ is contained in B. But by (34) and the definition of γ, ϵ is contained in A; this contradiction establishes (16).

We note that (15) and (16) might also have been established by proving first that

$$\lim_{\xi \to \gamma} f_\xi(x) = f_\gamma(x), \qquad a \leqq x \leqq b.$$

6. **The value of γ.** By (14) we have either

$$(35) \qquad \gamma = 0$$

or

$$(36) \qquad \gamma > 0.$$

If (36) holds, then by the definition of $f_\gamma(x)$ we have

$$f_\gamma(a) > g(a), \qquad f_\gamma(b) > g(b). \tag{37}$$

By the continuity of $f_\gamma(x)$ and of $g(x)$, there is a greatest number x_0 in $a \leqq x \leqq b$ satisfying (16); and by (37), this greatest x_0 satisfies $a < x_0 < b$. Then

$$f_\gamma(x) > g(x), \qquad x_0 < x \leqq b. \tag{38}$$

By hypothesis there is a positive number $h_0 = h_0(x_0)$, $a \leqq x_0 - h_0 < x_0 + h_0 \leqq b$, for which (3) is satisfied.

From (15) we obtain

$$f_\gamma(x_0 - h_0) \geqq g(x_0 - h_0),$$

and from (38) we get

$$f_\gamma(x_0 + h_0) > g(x_0 + h_0).$$

Hence by Lemma 1 we have

$$f_\gamma(x) > f[x;\, x_0 - h_0,\, g(x_0 - h_0);\, x_0 + h_0,\, g(x_0 + h_0)]$$

for $x_0 - h < x \leqq x_0 + h$; in particular,

$$f_\gamma(x_0) > f[x_0;\, x_0 - h_0,\, g(x_0 - h_0);\, x_0 + h_0,\, g(x_0 + h_0)]. \tag{39}$$

Now (39) contradicts (3) and (16), so that (36) does not hold, and therefore the value of γ is given by (35).

7. **Proof of the theorem.** We shall establish the theorem by showing that

$$g(x) \equiv f_0(x), \qquad a \leqq x \leqq b. \tag{40}$$

Suppose that (40) does not hold; that is, that we have

$$g(x) \not\equiv f_0(x), \qquad a \leqq x \leqq b. \tag{41}$$

We shall obtain a contradiction.

By (13) and (41) there is an $\bar{x}$, $a < \bar{x} < b$, such that

$$g(\bar{x}) \neq f_0(\bar{x});$$

and we can suppose that

$$g(\bar{x}) > f_0(\bar{x}), \tag{42}$$

since otherwise we could consider the function $-g(x)$ and the family $\{-f(x)\}$.

From (15) and (35) we obtain

$$f_0(x) \geqq g(x), \qquad a \leqq x \leqq b. \tag{43}$$

Since (43) contradicts (42), the assumption that (40) does not hold has led to a contradiction.

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THE UNIVERSITY OF TEXAS

NOTE ON A CERTAIN CONTINUED FRACTION

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The continued fraction

$$\cfrac{1}{1+\cfrac{az}{1+\cfrac{bz}{1+\cfrac{(a+1)z}{1+\cfrac{(b+1)z}{1+\cfrac{(a+2)z}{1+\cdots}}}}}} \tag{1}$$

is a limiting case of the continued fraction of Gauss, and is the formal expansion of the quotient $\Omega(a, b; z)/\Omega(a, b-1; z)$, where

$$\Omega(a, b; z) = 1 - ab\frac{z}{1!} + a(a+1)b(b+1)\frac{z^2}{2!} + \cdots . \tag{2}$$

If a and b are real and positive, then it follows from the work of Stieltjes that (1) converges in the domain Z exterior to the negative

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half of the real axis, and its value is an analytic function of z in this domain. It is easy to show that the same holds for a and b any real numbers, except that the function may have poles in the domain Z. By means of technique which has been developed in recent years, this result may be extended to arbitrary complex a and b. We shall first prove the following theorem.

THEOREM 1. *Let A and B denote two arbitrary bounded regions of the complex plane. Then, there exists a number $\delta > 0$, depending upon A and B, such that the continued fraction* (1) *converges uniformly for a in A, b in B and z in the real interval* $(0, \delta)$.

PROOF. We may evidently choose $\delta > 0$ sufficiently small in order that the numbers $(a+p)z$, $(b+p)z$, $p=0, 1, 2, \cdots$, will be in the parabolic region $|w| - R(w) \leqq 1/2$ for all z in the interval $(0, \delta)$. The convergence then follows from the *parabola theorem* [3, p. 166].[1] The uniformity of the convergence follows from the fact that the approximants are uniformly bounded: their values are all in the circle with center 1 and radius 1 [4, p. 581].

THEOREM 2. *Let a and b be arbitrary complex constants not* 0 *or a negative integer.*[2] *Let G be any bounded closed region within the domain Z defined above. The continued fraction* (1) *converges over G except possibly at certain isolated points, and uniformly over the region obtained from G by removing the interiors of small circles with centers at these points. The value of the continued fraction is an analytic function having these points as poles.*

PROOF. Let A and B of Theorem 1 be the single points a and b, and choose the number $\delta > 0$ accordingly. We may suppose that G is a connected region containing the interval $(\delta/2, \delta)$ on the interior. Let $h > 0$ be chosen sufficiently small in order that G will be contained within the cardioid region

$$| w | \leqq \frac{1}{2h^2}(1 + \cos \theta), \qquad w = | w | e^{i\theta}.$$

Next choose N so that for $n > N$ the numbers $(a+n)$, $(b+n)$ will be in the parabolic region

$$| w | - R(w) \leqq h^2/2.$$

If $n > N$, the continued fraction

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

[2] If a or b is 0 or a negative integer, the continued fraction breaks off and is a rational fraction.

$$\cfrac{1}{1+\cfrac{(a+n)z}{1+\cfrac{(b+n)z}{1+\cfrac{(a+n+1)z}{1+\cfrac{(b+n+1)z}{1+\cdots}}}}}$$

is then uniformly convergent over G by the *cardioid theorem* [1, pp. 367–368]. It follows that (1) converges over G except possibly at certain isolated points, or else diverges to the constant ∞. Inasmuch as it converges for z in the real interval $(\delta/2, \delta)$, the latter alternative is ruled out. The convergence is evidently uniform over the region obtained from G by deleting small circular neighborhoods of the aforementioned isolated points.

In order to express the analytic function represented by the continued fraction in terms of integrals, we write

$$\begin{aligned}
&\Omega(a, b; z)\\
&=1+\sum_{p=1}^{\infty}(-1)^p a(a+1)\cdots(a+p-1)b(b+1)\cdots(b+p-1)\frac{z^p}{p!}\\
&=\frac{1}{\Gamma(a)}\sum_{p=0}^{\infty}C_{-b,p}\Gamma(a+p)z^p\\
&=\frac{1}{\Gamma(a)}\sum_{p=0}^{\infty}C_{-b,p}\int_0^{\infty}e^{-u}u^{a+p-1}du\, z^p\\
&=\frac{1}{\Gamma(a)}\int_0^{\infty}\frac{e^{-u}u^{a-1}du}{(1+zu)^b}.
\end{aligned}$$

This *formal* procedure suggests the possibility that (1) has the value

$$\int_0^{\infty}\frac{e^{-u}u^{a-1}du}{(1+zu)^b}\Big/\int_0^{\infty}\frac{e^{-u}u^{a-1}du}{(1+zu)^{b-1}} \tag{3}$$

whenever these integrals converge, that is, for z in Z and $R(a)>0$. Now, this is known to be true for a, b and z real and positive [2, p. 492], and the extension to complex z in Z is immediate inasmuch as (1) and (3) are both analytic functions of z over Z. Regarding a and b as variables, and using Theorem 1, the same conclusion can be extended immediately to complex values of a and b.

In the special case $b=1$:

$$\frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-u}u^{a-1}du}{1+zu} = \cfrac{1}{1+\cfrac{az}{1+\cfrac{1\cdot z}{1+\cfrac{(a+1)z}{1+\cfrac{2\cdot z}{1+\cdot_{\ddots}}}}}}, \tag{4}$$

for $R(a)>0$ and z in Z. The left-hand member can be transformed into an integral which converges for *all* a, and we get

$$\int_0^\infty \frac{e^{-u}du}{(z+u)^a} = \cfrac{z^{1-a}}{z+\cfrac{a}{1+\cfrac{1}{z+\cfrac{(a+1)}{1+\cfrac{2}{z+\cfrac{(a+2)}{1+\cfrac{3}{z+\cdot_{\ddots}}}}}}}}, \tag{5}$$

valid for all a and for z in Z. Let $z=x$, real and positive, make the change of variable $v=u+x$ in the integral, and then replace a by $1-a$. This gives

$$\int_x^\infty e^{-v}v^{a-1}dv = \cfrac{e^{-x}x^a}{x+\cfrac{1-a}{1+\cfrac{1}{x+\cfrac{2-a}{1+\cfrac{2}{x+\cfrac{3-a}{1+\cfrac{3}{x+\cdot_{\ddots}}}}}}}}, \tag{6}$$

valid for all a and for $x>0$. The special cases $a=0$ and $a=1/2$ furnish expansions for the integrals

$$\int_0^{e^{-x}} \frac{du}{\log u} \quad \text{and} \quad \int_x^{\infty} e^{-u^2} du,$$

respectively. The latter gives immediately the expansion

$$2\int_0^x e^{-u^2}du = \pi^{1/2} - \cfrac{e^{-x^2}}{x + \cfrac{1}{2x + \cfrac{2}{x + \cfrac{3}{2x + \cfrac{4}{x + \cdots}}}}}, \tag{7}$$

valid for $x>0$. For a discussion of these formulas, with references, see [2, pp. 296–298].

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A NOTE ON THE ZEROS OF THE SECTIONS OF A PARTIAL FRACTION

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1. **Introduction.** If $f(z)$ is a rational function with a total of three distinct zeros and poles, the zeros of its logarithmic derivative may be located as points in the complex plane by aid of the following theorem.

THEOREM 1. *The zeros of the partial fraction*

$$F(z) = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3}, \qquad m_1 m_2 m_3 \neq 0,$$

where z_1, z_2 *and* z_3 *are three distinct, noncollinear points, lie at the foci of the conic which touches the line segments* (z_2, z_3), (z_3, z_1) *and* (z_1, z_2) *in the points* ζ_1, ζ_2 *and* ζ_3 *that divide these segments in the ratio* $m_2:m_3$, $m_3:m_1$, *and* $m_1:m_2$ *respectively. If* $n = m_1 + m_2 + m_3 \neq 0$, *this conic is an ellipse or hyperbola according as* $nm_1m_2m_3 > 0$ *or* < 0. *If* $n = 0$, *the conic is a parabola whose axis is parallel to the line joining the origin to the point* $\nu = m_1z_1 + m_2z_2 + m_3z_3$.

In the special case $m_1 = m_2 = m_3 = 1$, this theorem was proved geometrically by Bôcher and Grace.[1] In the general case it was first deduced by Linfield as a corollary to the following theorem which in turn was established by the use of line coordinates and polar forms.[2]

THEOREM 2. *The zeros of the partial fraction* $F(z) = \sum_{j=1}^{p} m_j/(z - z_j)$ *lie at the foci of the curve* $C(z_1, z_2, \cdots, z_p; m_1, m_2, \cdots, m_p)$ *of class* $p-1$ *which touches each of the* $p(p-1)/2$ *line-segments* (z_j, z_k) *in a point dividing it in the ratio* $m_j:m_k$.

In view, however, of the elementary character of Theorem 1, it

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[1] M. Bôcher, Ann. of Math. vol. 7 (1892) pp. 70–76; J. H. Grace, Proc. Cambridge Philos. Soc. vol. 11 (1901) pp. 352–357.

[2] For the case that all $m_j > 0$, see Siebeck, J. Reine Angew. Math. vol. 64 (1864) p. 175; M. Van den Berg, Niew Archief voor Wiskunde vol. 9 (1882) pp. 1–14, 60, vol. 11 (1884) pp. 153–186, vol. 15 (1899) pp. 100–164; J. Juhel-Renjoy, C. R. Acad. Sci. Paris vol. 142 (1906); P. J. Heawood, Quart. J. Math. vol. 38 (1907) pp. 84–107; and M. Fujiwara, Tôhoku Math. J. vol. 9 (1916) pp. 102–108. It is to be observed that, although priority for the theorem when all $m_j > 0$ is usually accorded to Van den Berg, it should rightfully be given to Siebeck. For arbitrary integral m_j, see B. Z. Linfield, Bull. Amer. Math. Soc. vol. 27 (1920) pp. 17–21 and Trans. Amer. Math. Soc. vol. 25 (1923) pp. 239–258.

seems desirable to furnish for it an elementary proof based upon some familiar property of the conics. In the next paragraph we shall construct such a proof, based upon the optical properties of the conics and upon some apparently new propositions concerning the relative positions of the zeros of the partial fraction $F(z)$, and those of its two sections, $F_1(z)=\sum_1^k m_j(z-z_j)^{-1}$ and $F_2(z)=\sum_{k+1}^p m_j(z-z_j)^{-1}$. These propositions, which will be derived in §§2 and 3 by the aid of very simple analysis, will also be used in §4 together with Theorem 2 to obtain some relations among the foci of the curves $C(z_1, z_2, \cdots, z_k; m_1, m_2, \cdots, m_k)$, $C(z_{k+1}, \cdots, z_p; m_{k+1}, \cdots, m_p)$ and $C(z_1, z_2, \cdots, z_p; m_1, m_2, \cdots, m_p)$. Finally, in §5, applications of the method will be indicated for nonlinear, partial fractions.

2. **Proof of Theorem 1.** Assuming $n\neq 0$ and $n_3=m_1+m_2\neq 0$, let us write

$$F(z) = \frac{n(z - Z_1)(z - Z_2)}{(z - z_1)(z - z_2)(z - z_3)}. \tag{2.1}$$

Since by definition

$$n_3\zeta_3 = m_2z_1 + m_1z_2, \tag{2.2}$$

it follows that ζ_3 is the zero of the partial fraction $F_1(z)=m_1/(z-z_1)+m_2/(z-z_2)$. Hence,

$$\frac{\zeta_3 - z_3}{n}F(\zeta_3) = \frac{m_3}{n} = \frac{(Z_1 - \zeta_3)(Z_2 - \zeta_3)}{(z_1 - \zeta_3)(z_2 - \zeta_3)},$$

$$\arg (Z_1 - \zeta_3)/(z_1 - \zeta_3) \equiv \arg (z_2 - \zeta_3)/(Z_2 - \zeta_3) \pmod{2\pi} \text{ if } nm_3 > 0, \tag{2.3}$$

$$\arg (Z_1 - \zeta_3)/(z_1 - \zeta_3) \equiv \arg (\zeta_3 - z_2)/(Z_2 - \zeta_3) \pmod{2\pi} \text{ if } nm_3 < 0. \tag{2.4}$$

By clearing equation (2.1) of fractions and equating the coefficients of z on both sides of the resulting equation, we learn that

$$(Z_1 + Z_2)/2 = (n_1z_1 + n_2z_2 + n_3z_3)/n$$

where $n_1=m_2+m_3$, $n_2=m_3+m_1$ and $n_3=m_1+m_2$. Since by hypothesis $n_3\neq 0$, the point $(Z_1+Z_2)/2$ cannot lie on the segment (z_1, z_2) and hence not both Z_1 and Z_2 may lie on this segment.

If ζ_3 is an interior division point of the line segment (z_1, z_2) and thus $m_1m_2>0$, we learn from equations (2.3) and (2.4) that the line (z_1, z_2) makes equal angles with lines (Z_1, ζ_3) and (Z_2, ζ_3) and does or does not separate the points Z_1 and Z_2 according as $nm_3<0$ or >0.

Likewise, if ζ_3 is an exterior division point of the line segment (z_1, z_2) and thus $m_1m_2<0$, we learn from (2.3) and (2.4) that the line (z_1, z_2) makes equal angles with the lines (Z_1, ζ_3) and (Z_2, ζ_3) and does or does not separate the points Z_1 and Z_2 according as $nm_3>0$ or <0. Hence, using the optical properties of the conics, we infer that at point ζ_3 the line (z_1, z_2) is tangent to a conic with foci at points Z_1 and Z_2 and that this conic is an ellipse or hyperbola according as $nm_1m_2m_3>0$ or <0.

Now, by merely permuting the subscripts 1, 2, and 3, we may complete the proof of the part of the theorem involving the ellipse and hyperbola.

To prove the part involving the parabola, let us, since $n=0$, write instead of (2.1) the expression

$$F(z) = \frac{\nu(z - Z_1)}{(z - z_1)(z - z_2)(z - z_3)}.$$

The quantity $\nu=m_1z_1+m_2z_2+m_3z_3\neq 0$ because, by hypothesis, the points z_1, z_2 and z_3 are noncollinear. Since now ν replaces the factor $(z-Z_2)$, a repetition of the above details would prove at once that line (z_1, z_2) is tangent at point ζ_3 to the parabola described in Theorem 1.

Remark. It is obvious from the proof that Theorem 1 holds not only, as usually asserted, when the m_j are integers but also when they are arbitrary real numbers. Using this fact, we may extend our theorem to the case that, for example, $n_3=m_1+m_2=0$ but $n\neq 0$. For, taking n_3 very small, we find that $nm_3>0$, $m_1m_2<0$, and, hence, $nm_1m_2m_3<0$. The corresponding conic is therefore a hyperbola which is tangent at ζ_3 to the line (z_1, z_2). Now, as $n_3\to 0$, the point ζ_3 goes to infinity and, hence, the line (z_1, z_2) becomes an asymptote of the hyperbola.

3. **Sections of a linear partial fraction.** In order to generalize Theorem 1, let us now assume that $n=m_1+m_2+\cdots+m_p\neq 0$, $n_1=m_1+m_2+\cdots+m_k\neq 0$, $n_2=n-n_1\neq 0$.

Let us write

$$F(z) = \sum_{j=1}^{p} \frac{m_j}{z - z_j} = \frac{n(z - Z_1)(z - Z_2)\cdots(z - Z_{p-1})}{(z - z_1)(z - z_2)\cdots(z - z_p)},$$

$$F_1(z) = \sum_{j=1}^{k} \frac{m_j}{z - z_j} = \frac{n_1(z - z_1')(z - z_2')\cdots(z - z_{k-1}')}{(z - z_1)(z - z_2)\cdots(z - z_k)},$$

$$F_2(z) = \sum_{j=k+1}^{p} \frac{m_j}{z - z_j} = \frac{n_2(z - z_1'')(z - z_2'')\cdots(z - z_{p-k-1}'')}{(z - z_{k+1})(z - z_{k+2})\cdots(z - z_p)}.$$

Since $F(z)=F_1(z)+F_2(z)$, it follows that

$$\begin{aligned} n(z-Z_1)&\cdots(z-Z_{p-1}) \\ &= n_1(z-z_1')\cdots(z-z_{k-1}')(z-z_{k+1})\cdots(z-z_p) \\ &\quad + n_2(z-z_1)\cdots(z-z_k)(z-z_1'')\cdots(z-z_{p-k-1}''). \end{aligned}$$

Hence, if z'' is any z_j'' or a point z_j with $1\leqq j\leqq k$, and if Z is any point Z_j, we find that

$$(3.1)\qquad \frac{(z''-Z_1)\cdots(z''-Z_{k-1})(z''-Z_k)\cdots(z''-Z_{p-1})}{(z''-z_1')\cdots(z''-z_{k-1}')(z''-z_{k+1})\cdots(z''-z_p)}=\frac{n_1}{n},$$

$$(3.2)\qquad \frac{(Z-z_1')\cdots(Z-z_{k-1}')(Z-z_{k+1})(Z-z_{k+2})\cdots(Z-z_p)}{(Z-z_1)\cdots(Z-z_{k-1})(Z-z_k)(Z-z_1'')\cdots(Z-z_{p-k-1}'')}=-\frac{n_2}{n_1}.$$

Now, taking the moduli and amplitudes of equations (3.1) and (3.2), we derive the following theorem.

THEOREM 3. *Given the partial fraction* $F(z)=\sum_1^p m_j(z-z_j)^{-1}$ *and its sections* $F_1(z)=\sum_1^k m_j(z-z_j)^{-1}$ *and* $F_2(z)=\sum_{k+1}^p m_j(z-z_j)^{-1}$, *where* $n_1=\sum_1^k m_j\neq 0$, $n_2=\sum_{k+1}^p m_j\neq 0$, *and* $n=n_1+n_2\neq 0$. *Denote by S the set of $p-1$ zeros of $F(z)$; by S_1 the set of $p-1$ points comprised of the $k-1$ zeros of $F_1(z)$ and the $p-k$ points $z_{k+1}, z_{k+2}, \cdots, z_p$; and, by S_2, the set of $p-1$ points comprised of the k points $z_1, z_2, \cdots, z_k$ and the $p-k-1$ zeros of $F_2(z)$.*

A. *Then, if z'' is any point of S_2, the product of its distances to the points of S is $|n_1/n|$ times the product of its distances to the points of S_1. Also, if each point in S is paired in any manner with a point in S_1, the sum of the angles subtended at z'' by the $p-1$ pairs of points is zero or* π (mod 2π) *according as $n_1n>0$ or <0.*

B. *Further, if Z is any point of S, the product of its distances to the points of S_1 is $|n_2/n_1|$ times the product of its distances to the points in S_2. Also, if each point in S_1 is paired in any manner with a point in S_2, the sum of the angles subtended at Z by the $p-1$ pairs of points is zero or* π (mod 2π) *according as $n_1n_2<0$ or >0.*[3]

For example, if $p=4$ and $k=2$, then $F_1(z)$ has as its only zero the point z' dividing the line segment (z_1, z_2) in the ratio $m_1:m_2$ and $F_2(z)$ has as its only zero the point z'' dividing the line segment (z_3, z_4) in

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[3] In the case $m_1=m_2=\cdots=m_p=k=1$, the set S_2 consists of the point z_1 and the $p-2$ zeros of $F_2(z)$. Bôcher (see footnote 1) states without proof Theorem 3, but takes the point z'' to be only the point z_1, making no mention of the fact that z'' may be also any of the other $p-2$ points of S_2. In Theorem 3, the sense in which an angle subtended by a line segment is to be measured may be determined from formulas (3.1) and (3.2).

the ratio $m_3:m_4$. Using part B of Theorem 3, we may prove that the sum of the angles subtended in any zero Z of $F(z)$ by the line segments (z', z_1) and (z_4, z'') is the supplement of the angle subtended in Z by the segment (z_2, z_3). Likewise, by using part A, we may prove that, if ψ denotes double the angle from the line (z_3, z_4) to the line (z_1, z_2), then $(\psi-\pi)$ equals the sum of the angles subtended by the line segment (z', z'') in the zeros Z_1, Z_2, and Z_3 of $F(z)$ (mod 2π).

4. **Geometric interpretations of Theorem 3.** First, Theorem 3 may be restated in terms of two types of generalizations of the circle which were studied by Darboux.[4] One is the so-called lemniscate, the locus of a point which moves so that the product of its distances from one set of fixed points (called poles) is a constant multiple of the product of its distances from another set of fixed points (also called poles). The other is the so-called stelloid, the locus of a point which moves so that the sum of the angles subtended in it by a set of pairs of fixed points (called poles) is constant. These two families are respectively the equipotential curves and lines of force in the field due to unit attractive particles at the points of the one set and unit repulsive particles at the points of the other set. Thus, according to Theorem 3, *the points of S_2 lie at the intersections of a certain lemniscate and a certain stelloid, both of which have the points of S and S_1 as poles*, whereas *the points of S lie at the intersections of a lemniscate and a stelloid, both of which have the points of S_1 and S_2 as poles.*

Secondly, Theorem 3 may be used in combination with Theorem 2. By Theorem 2, the points $Z_1, Z_2, \cdots, Z_{p-1}$ are the foci of the curve $C_0 \equiv C(z_1, \cdots, z_p; m_1, \cdots, m_p)$ of class $p-1$; the points $z_1', z_2', \cdots, z_{k-1}'$ the foci of the curve $C_1 \equiv C(z_1, \cdots, z_k; m_1, \cdots, m_k)$ of class $k-1$; and the points $z_1'', z_2'', \cdots, z_{p-k-1}''$ the foci of the curve $C_2 \equiv C(z_{k+1}, \cdots, z_p; m_{k+1}, \cdots, m_p)$ of class $p-k-1$. Thus, if $p=6$ and $k=3$ and all $m_j>0$, the points z_1' and z_2' are the foci of the ellipse which touches the line segments (z_1, z_2), (z_2, z_3) and (z_3, z_1) in points dividing these segments in the respective ratios $m_1:m_2$, $m_2:m_3$ and $m_3:m_1$, while the points z_1'' and z_2'' are the foci of the ellipse which touches the line segments (z_4, z_5), (z_5, z_6) and (z_6, z_4) in the points dividing these segments in the respective ratios $m_4:m_5$, $m_5:m_6$ and $m_6:m_4$.

The set S of Theorem 3 consists thus of the foci of the curve C_0. The set S_1 consists thus of the points $z_{k+1}, z_{k+2}, \cdots, z_p$ and the foci of the curve C_1 or, what amounts to the same, the foci of the degenerate curve $C' \equiv C(z_1, \cdots, z_k, z_{k+1}, \cdots, z_p; m_1, \cdots, m_k, 0, \cdots, 0)$ of class $p-1$. Similarly the set S_2 may be regarded as consisting of the foci of the degenerate curve $C'' = C(z_1, \cdots, z_k, z_{k+1}, \cdots, z_p; 0, \cdots, 0, m_{k+1}, \cdots, m_p)$, also of class $p-1$. Thus, *Theorem* 3 *expresses two geometric relations regarding the foci of the three curves* C_0, C' and C'' *of class* $p-1$.

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[4] G. Darboux, *Sur une classe remarquable de courbes et de surfaces algebrique*, Paris, 1873, pp. 66–73.

5. **Further generalizations.** It is obvious that Theorem 3 may be extended at once to the more general partial fractions

$$F(z) = \sum_{j=1}^{p} \frac{m_j(z - a_{j1}) \cdots (z - a_{jr})}{(z - b_{j1}) \cdots (z - b_{js})}.$$

For example, let us take all $m_j > 0$, and

$$F(z) = \sum_{j=1}^{3} \frac{m_j}{(z - z_j)^q}, \qquad F_1(z) = \frac{m_1}{(z - z_1)^q}, \qquad F_2(z) = F(z) - F_1(z).$$

The zeros of $F_2(z)$ are the roots of the equations

$$z - z_2/z - z_1 = (m_2/m_1)^{1/q} \exp\left[\pi(2k + 1)i/q\right],$$

where $k = 0, 1, 2, \cdots, q-1$. These zeros may therefore be located as the intersections of the circle $|z - z_2| = (m_2/m_1)^{1/q}|z - z_1|$ with the q arcs of circles $\arg (z - z_2)/(z - z_1) = \pi(2k+1)/q$, $k = 0, 1, 2, \cdots, q-1$. On the other hand, if ζ is any zero of $F_2(z)$,

$$(5.1) \qquad \frac{(\zeta - z_1)^q}{n} F(\zeta) = \frac{m_1}{n} = \frac{(\zeta - Z_1) \cdots (\zeta - Z_{2q})}{(\zeta - z_2)^q(\zeta - z_3)^q},$$

where $Z_1, Z_2, \cdots, Z_{2q}$ are the zeros of $F(z)$ and $n = m_1 + m_2 + m_3$. From equation (5.1) we may thus conclude that *the sum of the angles subtended in* ζ *by the segments from* z_2 *to any* q *of the points* Z_j *equals* (modulo 2π) *the sum of the angles subtended in* ζ *by the segments from the remaining* Z_j to z_3.

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ASYMPTOTIC DISTRIBUTION OF SUMS OF RADEMACHER FUNCTIONS

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1. **Introduction.** The central limit theorem of the calculus of probability applied to the special case of the Rademacher functions $r_k(t) = \text{sign}\,(\sin 2^k\pi t)$ can be stated as follows:

$$\lim_{n\to\infty}\left| \underset{t}{E}\left\{\frac{1}{(2n)^{1/2}}\left|\sum_{k=1}^{n} r_k(t)\right| < a\right\}\right| = \frac{1}{\pi^{1/2}}\int_{-a}^{a} e^{-u^2}du.$$

This form of the theorem suggests two generalizations, the first of which consists in replacing the constant a by a function $f(t)$. The second generalization is obtained by replacing the constant upper limit of summation n by a sequence of integral-valued functions $N_n(t)$, so that the number of Rademacher functions in the sum varies with t. These are proved by combining the standard techniques of the calculus of probability with the methods of orthogonal series. Both proofs make use of the continuity theorem of Fourier-Stieltjes transforms:

If $f(t) = \lim_{n\to\infty} f_n(t)$, *and if*

$$\lim_{n\to\infty}\int_0^1 \exp\,[ixf_n(t)]dt = e^{-x^2/4}$$

uniformly in x, *then*

$$\lim_{n\to\infty}\left| \underset{t}{E}\left\{\,|f_n(t)| < a\right\}\right| = \frac{1}{\pi^{1/2}}\int_{-a}^{a} e^{-u^2}du.$$

Essential use is made of the Walsh-Kaczmarz system[1] of orthonormal functions, which have two useful properties: (1) each of them is equal to a finite product of Rademacher functions, and (2) any Lebesgue-square integrable function can be expanded in a series of these functions, which series will converge in the mean with index two.

2. **First generalization.** The first theorem expresses in asymptotic form the measure of the point set over which the sum of the Rademacher functions is bounded by a given function.

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[1] Cf. S. Kaczmarz and H. Steinhaus, *Le système orthogonal de M. Rademacher*, Studia Mathematica vol. 2 (1930) and J. L. Walsh, *A closed set of normal orthogonal functions*, Amer. J. Math. vol. 45 (1923) pp. 5–24.

THEOREM. *If $f(t)$ is a non-negative, measurable function in the interval* (0, 1), *then*

$$\lim_{n\to\infty}\left|\mathop{E}_{t}\left\{\frac{1}{(2n)^{1/2}}\left|\sum_{k=1}^{n}r_k(t)\right|<f(t)\right\}\right|=\frac{1}{\pi^{1/2}}\int_0^1\int_{-f(t)}^{f(t)}e^{-u^2}du\,dt.$$

PROOF. Let $F_n(t)$ be $(2n)^{-1/2}\sum_{k=1}^{n}r_k(t)$ and let $S_r(t)$ be the partial sum of the expansion of a Lebesgue-square integrable function $h(t)$ in a series of Walsh-Kaczmarz functions. It follows from the first property of these functions that

$$\lim_{n\to\infty}\int_0^1 S_r(t)\exp\left[ixF_n(t)\right]dt=e^{-x^2/4}\int_0^1 h(t)dt.$$

In view of the second property of the Walsh-Kaczmarz functions, this implies that

$$\lim_{n\to\infty}\int_0^1 h(t)\exp\left[ixF_n(t)\right]dt=e^{-x^2/4}\int_0^1 h(t)dt.$$

If $h(t)$ be the characteristic function of an interval E, then

$$\lim_{n\to\infty}\int_E \exp\left[ixF_n(t)\right]dt=|E|\cdot e^{-x^2/4}.$$

By virtue of the continuity theorem of Fourier-Stieltjes transforms, it can be demonstrated that

$$\lim_{n\to\infty}\left|\mathop{E}_{t}\left\{t\in E;\ |F_n(t)|<a\right\}\right|=\frac{|E|}{\pi^{1/2}}\int_{-a}^{a}e^{-u^2}du.$$

Hence, if $g(t)$ be a non-negative step-function having the constant value a_k over the interval E_k, then

$$\lim_{n\to\infty}\left|\mathop{E}_{t}\left\{|F_n(t)|<g(t)\right\}\right|=\lim_{n\to\infty}\sum_{k=1}^{\infty}\left|\mathop{E}_{t}\left\{t\in E_k;\ |F_n(t)|<a_k\right\}\right|.$$

Inspection shows that the limit on the right is equal to

$$\frac{1}{\pi^{1/2}}\int_0^1\int_{-g(t)}^{g(t)}e^{-u^2}du\,dt.$$

The theorem now follows by considering $f(t)$ to be the limit of properly chosen step-functions.

3. **Second generalization.** In order to obtain the generalization of the central limit theorem in which the upper limit of summation of

the Rademacher functions is a given function, we must properly qualify this function.

THEOREM. *Let* $N_n(t)=nN(t)+Q_n(t)$, *where* $N_n(t)$ *and* $N(t)$ *assume as values only zero and the positive integers and where* $N(t)$ *is Lebesgue integrable, with* $p=\int_0^1 N(t)dt$. *If* $\lim_{n\to\infty} n^{-1/2}|Q_n(t)|=0$, *then for any non-negative* a,

$$\lim_{n\to\infty}\left| \underset{t}{E}\left\{\frac{1}{(2np)^{1/2}}\left|\sum_{k=1}^{N_n(t)} r_k(t)\right| < a\right\}\right| = \left(\frac{p}{\pi}\right)^{1/2}\sum_{M=0}^{\infty}\frac{|E_M|}{M^{1/2}}\int_{-a}^{a} e^{-px^2/M}dx$$

where

$$E_M = \underset{t}{E}\left\{N(t) = M\right\}.$$

PROOF. Since the conditions of the theorem imply that

$$\sum_{k=1}^{N_n(t)} r_k(t) = \sum_{k=1}^{nN(t)} r_k(t) + S_n(t)$$

where $S_n(t)$ is such that $|S_n(t)| \leqq |Q_n(t)|$, it follows that

$$\lim_{n\to\infty}\int_{E_M}\exp\left[\frac{ix}{(2np)^{1/2}}\sum_{k=1}^{N_n(t)} r_k(t)\right]dt$$
$$= \lim_{n\to\infty}\int_{E_M}\exp\left[\frac{ix}{(2np)^{1/2}}\sum_{k=1}^{nN(t)} r_k(t)\right]dt.$$

Since $N(t)$ is constant over E_M, the second limit equals $|E_M|\cdot e^{-Mx^2/4p}$. By virtue of the continuity theorem of Fourier-Stieltjes transforms, it follows that

$$\lim_{n\to\infty}\left|\underset{t}{E}\left\{t\in E_M;\ \frac{1}{(2np)^{1/2}}\left|\sum_{k=1}^{N_n(t)} r_k(t)\right| < a\right\}\right|$$
$$= |E_M|\cdot\left(\frac{p}{M\pi}\right)^{1/2}\int_{-a}^{a} e^{-px^2/M}dx,$$

and hence the theorem is proved.

The conclusion of the theorem obviously holds under the stronger condition that there exists an $N(t)$ and an A such that $|N_n(t)-nN(t)| < A$ for all n.

4. **Sums of cosines with big gaps.** By a similar method, using a Fourier series instead of a series of Walsh-Kaczmarz functions, it can be proved[2] that if $f(t)$ is a positive, measurable function in the inter-

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[2] Professor M. Kac has proved the central limit theorem for cosines with big gaps in *Note on power series with big gaps*, Amer. J. Math. vol. 61 (1939) pp. 473–476.

val (0, 1) and if $\{n_k\}$ is a sequence of positive integers satisfying the gap condition: $\lim_{k\to\infty} n_k/n_{k-1} = \infty$, then

$$\lim_{m\to\infty} \left| \underset{t}{E} \left\{ \frac{1}{m^{1/2}} \left| \sum_{k=1}^{m} \cos 2\pi n_k t \right| < f(t) \right\} \right| = \frac{1}{\pi^{1/2}} \int_0^1 \int_{-f(t)}^{f(t)} e^{-u^2} du dt.$$

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ON THE HAMILTON DIFFERENTIAL

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1. **Introduction.** In the absolute geometric development of vector analysis Hamilton found it necessary to formulate a definition for the differential of a point function, since division by a vector is excluded in vector analysis. It is the purpose of this note to relate a restricted form of the Hamilton differential to that of Stolz and another modified form to a differential defined by Rainich.

2. **The Hamilton differential.** The definition of Hamilton for the differential ϕ' of a point function $\phi(P)$ may be expressed by

$$\phi'(P, dP) = \operatorname*{limit}_{\lambda\to 0} \frac{\phi(P + \lambda dP) - \phi(P)}{\lambda}. \tag{2.1}$$

This definition is not entirely satisfactory. For some functions it furnishes differentials which are usually considered as nonexistent. For example, $(dxdy)^{1/2}$ is the differential of $(xy)^{1/2}$ at the origin according to definition (2.1). That such situations arise is due to the fact that the differential here defined does not necessarily possess the linearity property which will be defined later. This defect was recognized by Rainich[1] who proposed another form of the Hamilton differential which possesses this desired property. This is essential if the differential of a tensor point function is to be again a tensor.

3. **The Rainich differential.** Instead of making a single point $P+\lambda dP$ approach P as λ goes to zero Rainich makes each of two points Q_λ and P_λ approach P as λ goes to zero. His definition may be formulated as follows:

DEFINITION (3.1).

$$\text{(a)} \qquad \phi'(P, dP) = \operatorname*{limit}_{\lambda\to 0} \frac{\phi(Q_\lambda) - \phi(P_\lambda)}{\lambda},$$

which limit must exist for all modes of approach of Q_λ and P_λ to P for which

$$\text{(b)} \qquad \operatorname*{limit}_{\lambda\to 0} \frac{Q_\lambda - P_\lambda}{\lambda} = dP.$$

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[1] G. Y. Rainich, Amer. J. Math. vol. 46 (1924) p. 78.

4. **The linearity property.** The linearity property may be expressed by the formula

$$\phi'(P, \tau dP + \sigma\delta P) = \tau\phi'(P, dP) + \sigma\phi'(P, \delta P) \tag{4.1}$$

where τ and σ are arbitrary scalars.

We wish to show that the Rainich differential possesses this property. In doing this we make a direct use of (3.1); we may thus write

$$\operatorname*{limit}_{\lambda\to 0}\frac{Q_\lambda - R_\lambda}{\tau\lambda} = dP; \qquad \operatorname*{limit}_{\lambda\to 0}\frac{R_\lambda - P_\lambda}{\sigma\lambda} = \delta P,$$

so that

$$\operatorname*{limit}_{\lambda\to 0}\frac{Q_\lambda - P_\lambda}{\lambda} = \tau dP + \sigma\delta P.$$

But

$$\begin{aligned}\operatorname*{limit}_{\lambda\to 0}\frac{\phi(Q_\lambda) - \phi(P_\lambda)}{\lambda} &= \operatorname*{limit}_{\lambda\to 0}\frac{\phi(Q_\lambda) - \phi(R_\lambda)}{\lambda}\\ &\quad + \operatorname*{limit}_{\lambda\to 0}\frac{\phi(R_\lambda) - \phi(P_\lambda)}{\lambda},\end{aligned} \tag{4.2}$$

or

$$\phi'(P, \tau dP + \sigma\delta P) = \phi'(P, \tau dP) + \phi'(P, \sigma\delta P).$$

Also

$$\operatorname*{limit}_{\lambda\to 0}\frac{\phi(Q_\lambda) - \phi(R_\lambda)}{\tau\lambda} = \phi'(P, dP),$$

or multiplying by τ we obtain $\tau\phi'(P, dP)$, the first term in the right member of (4.2), while the second term is obtained in the same way. This establishes (4.1).

5. **The modified Hamilton.** The modified Hamilton differential will be defined to be *the Hamilton differential* (2.1) *upon which is imposed the linearity property* (4.1) *together with the continuity property,*

$$\operatorname*{limit}_{\lambda\to 0}\phi'\left(R_\lambda, \frac{Q_\lambda - P_\lambda}{\lambda}\right) = \phi'(P, dP). \tag{5.1}$$

R_λ is a neighboring point to P which approaches P as $\lambda\to 0$.

THEOREM 1. *The existence of the modified Hamilton differential is a sufficient condition for the existence of the Rainich differential* (3.1).

To prove this theorem we deduce the Rainich differential (3.1) from the modified Hamilton as follows: we define $f(\xi)$ by the equation

$$f(\xi) = \phi[P_\lambda + \xi(Q_\lambda - P_\lambda)] \tag{5.2}$$

where ξ is an arbitrary parameter. From (5.2) it follows that

$$\begin{aligned} f'(\xi) &= \lim_{\Delta\xi\to 0} \frac{\phi[P_\lambda + \xi(Q_\lambda - P_\lambda) + \Delta\xi(Q_\lambda - P_\lambda)] - \phi[P_\lambda + \xi(Q_\lambda - P_\lambda)]}{\Delta\xi} \\ &= \phi'[P_\lambda + \xi(Q_\lambda - P_\lambda), Q_\lambda - P_\lambda] = \phi'(R_\lambda, Q_\lambda - P_\lambda), \end{aligned}$$

where for $0 \leqq \xi \leqq 1$, R_λ plays the role of an intermediate point. This is by definition (2.1) the Hamilton differential. From (5.2), $f(1) = \phi(Q_\lambda)$ and $f(0) = \phi(P_\lambda)$ follow while the mean value theorem asserts the existence of a value ξ such that $f(1) - f(0) = f'(\xi) = \phi'(R_\lambda, Q_\lambda - P_\lambda)$. We then have

$$\lim_{\lambda\to 0} \frac{\phi(Q_\lambda) - \phi(P_\lambda)}{\lambda} = \lim_{\lambda\to 0} \phi'\left(R_\lambda, \frac{Q_\lambda - P_\lambda}{\lambda}\right) \tag{5.3}$$

(we have here used the linearity property) while if Q_λ and P_λ are made to go to P in any manner whatever provided that $\lim_{\lambda\to 0}(Q_\lambda - P_\lambda)/\lambda = dP$ and if the Hamilton differential is continuous at P it follows that

$$\lim_{\lambda\to 0} \phi'\left(R_\lambda, \frac{Q_\lambda - P_\lambda}{\lambda}\right) = \phi'(P, dP).$$

6. **The restricted Hamilton.** By the restricted Hamilton differential we shall mean *the Hamilton differential* (2.1) *restricted by the linearity condition* (4.1) *only.*

In the sequel we shall write Q_λ for $P + \lambda dP$.

THEOREM 2. *For a point function $\phi(Q_\lambda)$ to possess a finite restricted Hamilton differential, it is necessary and sufficient that $\phi(Q_\lambda)$ can be represented in the form*

$$\phi(Q_\lambda) = \phi(P) + \phi'(P, Q_\lambda - P) + \eta \tag{6.1}$$

where η is an infinitesimal function of λ of higher order than the first, and ϕ' is linear in the second argument.

To prove the necessity of the condition we introduce the notation

$$\eta \equiv \phi(Q_\lambda) - \phi(P) - \phi'(P, Q_\lambda - P)$$

and we write, using the linearity of ϕ',

$$\frac{\eta}{\lambda} = \frac{[\phi(Q_\lambda) - \phi(P)}{\lambda} - \phi'(P, dP) - \phi'\left(P, \frac{Q_\lambda - P}{\lambda} - dP\right).$$

It only remains to show that η/λ goes to zero with λ.

The difference of the first two terms goes to zero with λ, according to (2.1) of the definition, and the last term goes to zero because of (3.1b) and the fact that a linear function in a finite-dimensional vector space goes to zero with its argument; this means that η/λ goes to zero. The same relation may be used to prove the sufficiency by remarking that in this case it is given that η/λ goes to zero with λ.

In passing we may note that in euclidean 2-space (6.1) becomes

$$\phi(x', y') = \phi(x, y) + \phi'[P, (x' - x)i + (y' - y)j] \times \eta$$

where i and j are coordinate vectors. The linearity condition permits us to write this equation in the form

$$\phi(x', y') = \phi(x, y) + \phi'(P, i)(x' - x) + \phi'(P, j)(y' - y) + \eta \tag{6.2}$$

while it is easy to prove that $\phi'(P, i)$ and $\phi'(P, j)$ are the partial derivatives of ϕ at P. But this form of the function indicates that the restricted Hamilton is similar to a Stolz differential.[2] In fact when

$$(x', y') = (x + \lambda dx, y + \lambda dy)$$

is introduced into (6.2) and if the restricted Hamilton exists at P, it follows that for every $\epsilon > 0$ there exists a δ such that for $|\lambda| < \delta$, $|\eta/\lambda| < \epsilon$.

In conclusion it may be noted that the general notions arising here are not restricted to euclidean 3-space.

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[2] Otto Stolz, *Differential and Integral Rechnung*, vol. 1, p. 132.

MEASURE PRESERVING HOMEOMORPHISMS AT FIXED POINTS

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In an article of a few years ago [2][1] Kerékjártó obtained interesting results about certain types of transformations which he called similitudes. With a few modifications and extensions his methods can be used to gain information about the structure of measure preserving transformations at fixed points. For simplicity the results are formulated for Euclidean n-space although they could easily be given a much more general setting and in particular the relevant ones apply to any n-dimensional manifold on which there is defined a measure satisfying light restrictions. Actually, as in most topological investigations of measure preserving transformations, the main property needed is that a bounded open set can not be carried into a subset of itself such that the difference of the two sets contains interior points.

It is shown that there are compact continua of assorted sizes which contain the fixed point and which are carried into themselves by the transformation. Such continua might, for example, be solid spheres $x_1^2+ \cdots +x_n^2 \leqq r^2$ as in the case of an orthogonal transformation. On the other hand they might be arcs as in the case of the transformation

$$x_1' = 2x_1, \qquad x_{n-1}' = 2x_{n-1}, \qquad x_n' = 1/2^{n-1}x_n,$$

where continua of the type described are intervals on the x_n-axis which include the origin.

The results also show that there are certain points near the fixed point which remain near it under indefinite positive iteration of the transformation. We use the symbol U^{-1} for the set $T^{-1}(U)$, and so on.

THEOREM 1. *Let T be a measure preserving homeomorphism of E_n onto itself, and let A be a compact connected set such that $T(A) \subset A$. Then if U is an open set with compact closure which includes A, there exists a compact connected set K of which A is a proper subset and such that K is in $\overline{U}^{-1}$ and $T(K) \subset K$.*

The theorem applies to the particular case where A consists of a single fixed point. We divide the proof into two cases.

Case I. Assume that there exists an open connected set V in U^{-1}

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.

which includes A and which is such that $T(V)\subset V$.

In this case $\overline{V}$ may be taken as the desired continuum because if either $T(V)$ is in V or $T^{-1}(V)$ is in V, then $T(\overline{V})=\overline{V}$. In order to see this suppose for example that $T(V)$ is in V so that we have immediately $T(\overline{V})\subset\overline{V}$. If $T(\overline{V})$ did not include all of $\overline{V}$, then because of the nature of $\overline{V}$ it would have to omit some interior points of $\overline{V}$ and hence could not have the same measure as $\overline{V}$. When $T^{-1}(V)$ is in V, the procedure is similar and in either event we have $T(\overline{V})=\overline{V}$ which is even stronger than the stated conclusion.

Case II. Assume that there exists no open connected set V in U^{-1} with the properties described in case I. Both U^{-1} and U include A. Let

$$U_1^1 = C(U^{-1}\cap U)$$

where $C(U^{-1}\cap U)$ is used to designate the component of $U^{-1}\cap U$ which includes A. We also have

$$U_1 = C(U^{-2}\cap U^{-1}),$$

where, as usual, superscripts are used to indicate positive or negative iterates of a set, and subscripts merely distinguish between different sets. We see that U_1 is not contained in U, because U_1 in U implies U_1 is in U_1^1. But this is impossible because U_1 includes A and is in U^{-1}, and we have assumed there is no such set which goes into itself. Let

$$U_2^1 = C(U_1\cap U_1^1)$$

so that

$$U_2 = C(U_1^{-1}\cap U_1).$$

Then U_2 is not in U, because U_2 is in $U^{-3}\cap U^{-2}\cap U^{-1}$ and if it were also in U then U_2 would also be in U_1^1 and in U_2^1 which is impossible because U_2 includes A and is in U^{-1}. Continuing step by step let

$$U_k^1 = C(U_{k-1}\cap U_{k-1}^1)$$

so that

$$U_k = C(U_{k-1}^{-1}\cap U_{k-1}).$$

Then U_k is not in U and we see this as follows: we have

$$U_k\subset U^{-(k+1)}\cap\cdots\cap U^{-1}$$

and

$$U_k^1\subset U^{-k}\cap\cdots\cap U^{-1}\cap U,$$

so that if U_k were in U then U_k would also be in U_k^1 because it will then be in U_1, U_1^1, U_2, U_2^1, and so on, which is impossible as before.

Now let

$$K = \overline{U}_1 \cap \overline{U}_2 \cap \cdots .$$

Since it is the intersection of a monotone decreasing sequence of connected compact sets including A, it follows that K also has these properties. It is clear that K is in $\overline{U^{-1}}$, and since

$$K^1 = \overline{U_1^1} \cap \overline{U_2^1} \cap \cdots$$

it also follows that $T(K) \subset K$ because $U_k^1 \subset U_{k-1}$. Each of the sets $\overline{U}_i$ includes a point not in U and consequently K must do the same so that K must contain points not in A. The proof of Theorem I is thus complete.

COROLLARY 1. *If T is a measure preserving homeomorphism of E_n onto itself with a fixed point p, then there exist arbitrarily small compact connected closed sets, which contain p, and which are carried into themselves by T, and which contain more than one point.*

COROLLARY 2. *If T is a measure preserving homeomorphism of E_n onto itself with a fixed point p, then there exist arbitrarily large compact connected closed sets which contain p and which are carried into themselves by T.*

The proof of Corollary 1 follows immediately from the proof of Theorem 1. We turn to the proof of Corollary 2. Let A_1 be any compact connected set which includes p and which is carried into itself by T. From Theorem I we know that we can find a larger set with similar properties and the proof of the corollary follows by transfinite induction or any similar technique.

COROLLARY 3. *Let T be a measure preserving homeomorphism of E_n onto itself and let A be a compact connected set such that $T(A) \subset A$. Further let U be an open set, including A and having compact closure. Then there exists a compact connected set K such that $A \subset K \subset \overline{U^{-1}}$, K is not in U, and $T(K) \subset K$.*

For the proof we must keep in mind the proof of Theorem 1 which has the same hypothesis. Notice that in either Case I or Case II we can assert that there exists a compact connected set larger than A which goes into itself and which is in $\overline{U^{-1}}$. If this set is not in U we may stop. If it is in U we may proceed by transfinite induction to reach the conclusion.

The first two lemmas show the existence of both small and large continua of the type under discussion, but the third contains a slightly sharper result useful in the next theorem.

If $T(t; x)$ is a one-parameter group of homeomorphisms of E_n onto itself, then a point p fixed under every element of the group is called a stationary point or a singular point of the group. The function $T(t; x)$ is always assumed to be simultaneously continuous in t and x.

THEOREM 2. *Let $T(t; x)$ be a one-parameter group of homeomorphisms of E_n into itself and let A be a compact connected set having the property that for all positive values of t, A is mapped into itself. Then there exists a larger set with the same properties.*

Let U be an open connected set which includes A and has a compact closure. For each n we may find a set K_n which is not in U but is in $\overline{U^{-1}}$ and such that $T(1/n; K_n) \in K_n$. Some subsequence of this set of compact connected sets approaches (in the sense of Hausdorff distance [1]) a compact connected set K which is in $\overline{U^{-1}}$ but not in U. We show now that for all positive values of t this set is carried into itself. Let x be any point of K and let x_n in K_n be a sequence of points approaching x. It is clear that for all positive k and n

$$T(k/n; x_n) \in K_n.$$

Letting t be any positive number it will be sufficient to prove that

$$T(t; x) \in K.$$

Let k_n be the largest non-negative integer such that $k_n/n < t$, so that as n approaches infinity k_n/n approaches t. We know that

$$T(k_n/n; x_n) \in K_n.$$

Since x_n approaches x, k_n/n approaches t, and T is simultaneously continuous we may conclude that $T(t; x)$ is in K which therefore ends the proof.

It follows from these considerations that we could find sets similar to those described in Theorem 1 and its corollaries for the negative iterates simply by letting T^{-1} be the transformation. There need not be such sets which are carried into themselves by both T and T^{-1} as we can see for instance in the second example cited in the introduction. We may also make analogous remarks about the case of a one-parameter group.

Remark. It is well known that any continuous transformation of a closed disc into itself has a fixed point. An orientation preserving homeomorphism of the interior of a unit disc does not necessarily have a fixed point but if this homeomorphism is measure preserving then it must have a fixed point as we can see from the Brouwer translation theorem. We see from the above that there must be compact continua which include the fixed point, go into themselves and which extend as near as we please to the boundary of the disc. This fact may be of use in studying measure preserving homeomorphisms of the interior of a disc onto itself.

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GENERALIZATIONS OF TWO THEOREMS OF JANISZEWSKI

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Janiszewski proved [2][1] the following for the plane.

THEOREM A. *The sum of two compact continua does not cut the point A from the point B provided neither cuts A from B and their common part is connected or does not exist.*

THEOREM B. *The sum of two compact continua cuts the plane provided their common part is not connected.*

Many generalizations and modifications of these theorems appear in the literature (for example, see [**1, 3, 4, 6, 8, 9, 10, 11, 12**]). This paper generalizes the two theorems by considering, instead of continua, sets that are neither open nor closed. All sets referred to in this paper are in the plane.

DEFINITION. The set R cuts the point A from the point B in the set W provided some continuum in W contains A and B and R intersects neither A nor B but it intersects every continuum in W containing A and B. If R cuts A from B in the plane, we simply say that it cuts A from B. Thus we use *cut* in the sense that *coupe* was used in Fundamenta Mathematicae (for example, see [**1**, p. 75, and **10**, p. 15]), and not in the sense used by some writers to mean separate. If a closed set cuts two points from each other in the plane, it separates them from each other; this is not true for more general sets.

Generalizations of Theorem A. We shall make use of the following known result [**12**, p. 129, and **4**, pp. 35–36].

THEOREM 1. *If neither of the closed point sets H and K separates the point A from the point B, the common part of H and K is connected or does not exist and the part of H in the complement of K is compact, then the sum of H and K does not separate A from B.*

THEOREM 2. *If neither of the point sets H and K separates the point A from the point B, each of the sets is closed in their sum,[2] one of the sets is compact[3] and their common part is a continuum or does not exist, then the sum of H and K does not separate A from B.*

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] The set H is closed in the set S if $\bar{H}\cdot S$ is equal to H.

[3] In Theorems 2, 5, 6, 7, 8, 9, 10 instead of assuming that one of the sets is compact, we can assume that the part of it in the complement of the other is compact.

PROOF. Assume that $H+K$ separates A from B. Then $H+K$ contains a continuum M separating A from B [8, Theorem 73, p. 150, and Theorem 24, p. 194]. However, neither of the closed point sets $M\cdot H+H\cdot K$ and $M\cdot K+H\cdot K$ separates A from B because neither H nor K does. Hence, by Theorem 1 their sum does not. But their sum contains M.

Example. Theorem 2 would not be true if instead of requiring that $H\cdot K$ be a continuum we only required that it be connected. Let $H\cdot K$ be the sum of the interval from (0, 0) to (−1, 2), the interval from (−1, 2) to (1, 1) and the image of

$$y = \sin(\pi/2x), \qquad 0 < x \leqq 1;$$

let $H-H\cdot K$ be the image of

$$0 \leqq y < \sin(\pi/2x), \qquad 0 < x \leqq 1;$$

let $K-H\cdot K$ be the image of

$$0 \geqq y > \sin(\pi/2x), \qquad 0 < x \leqq 1.$$

THEOREM 3. *If the sum of the simple closed curve J and the arc PQ separates the point A from the point B then their sum contains a simple closed curve J' such that J' minus the common part of J' and PQ is a connected subset of J.*

PROOF. There is a subset M of J irreducible with respect to $M+PQ$ separating A from B. Also, $M+PQ$ contains a continuum J' which is irreducible with respect to separating A from B. This continuum is a simple closed curve containing M. If M were the sum of two mutually separated point sets M_1 and M_2, neither of the closed sets M_1+PQ and M_2+PQ would separate A from B and their sum would not. As M is irreducible with respect to $M+PQ$ separating A from B, it contains no point of PQ and is therefore equal to $J'-J'\cdot PQ$.

THEOREM 4. *If the points A and B belong to the same component of the complement of the connected domain D and J is a simple closed curve separating A from B then J contains a continuum which contains no point of D and which cuts A from B in the complement of D.*[4]

PROOF. Assume that J intersects D; the theorem is obviously true if it does not. Let $O_1, O_2, \cdots$ be a sequence of points dense in $D\cdot J$. There is an arc O_1O_2 in D and by Theorem 3 there is a simple closed curve J' separating A from B and such that $J'-J'\cdot O_1O_2$ is an open arc (P_1Q_1) of J containing neither O_1 nor O_2. If P_1 is not O_3 there is an arc P_1O_3 in D and there is a simple closed curve J'' separating A from B such that $J''-J''\cdot(O_1O_2+P_1O_3)$ is an open arc (P_2Q_2) of (P_1Q_1). This process is continued to get an infinitude of (P_nQ_n)'s where (P_nQ_n) is a subset of $(P_{n-1}Q_{n-1})$ containing neither $O_1, O_2, \cdots$, nor O_{n+1} but separating A from B in the complement of D.

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[4] By Theorem 7 it follows that J contains a continuum cutting A from B in the complement of D and intersecting $\overline{D}$ in no more than two points.

The common part M of $(P_1Q_1), (P_2Q_2), \cdots$ is the same as the common part of $P_1Q_1, P_2Q_2, \cdots$ and is therefore a continuum. As no point of D belongs to each P_nQ_n, M does not intersect D. If C is a continuum containing A and B but no point of D, then for each positive integer n it contains a point of P_nQ_n and therefore contains a point of M.

Example. Theorem 4 would not be true if the word "cuts" were replaced by "separates." Consider the domain D which is the image of

$$\sin(\pi/x) < y < 1 + \sin(\pi/x), \qquad 0 < x < 1,$$

and a circle J with center at (0, 0) and radius 1. Now J contains no continuum which does not intersect D but which separates (0, 0) from (2, 0) in the complement of D.

THEOREM 5. *The sum of two domains does not separate the point A from the point B provided neither of the domains separates A from B, one of them is compact and their common part is connected or does not exist.*

PROOF. First, assume that H and K are two compact domains satisfying the above hypotheses whose sum separates A from B. Then $H+K$ contains a simple closed curve [8, Theorem 9, p. 184] separating A from B. By Theorem 4, this simple closed curve contains a continuum M which contains no point of $H\cdot K$ but which cuts A from B in the complement of $H\cdot K$. As M contains no point of $H\cdot K$, it is a subset of either H or K, say H. Then the domain H cuts A from B and therefore separates A from B.

The theorem may be proved in the case in which only one of H, K is compact by regarding the plane as a sphere minus one point.

THEOREM 6. *Suppose that H and K are sets such that one is compact, neither cuts the point A from the point B, their common part is connected and on the subtraction of their common part from each, the remainders are mutually separated. Then the sum of H and K does not cut A from B.*[5]

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[5] Using other methods, Eilenberg showed [1, Theorem 8, p. 78] that the sum of the two compact sets H and K does not cut A from B provided neither cuts A from B, each is open or closed in $H+K$ and $H\cdot K$ is connected.

PROOF. Suppose that C_H and C_K are continua such that each contains $A+B$, C_H does not intersect H and C_K does not intersect K. There exist domains D_0, D_1, D_2 such that D_0 is connected and covers $H\cdot K$ but no point of C_H+C_K, D_1 covers H but no point of C_H, D_2 covers K but no point of C_K, $D_1-D_0\cdot D_1$ and $D_2-D_0\cdot D_2$ are mutually separated and one of D_1, D_2 is compact. Considering D_0+D_1 and D_0+D_2 as the domains of Theorem 5, we find that $D_0+D_1+D_2$ does not cut A from B. Neither does $H+K$ which is a subset of $D_0+D_1+D_2$.

THEOREM 7. *If H is a compact closed set cutting the point A from the point B in the complement of the connected set K, then H contains a subset H' irreducible with respect to being a closed set cutting A from B in the complement of K. Also, H' is a continuum that is not separated by any subset of the closure of K.*

PROOF. Let D_1, D_2, $\cdots$ be a sequence of domains such that if D is a domain containing a point P of H, there exists an integer i such that D_i is a subset of D containing P. Let H_1 be a closed subset of H such that (1) H_1 cuts A from B in the complement of K and (2) H_1 intersects D_1 only if every closed subset of H satisfying (1) intersects D_1. In general, let H_n be a closed subset of H_{n-1} such that (1) H_n cuts A from B in the complement of K and (2) H intersects D_n only if every closed subset of H_{n-1} satisfying (1) intersects D_n.

The common part H' of H_1, H_2, $\cdots$ is a closed subset of H. It cuts A from B in the complement of K for each continuum in the complement of K containing A and B intersects each of H_1, H_2, $\cdots$. If H' contained a proper closed subset M cutting A from B in the complement of K, then there would be an integer i such that D_i contained a point P of H' but no point of M. Then H_i would not contain P.

Assume that $\overline{K}$ contains a subset R such that $H'-R$ is the sum of two mutually separated point sets E_1 and E_2. Then there exist mutually exclusive domains F_1 and F_2 containing E_1 and E_2 respectively. However, as neither $K+(H'-H'\cdot F_1)$ nor $K+(H'-H'\cdot F_2)$ cuts A from B, then by Theorem 6, $K+H'$ could not.

Generalizations of Theorem B. We shall now consider some generalizations of Theorem B.

DEFINITION. The *junction* [5, p. 206] of the sets H and K is the common part of H and the closure of K plus the common part of K and the closure of H, that is $H\cdot\overline{K}+\overline{H}\cdot K$.

THEOREM 8. *If H and K are connected sets one of which is compact,* *and such that H is continuumwise connected and the junction of H and K is the sum of two mutually separated sets each of which contains a point of H, then the sum of H and K cuts the plane.*[6]

PROOF. Let the junction M of H and K be the sum of two mutually separated point sets M_A and M_B which contain points A and B respectively of H. There exists [7, Corollary 2, p. 221] a compact continuum J irreducible with respect to separating A from B and such that if the components of $\overline{M}_A \cdot \overline{M}_B$ are regarded as points, J is a simple closed curve (or a point) containing no point of $\overline{M} - \overline{M}_A \cdot \overline{M}_B$. Let T be the set of all continua t such that t is either a point of J not belonging to $\overline{M}$ or a component of $\overline{M}_A \cdot \overline{M}_B$.

Let C be a continuum in H containing A and B. No point of $C \cdot J$ is a limit point of K, for J contains no point of M. Using an argument similar to that employed by Kuratowski [7, Theorem 11, pp. 222–223] we find that T contains a finite number of elements $t_1, t_2, \cdots, t_n$ none of which intersect $C+K$ and each of which belongs both to the boundary of a component of $J-(t_1+\cdots+t_n)$ which contains a point of C but no point of K and also to the boundary of a component of $J-(t_1+\cdots+t_n)$ which contains a point of K but no point of C. Let W_C be the sum of those components containing a point of C and let W_K be the sum of those containing a point of K.

Assume that $C+K$ does not cut between any two elements of $t_1, t_2, \cdots, t_n$. Then there is a continuum V containing $t_1+t_2+\cdots+t_n$ but no point of $C+K$. As neither the continuum $V+W_C$ nor the continuum $V+W_H$ separates A from B, then by Theorem 1 their sum does not. However, their sum contains J.

Let i and j be integers such that $C+K$ cuts t_i from t_j and let α_i and α_j be maximal connected subsets of J containing t_i and t_j respectively but no point belonging to an element of T containing a point of $C+K$. Now $C+K$ cuts every point of α_i from every point of α_j. But α_m $(m=i, j)$ contains a point P_m not belonging to H for J contains no point of M. Therefore $H+K$ cuts P_i from P_j.

Example. Theorem 8 would not be true if we did not require that H be continuumwise connected. Let H be the sum of the intervals having end points at $(-3, 0)$ and $(-3, 1)$, at $(-3, 0)$ and $(-1, 0)$, at $(1,0)$ and $(2, 0)$ and at $(-3, 1/2^n)$ and $(2, 1/2^n)$ where $n=1, 2, \cdots$. Let K be the sum of the intervals having end points at $(3, 0)$ and $(3, 1)$, at $(1, 0)$ and $(3, 0)$, at $(-2, 0)$ and $(-1, 0)$ and at $(-2, 1/3^n)$ and $(3, 1/3^n)$.

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[6] Nicodym proved this theorem in the case where H is a bounded continuum [10, Theorem 2, p. 20].

THEOREM 9. *If H and K are connected sets one of which is compact, the common part of the closure of H and the closure of K is the sum of two mutually separated point sets each of which contains a point of the junction of H and K, then $H+K$ cuts the plane.*

PROOF. There exists a simple closed curve containing no point of $\overline{H}\cdot\overline{K}$ and separating some two points of the junction of H and K. Using an argument similar to that used in Theorem 8 we find that this simple closed curve contains two points which $H+K$ cuts from each other.

THEOREM 10. *The sum of two continuumwise connected sets separates the plane provided one of them is compact and their common part exists but does not belong to any continuumwise connected subset of their junction.*

PROOF. Suppose that H and K are sets satisfying the hypotheses of Theorem 10 and that the complement of $H+K$ is connected. Neither the complement of $H+\overline{H}\cdot K$ nor the complement of $K+\overline{K}\cdot H$ cuts between any two points of $H\cdot K$ and, by Theorem 6, the complement of $\overline{H}\cdot K+H\cdot\overline{K}$ does not.

This type of argument gives the following result.

THEOREM 11. *The sum of two connected domains one of which is compact separates the plane provided their common part is not connected.*

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THE UNIVERSITY OF TEXAS

ON ISOMETRIES OF SQUARE SETS

PAUL J. KELLY

1. **Introduction.** It is not fully known under what conditions the isometry of two square, metric sets, say E^2 and F^2, implies the isometry of E and F. Using the notion of order two self-isometries, this paper gives conditions sufficient to imply E isometric to F when E^2 and F^2 are finite and are metrized under any one of a fairly extensive class of functions. The basic ideas are first applied to non-square sets to yield a more general theorem which is then applied to the inverse square problem.

2. **Definitions.** A set is called metric if to every pair of its elements, a and b, there corresponds a real, non-negative number, which is independent of the order of a and b, zero if and only if a equals b, and which satisfies the triangle law.

Two metric sets are isometric (written "$\equiv$") if there is a one-to-one transformation of one set on the other in which the metric number associated with any pair is the same as that associated with the transformed pair.

A non-identity mapping of a set on itself, which is an isometry, and which leaves each element of the set invariant or else interchanges it with another, is called a self-isometry of order two. Any subset on which the self-isometry is the identity is said to be left pointwise invariant.

THEOREM 1. *Assume $A \equiv B$ under a mapping T, where A and B are finite metric sets. Let A and B have self-isometries of order two under mappings R and S respectively and let A_1 and B_1 denote respectively the maximum subsets left pointwise invariant. If A_1 has no self-isometry of order two, and has at least as many elements as E_1, then $A_1 \equiv B_1$ and there*

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exists a composition of R, S, T *and* T^{-1} *which maps* A *isometrically on* B *and carries* A_1 *into* B_1.

PROOF. Starting with the set A_1 the following sequence of sets outlined is obtained by transforming A_1 by T, the set obtained by S, this set by T^{-1}, and this set by R, and so on repeating cyclically the transformations T, S, T^{-1}, R.

$$\begin{array}{lllcrll}
 & & \text{Column 1} & & \text{Column 2} & & \\
 & & A_1 & \longrightarrow & T(A_1) & \big\} & (1,a) \\
(2,a) & \big\{ & T^{-1}ST(A_1) & \longleftarrow & ST(A_1) & & (1,b) \\
(2,b) & & RT^{-1}ST(A_1) & \longrightarrow & TRT^{-1}ST(A_1) & \big\} & (3,a) \\
 & & & \longleftarrow & STRT^{-1}ST(A_1) & & (3,b) \\
(2n,a) & \big\{ & & & & & \\
(2n,b) & & & \longrightarrow & & \big\} & (2n+1,a) \\
 & & & & & & (2n+1,b)
\end{array}$$

The notation at the side is such that set (n, x), $x=a$ or b, is in B if n is odd and in A if n is even. From the construction and the nature of R and S, the following relations are easily verified: $R(2n, a)=(2n, b)$, $R(2n, b)=(2n, a)$, $S(2n+1, a)=(2n+1, b)$, $S(2n+1, b)=(2n+1, a)$, $T(2n, b)=(2n+1, a)$, $T^{-1}(2n+1, b)=(2n+2, a)$.

(1) Assume no set in column 2 is the set B_1.

(2) Since all sets in both columns are isometric to A_1, isometry being transitive, and since A_1 has as many elements as B_1, (1) implies that no set in column 2 is a subset of B_1.

(3) For any n, $x=a$ or b, $S(2n+1, x)\neq(2n+1, x)$. Since S is the identity mapping only on B_1 and since, from (1) and (2), $(2n+1, x)$ is not B_1 or a subset of it, $S(2n+1, x)=(2n+1, x)$ would mean that $(2n+1, x)$ had a self-isometry of order two. This, together with $A_1\equiv(2n+1, x)$, would imply A_1 had a self-isometry of order two, contradicting the given conditions.

(4) For any n, no two sets of column 1 up to and including $(2n, a)$ are identical. The proof is by induction.

(4.1) Statement (4) holds for $n=1$, since $A_1=(2, a)$ would give $T(A_1)=T(2, a)$ or $(1, a)=(1, b)$, contradicting (3).

(4.2) Assume (4) holds for $n=k$.

(4.3) Since R is the identity only on A_1 and since $(2k, a)$ is not a subset of A_1, being isometric to it, and is not equal to A_1, from (4.2), then $R(2k, a)=(2k, a)$ would imply that $(2k, a)$ had a self-isometry of order two, and hence that A_1 did also. Therefore $R(2k, a) \neq (2k, a)$, that is $(2k, b) \neq (2k, a)$. This, in turn, implies $(2k, b) \neq A_1$.

(4.4) For $i<k$, $x=a$ or b, $(2k, b) \neq (2i, x)$. From $(2k, b)=(2i, x)$ would follow $R(2k, b)=R(2i, x)$, that is $(2k, a)=R(2i, x)$, which for $i<k$ would contradict (4.2).

(4.5) From (4.2), (4.3), and (4.4) no two sets of column 1 up to and including $(2k, b)$ are identical. This, with the one-to-oneness of T, implies that no two sets of column 2 up to and including $(2k+1, a)$ are identical.

(4.6) From (3), $(2k+1, b) \neq (2k+1, a)$.

(4.7) For $i<k$, $x=a$ or b, $(2k+1, b) \neq (2i+1, x)$. For, from $(2k+1, b)=(2i+1, x)$ would follow $S(2k+1, b)=S(2i+1, x)$, that is $(2k+1, a)=S(2i+1, x)$, which for $i<k$ would contradict (4.5).

(4.8) From (4.6) and (4.7) no two sets of column 2 up to and including $(2k+1, b)$ are identical. This, with the one-to-oneness of T^{-1}, implies that no two sets of column 1 up to and including $(2(k+1), a)$ are identical, and completes the induction establishing (4).

(5) Since (4) implies the existence of an unlimited number of distinct subsets in the finite set A, it is clearly a contradiction reached through assuming (1). Therefore (1) is false and B_1 must occur in column 2 and be isometric to A_1. The remainder of the theorem follows from the fact that the sequence of sets can be started with A rather than A_1.

If A and B are the same set and T is replaced by the identity, Theorem 1 reduces to the following result:

THEOREM 2. *Let A be a finite metric set and let A_1 and B_1 be the maximum subsets left pointwise invariant under two distinct self-isometries, R and S, of order two. If A_1 has no self-isometry of order two and has at least as many elements as B_1, then $A_1 \equiv B_1$ and there is a composition of R and S which maps A isometrically on itself and carries A_1 into B_1.*

3. **Definitions concerning square sets.** Let E be a finite metric set with elements $x_1, x_2, \cdots, x_n$ and metric ρ_E. By E^2 is meant the set of couples obtained from the cartesian product of E with itself.

In E^2 the subset of couples (x_i, x_i), $i=1, 2, \cdots, n$, is called the diagonal set.

The reflection mapping, R, of E^2 on itself is defined by $R(x_i, x_j) = (x_j, x_i)$.

If a metric ρ_{E^2} is defined on the elements of E^2 it is called a metric of class α if, in addition to making E^2 a metric set, it has the following properties:

(1) For any two points of E^2, $P_1\colon(x_i, x_j)$, $P_2\colon(x_k, x_l)$, $\rho_{E^2}(P_1, P_2) = f(X_1, X_2)$ where $X_1 = \rho_E(x_i, x_k)$, $X_2 = \rho_E(x_j, x_l)$.

(2) $f(X_1, X_2) = f(X_2, X_1)$.

(3) There exists a constant M associated with f, such that whenever $X_1 = X_2$, then $f(X_1, X_2) = MX_1$.

THEOREM 3. *Let E and F be finite metric sets, and let E^2 and F^2 be metrized under the same class α metric. If either the diagonal set of E^2 or that of F^2 has no self-isometry of order two, then $E^2 \equiv F^2$ implies $E \equiv F$.*

PROOF. Let R and S denote respectively the reflection mappings of E^2 and F^2 on themselves. From the definition of reflection and from property 2 of a class α metric, the mappings R and S establish self-isometries of order two in which the diagonal sets alone are left pointwise invariant. The two diagonal sets also have the same number of elements because $E^2 \equiv F^2$. From Theorem 1, then, with E^2 and F^2 playing the roles of A and B, and with the diagonal sets as A_1 and B_1, it follows that the diagonal set of E^2 is isometric to that of F^2. This isometry together with property 3 of a class α metric implies $E \equiv F$.

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ON RIEMANNIAN MANIFOLDS OF FOUR DIMENSIONS[1]

SHIING-SHEN CHERN

Introduction. It is well known that in three-dimensional elliptic or spherical geometry the so-called Clifford's parallelism or parataxy has many interesting properties. A group-theoretic reason for the most important of these properties is the fact that the universal covering group of the proper orthogonal group in four variables is the direct product of the universal covering groups of two proper orthogonal groups in three variables. This last-mentioned property has no analogue for orthogonal groups in n (>4) variables. On the other hand, a knowledge of three-dimensional elliptic or spherical geometry is useful for the study of orientable Riemannian manifolds of four dimensions, because their tangent spaces possess a geometry of this kind. It is the purpose of this note to give a study of a compact orientable Riemannian manifold of four dimensions at each point of which is attached a three-dimensional spherical space. This necessitates a more careful study of spherical geometry than hitherto given in the literature, except, so far as the writer is aware, in a paper by E. Study [2].[2] Our main result consists of two formulas, which express two topological invariants of a compact orientable differentiable manifold of four dimensions as integrals over the manifold of differential invariants constructed from a Riemannian metric previously given on the manifold. These two topological invariants have a linear combination which is the Euler-Poincaré characteristic.

1. **Three-dimensional spherical geometry.** We consider an oriented Euclidean space of four dimensions E^4 with the coordinates x_0, x_1, x_2, x_3. In E^4 let S^3 be the oriented unit hypersphere defined by the equation

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1. \tag{1}$$

Three-dimensional spherical geometry is concerned with properties on S^3 which remain invariant under the rotation group (that is, the proper orthogonal group) of E^4 leaving the origin fixed.

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[1] The content of this paper was originally intended to be an illustration in the author's article, *Some new viewpoints in differential geometry in the large*, which is due to appear in this Bulletin. Later it appeared more advisable to publish these results separately, but a comparison with the above-mentioned article, in particular §7, is recommended.

[2] Numbers in brackets refer to the references cited at the end of the paper.

We call a frame an ordered set of four mutually perpendicular unit vectors $\mathfrak{e}_0$, $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$. There exists one and only one rotation carrying one frame to another. The coordinates x_0, x_1, x_2, x_3 of a point $\mathfrak{x} \in S^3$ with respect to the frame $\mathfrak{e}_0$, $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$ are defined by the equation

$$\mathfrak{x} = x_0\mathfrak{e}_0 + x_1\mathfrak{e}_1 + x_2\mathfrak{e}_2 + x_3\mathfrak{e}_3. \tag{2}$$

Let $\mathfrak{e}_0^*$, $\mathfrak{e}_1^*$, $\mathfrak{e}_2^*$, $\mathfrak{e}_3^*$ be a frame related to $\mathfrak{e}_0$, $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$ by means of the relations

$$\mathfrak{e}_\alpha^* = \sum_{\beta=0}^{3} a_{\alpha\beta}\mathfrak{e}_\beta, \qquad \alpha = 0, 1, 2, 3, \tag{3}$$

where $(a_{\alpha\beta})$ is a proper orthogonal matrix, and let x_0^*, x_1^*, x_2^*, x_3^* be the coordinates of the same point x with respect to the frame $\mathfrak{e}_0^*$, $\mathfrak{e}_1^*$, $\mathfrak{e}_2^*$, $\mathfrak{e}_3^*$. Then we have

$$x_\alpha^* = \sum_{\beta=0}^{3} a_{\alpha\beta}x_\beta, \qquad \alpha = 0, 1, 2, 3. \tag{3a}$$

The properties of spherical geometry are those which, when expressed in terms of coördinates with respect to a frame, remain invariant under change of the frame.

Let x_0, x_1, x_2, x_3 be the coordinates of a point $\mathfrak{x}$ with respect to a frame $\mathfrak{e}_0$, $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$, as defined by (2). To these coordinates we associate a unit quaternion

$$X = x_0 + x_1i + x_2j + x_3k, \qquad N(X) = 1, \tag{4}$$

where $N(X)$ denotes the norm of X. Let

$$X^* = x_0^* + x_1^*i + x_2^*j + x_3^*k. \tag{4a}$$

Then the following theorem is well known [1]:

THEOREM 1. *The proper orthogonal group* (3a) *can be expressed in the quaternion notation in the form*

$$X^* = AXB, \tag{5}$$

where A, B are unit quaternions. It contains the two subgroups

$$X^* = AX, \tag{6a}$$

$$X^* = XB, \tag{6b}$$

called the subgroups of left and right translations respectively. A left translation is a right translation when and only when it is $X^* = \pm X$.

It is important to give a distinction between the left and right translations. We shall show that this is possible when S^3 is oriented. When X_i, $i=1, 2, 3, 4$, are four quaternions, we introduce the notation $|X_1X_2X_3X_4|$ to denote the value of the determinant formed by their components. Then we have the following theorem:

THEOREM 2. *For any three quaternions* X, X', A *the inequalities*

$$|XX', AX, AX'| \leqq 0, \qquad |XX', XA, X'A| \geqq 0 \tag{7}$$

hold. Equality sign holds only when $A = \pm 1$ *or when* X, X', A *are linearly dependent.*

To prove this theorem we assume the quaternions to be different from zero and hence, without loss of generality, to be unit quaternions. By a change of frame, which does not affect the values of the determinants in question, we can suppose $X=1$. Let

$$A = a_0 + a_1i + a_2j + a_3k, \qquad X' = x_0' + x_1'i + x_2'j + x_3'k.$$

We get then

$$\begin{aligned} |XX', AX, AX'| &= -(a_1x_2' - a_2x_1')^2 - (a_2x_3' - a_3x_2')^2 \\ &\quad - (a_3x_1' - a_1x_3')^2 \leqq 0, \\ |XX', XA, X'A| &= (a_1x_2' - a_2x_1')^2 + (a_2x_3' - a_3x_2')^2 \\ &\quad + (a_3x_1' - a_1x_3')^2 \geqq 0. \end{aligned}$$

From these equations the theorem follows.

This theorem gives a criterion to distinguish between the left and the right translations. A reversal of orientation of S^3 interchanges the left and right translations.

We now put

$$A = a_0 + a_1i + a_2j + a_3k, \quad B = b_0 + b_1i + b_2j + b_3k, \tag{8}$$

and write out in detail the transformations (6a) and (6b):

$$\left\{\begin{aligned} x_0^* &= a_0x_0 - a_1x_1 - a_2x_2 - a_3x_3, \\ x_1^* &= a_1x_0 + a_0x_1 - a_3x_2 + a_2x_3, \\ x_2^* &= a_2x_0 + a_3x_1 + a_0x_2 - a_1x_3, \\ x_3^* &= a_3x_0 - a_2x_1 + a_1x_2 + a_0x_3, \end{aligned}\right. \tag{9a}$$

$$\left\{\begin{aligned} x_0^* &= b_0x_0 - b_1x_1 - b_2x_2 - b_3x_3, \\ x_1^* &= b_1x_0 + b_0x_1 + b_3x_2 - b_2x_3, \\ x_2^* &= b_2x_0 - b_3x_1 + b_0x_2 + b_1x_3, \\ x_3^* &= b_3x_0 + b_2x_1 - b_1x_2 + b_0x_3. \end{aligned}\right. \tag{9b}$$

Let $\mathfrak{A}$ and $\mathfrak{B}$ be respectively the matrices of these transformations. The associativity of quaternion multiplication implies that

$$\mathfrak{A}\mathfrak{B} = \mathfrak{B}\mathfrak{A}. \tag{10}$$

We then define

$$\mathfrak{E} = \begin{pmatrix} \mathfrak{e}_0 \\ \mathfrak{e}_1 \\ \mathfrak{e}_2 \\ \mathfrak{e}_3 \end{pmatrix}, \qquad \mathfrak{E}^* = \begin{pmatrix} \mathfrak{e}_0^* \\ \mathfrak{e}_1^* \\ \mathfrak{e}_2^* \\ \mathfrak{e}_3^* \end{pmatrix}, \tag{11}$$

so that $\mathfrak{E}$ and $\mathfrak{E}^*$ are one-columned matrices whose elements are vectors. From Theorem 1 we see that the rotation (3) can always be written

$$\mathfrak{E}^* = \mathfrak{A}\mathfrak{B}\mathfrak{E} = \mathfrak{B}\mathfrak{A}\mathfrak{E}. \tag{12}$$

We call two frames $\mathfrak{E}$ and $\mathfrak{E}^*$ left (right) equivalent if

$$\mathfrak{E}^* = \mathfrak{A}\mathfrak{E} \qquad (\mathfrak{E}^* = \mathfrak{B}\mathfrak{E}). \tag{13}$$

It is evident that this equivalence relation is reflexive, symmetric, and transitive. We now introduce as new figures in spherical geometry classes of left (right) equivalent frames and shall call them the left (right) Clifford figures.

With respect to a given frame $\mathfrak{E}_0$ a left (right) Clifford figure consists of all frames $\mathfrak{A}\mathfrak{B}\mathfrak{E}_0$, where $\mathfrak{B}$ ($\mathfrak{A}$) remains fixed and $\mathfrak{A}$ ($\mathfrak{B}$) varies. As its coordinate we can therefore take the matrix $\mathfrak{B}$ ($\mathfrak{A}$), or the corresponding unit quaternion B (A). From Theorem 1 it follows that two quaternions determine the same Clifford figure, when and only when one is the negative of the other. For our purpose we shall, however, make a distinction between the Clifford figures determined by $+B$ and $-B$ ($+A$ and $-A$). The new figure will then be called an oriented left (right) Clifford figure. With a natural topology the space of all Clifford figures is homeomorphic to the real projective space of three dimensions and the space of all oriented Clifford figures to the three-dimensional sphere.

With a differentiable family of frames we define

$$\omega_{\alpha\beta} = d\mathfrak{e}_\alpha \cdot \mathfrak{e}_\beta, \qquad \alpha, \beta = 0, 1, 2, 3, \tag{14}$$

where

$$\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0. \tag{14a}$$

The differential forms $\omega_{\alpha\beta}$ satisfy the following equations of structure of our spherical space:

$$d\omega_{\alpha\beta} = \sum_{\gamma=0}^{3} \omega_{\alpha\gamma}\omega_{\gamma\beta}, \qquad \alpha, \beta = 0, 1, 2, 3. \tag{15}$$

It is easy to verify that when the frames of the family belong to a left equivalence class we have

$$\phi_1 \equiv \omega_{01} - \omega_{23} = 0, \quad \phi_2 \equiv \omega_{02} - \omega_{31} = 0, \quad \varphi_3 \equiv \omega_{03} - \omega_{12} = 0. \tag{16}$$

Similarly, when the frames of the family belong to a right equivalence class, we have

$$\begin{gathered} \psi_1 \equiv -\omega_{01} - \omega_{23} = 0, \qquad \psi_2 \equiv -\omega_{02} - \omega_{31} = 0, \\ \psi_3 \equiv -\omega_{03} - \omega_{12} = 0. \end{gathered} \tag{17}$$

Using the forms ϕ_i, ψ_i, $i=1, 2, 3$, instead of $\omega_{\alpha\beta}$, $\alpha, \beta=0, 1, 2, 3$, the equations of structure (15) can be replaced by the equations

$$\begin{gathered} d\phi_1 = \phi_2\phi_3, \qquad d\phi_2 = \phi_3\phi_1, \qquad d\phi_3 = \phi_1\phi_2, \\ d\psi_1 = \psi_2\psi_3, \qquad d\psi_2 = \psi_3\psi_1, \qquad d\psi_3 = \psi_1\psi_2. \end{gathered} \tag{18}$$

The form $\phi_1\phi_2\phi_3$ is a volume element in the space R_l of oriented left Clifford figures and the same is true of the form $\psi_1\psi_2\psi_3$ in the space R_r of oriented right Clifford figures. Let R_l, R_r denote also the fundamental cycles on the manifolds R_l, R_r. We easily find

$$\int_{R_l} \phi_1\phi_2\phi_3 = \int_{R_r} \psi_1\psi_2\psi_3 = 2\pi^2. \tag{19}$$

It follows that if Z is a three-dimensional cycle of R_l which is homologous to mR_l, we shall have

$$\frac{1}{2\pi^2}\int_Z \phi_1\phi_2\phi_3 = m. \tag{20}$$

A similar statement holds for R_r.

2. **Riemannian manifolds of four dimensions.** Let M be a compact orientable Riemannian manifold of four dimensions. With M as base space is defined the tangent sphere bundle of unit vectors, on each fibre of which operates the rotation group in four variables. The geometry of a fibre of M is the three-dimensional spherical geometry. The construction of the last section defines two new fibre bundles $\mathfrak{F}_l$ and $\mathfrak{F}_r$ over M, namely the bundles of oriented left and right Clifford figures.

The equations of structure of M are

$$d\omega_{\alpha\beta} = \sum_{\gamma=0}^{3} \omega_{\alpha\gamma}\omega_{\gamma\beta} + \Omega_{\alpha\beta}, \qquad \alpha, \beta = 0, 1, 2, 3. \tag{21}$$

We put

$$\phi_1 = \omega_{01} - \omega_{23}, \qquad \phi_2 = \omega_{02} - \omega_{31}, \qquad \phi_3 = \omega_{03} - \omega_{12}, \tag{22a}$$

and

$$\psi_1 = -\omega_{01} - \omega_{23}, \quad \psi_2 = -\omega_{02} - \omega_{31}, \quad \psi_3 = -\omega_{03} - \omega_{12}. \tag{22b}$$

Equations (21) then become

$$\begin{aligned} d\phi_1 &= \phi_2\phi_3 + \Phi_1, & d\phi_2 &= \phi_3\phi_1 + \Phi_2, & d\phi_3 &= \phi_1\phi_2 + \Phi_3, \\ d\psi_1 &= \psi_2\psi_3 + \Psi_1, & d\psi_2 &= \psi_3\psi_1 + \Psi_2, & d\psi_3 &= \psi_1\psi_2 + \Psi_3, \end{aligned} \tag{23}$$

where

$$\begin{aligned} \Phi_1 &= \Omega_{01} - \Omega_{23}, & \Phi_2 &= \Omega_{02} - \Omega_{31}, & \Phi_3 &= \Omega_{03} - \Omega_{12}, \\ \Psi_1 &= -\Omega_{01} - \Omega_{23}, & \Psi_2 &= -\Omega_{02} - \Omega_{31}, & \Psi_3 &= -\Omega_{03} - \Omega_{12}. \end{aligned} \tag{24}$$

Consider the fibre bundle $\mathfrak{F}_l$. Each fibre being homeomorphic to a three-dimensional sphere, the characteristic class of $\mathfrak{F}_l$ in M is an integral cohomology class W_l of dimension four. Similarly, $\mathfrak{F}_r$ gives rise also to an integral cohomology class W_r of dimension four. By the multiplication of the homology group and the cohomology group, we put

$$I_l = W_l \cdot M, \qquad I_r = W_r \cdot M, \tag{25}$$

which are topological invariants of M (they are in fact invariants of the differential structure of M under consideration). We shall express I_l, I_r as integrals over M of differential invariants of the Riemannian metric in M.

THEOREM 3. *Let M be a compact orientable Riemannian manifold of dimension four and let $\Omega_{\alpha\beta}$, $\alpha, \beta = 0, 1, 2, 3$, and Φ_i, Ψ_i, $i = 1, 2, 3$, be the curvature forms of the Riemannian metric constructed according to the equations* (21), (24). *Then the following formulas hold*:

$$\frac{1}{2\pi^2}\int_M \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = I_l, \qquad \frac{1}{2\pi^2}\int_M \Psi_1^2 + \Psi_2^2 + \Psi_3^2 = I_r. \tag{26}$$

If χ is the Euler-Poincaré characteristic of M, we have

$$-I_l + I_r = 8\chi. \tag{27}$$

To prove this theorem we find from (23)

$$d(\phi_1\phi_2\phi_3) = \Phi_1\phi_2\phi_3 + \Phi_2\phi_3\phi_1 + \Phi_3\phi_1\phi_2,$$

$$d\left(\sum_{i=1}^{3}\phi_i\Phi_i\right) = -(\Phi_1\phi_2\phi_3 + \Phi_2\phi_3\phi_1 + \Phi_3\phi_1\phi_2) + \sum_{i=1}^{3}\Phi_i^2,$$

which gives

$$d\left(\phi_1\phi_2\phi_3 + \sum_{i=1}^{3}\phi_i\Phi_i\right) = \sum_{i=1}^{3}\Phi_i^2. \tag{28}$$

By a standard procedure in the theory of fibre bundles we see that it is always possible to define a mapping of M into $\mathfrak{F}_l$, with a finite number of singular points, such that the image of a point P of M is an oriented left Clifford figure attached to that point. The sum of the indices at the singular points is equal to I_l. The first formula in (26) then follows immediately from (20) and (28). A similar argument applies to the second formula.

From (26) we get, by subtraction,

$$I_l - I_r = -\frac{2}{\pi^2}\int_M \Omega_{01}\Omega_{23} + \Omega_{02}\Omega_{31} + \Omega_{03}\Omega_{12}.$$

By the formula of Allendoerfer-Weil we have

$$\chi = \frac{1}{4\pi^2}\int_M \Omega_{01}\Omega_{23} + \Omega_{02}\Omega_{31} + \Omega_{03}\Omega_{12}.$$

Consequently, we have

$$-I_l + I_r = 8\chi.$$

Hence the theorem is proved.

The theorem shows that by the consideration of the fibre bundles $\mathfrak{F}_l$, $\mathfrak{F}_r$ over M we have introduced, besides the Euler-Poincaré characteristic, essentially one new topological invariant of M. We denote this invariant by

$$\eta = (I_l + I_r)/2, \tag{29}$$

which is an integer. Then we have

$$2\pi^2\eta = \int_M \Omega_{01}^2 + \Omega_{02}^2 + \Omega_{03}^2 + \Omega_{12}^2 + \Omega_{23}^2 + \Omega_{31}^2. \tag{30}$$

If M is the toroidal space, then $\eta = 0$, for a locally Euclidean Riemannian metric can then be defined in M. If M is the four-dimensional sphere, the induced metric of M obtained by imbedding it as a unit hypersphere in the five-dimensional Euclidean space is of constant positive Riemannian curvature. We have then

$$\Omega_{\alpha\beta} = -\,\omega_\alpha\omega_\beta, \qquad \alpha, \beta = 0, 1, 2, 3.$$

It follows that $\eta(M)=0$, if M is a sphere.

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A 5 CURVE THEOREM GENERALIZING THE THEOREM OF CARNOT

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Because of the 2-dimensionality of the plane a relation holds between any 3 quantities determining a point or a line. The relation between the meets of a line with the sides of a fixed triangle is given by the theorem of Menelaos. This is generalized to the meets of an algebraic curve with a triangle by the theorem of Carnot. I give here a further generalization to the meets of an algebraic curve with a triangle *whose sides are arbitrary algebraic curves*. The formulation of the theorem requires the introduction of the new notion of curve cross ratio; it will be verified by a simple algebraic method allowing us to prove a wide class of allied theorems. The theorem itself is already very extensive; of the numerous special cases, which include many known theorems, I can, in this place, only give a few examples, but I give an outline of the main procedures in obtaining, step by step, special cases and related theorems.

The theorem of Carnot states that the cyclic product of the ratios in which the sides of a polygon are divided by an algebraic curve is 1: it may be written $\prod(AP:BP)=1$, if A is a vertex, B the next vertex, and P a meet of the curve and AB. Carnot's theorem, which Poncelet in his epoch-making treatise *Propriétés projectives des figures*, a great part of which is based upon it, calls "le principe fondamental," expresses, as I shall show elsewhere, a characteristic property of algebraic curves; it may also be extended to a class of limiting cases.

The usual proofs of Carnot's theorem are by reduction to one of two special cases. The polygon is decomposed either into triangles or into quadrangles with two opposite vertices in the infinite, and these cases (the second of which is a theorem of Newton) are proved by writing the free term of a polynomial as product of its roots.

Menelaos' theorem is usually proved by homothetic triangles (the simplest case of Newton's theorem); it may also be obtained by dualizing the theorem of Ceva. The latter is a trivial corollary of the representation of a point P by barycentric coordinates, with respect to a triangle ABC: if AP meets BC at A', and so on, we have $AC':BC'=ACC':BCC'$ (areas) $=ACP:BCP=ACP:-CBP$; and, evidently, the cyclic product of the ratios analogous to the last is -1. Menelaos' theorem, and its exact dual equivalent, may likewise be expressed as identities between areas of triangles, that is, between

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determinants. Now a determinant is only the simplest case of the resultant of 3 quantics, and this leads us to the formulation and proof of the generalized theorem.

We consider Möbius' 5 line theorem. If each ratio in Menelaos' theorem is considered as a cross ratio with one point at the infinite, we obtain a projective theorem on two transversals of a triangle: the cyclic product of the cross ratios on the sides is 1. Exactly the same statement holds for 5 general algebraic curves (here the 4 curve theorem obtained by choosing the line at infinity as 5th curve is a very special case from which the general theorem cannot be deduced by projective generalization). This statement will make sense as soon as we define the cross ratio determined on a curve (side) by 4 curves (2 sides and 2 transversals).

Let the given side (and the quantic vanishing there) be a, the adjacent sides b and c, the transversals p and q. Denoting by abp the resultant of the 3 quantics, the required cross ratio $a(bp/cq)$ shall be defined as $(abp:abq):(acp:acq)$. It has certain interesting properties, and its definition is a natural one, as seen by the following consideration. Denote the set of meets of a and p by ap, and so on. In case of lines, the cross ratio of the mutual distances of ab, ac, ap, aq equals the cross ratio of the distances of ab and ac from p and q. The last cross ratio may be obtained by substituting the values of ab in p, and so on. In the general case, substitution of every point x of ab in p and multiplication $\prod p(x)$ of the results gives abp.

Remarks. 1. In the definition of the curve cross ratio, as throughout the note, we make the tacit assumption that no resultant which is explicitly mentioned is 0, that is, that the corresponding 3 curves do not have a common point. 2. If b and c, and if p and q are of equal degree, the curve cross ratio is not affected by multiplication of the quantics by constants, and thus depends only on the given curves. The simple ratio $r=abp:abq=\prod(p(x):q(x))$, on the other hand, may differ for proportional quantics. For p and q of equal degree, all x with a fixed $r=-\mu:\lambda$ form a curve $\lambda p+\mu q=0$.

The 5 curve theorem follows at once from the definition: $a(bp/cq)\cdot b(cp/aq)\cdot c(ap/bq)=\pm 1$, as all factors cancel out, or in words:

The cyclic product of the cross ratios determined on the sides of a curvilinear triangle by two curvilinear transversals is ± 1.

The same relation $\prod a_k(a_{k-1}p/a_{k+1}q)=\pm 1$ holds for polygons.

We now prove the 4 curve theorem. In the preceding identities, already the product of the terms containing p equals ± 1, and the second transversal is needed only to complete the cross ratios. If a, b, c are given and if a fixed q is chosen, we obtain a relation between the meets of p with a, b, c. For example, if the meets ap and bp are given, we obtain a certain value v of $acp:bcp$; that is, the product v of the ratios of the "distances" of the points cp from a and b (as $-\mu:\lambda$ in Remark 2) remains unaltered if we change the curve p, as long as ap and bp remain fixed. Exchanging the rôle of p and c, we see that v also retains its value if c is modified without changing ac and bc. so we have:

The meets of two curves c and d with two given curves a and b of equal degrees determine the product acd:bcd of the distances of the meets of c and d from a and b.

If the degrees α and β of a and b are different, the value of $acd^{\beta}:bcd^{\alpha}$ is determined.

Examples of kindred theorems. Quotients of products of resultants in which all terms cancel (except the sign) can, of course, be formed in various ways. As examples we formulate the 2 following theorems:

I. *The cyclic product of the ratios determined on each side of a curvilinear polygon by all non-adjacent sides is* 1.

II. *The cyclic product of the ratios of the "segment" determined on each side of a curvilinear polygon by the following two sides to that determined on the same side by the two preceding sides is* 1.

Indeed, let the sides in their order be $a, b, c, \cdots, p, \cdots$. The product in I is composed of ratios $abp:bcp$ with $p\neq a, b, c$. All factors cancel out (the form abc is cancelled by cab). The factors of the product in II are of the form $cab:cde$ and cancel already on alternating sides; but to complete them to cross ratios (for use, for example, in the linear case) they must be written $cab:cbd\cdot cbd:cde:(caq:cqd\cdot cbq:cqe)$, and the additional terms will cancel only if all the sides are considered.

Other notions related to the curve cross ratio which may occur in theorems of the same type are:

(1) The *cyclic ratio on a curvilinear polygon*, that is, the cyclic product of the ratios determined on each side s_k by a corresponding curve p_k;

(2) The *cyclic ratio* $ab_jc_j:ac_jb_{j+1}$ *on a curve a* (this generalization of the cross ratio is a product of cross ratios);

(3) The *triple cross ratio* $a_jb_kc_l^{(-1)^{j+k+l}}$, $j, k, l=1, 2$, of 6 curves.

We now discuss methods for deriving other theorems and special cases. Starting with the 5-curve theorem (or a related one), we proceed by one or more of the following steps:

1. Dualization.
2. (a) Affine specialization by a suitable choice of the line at infinity; (b) metrical specialization by a suitable choice of the isotropic points I, J.
3. (a) Specialization of the polygon; (b) composition of polygons.
4. (a) Specialization of curves to cubics, conics and lines; (b) specialization to reducible curves.
5. Passing to limiting cases with one or more vanishing resultants.

Example: orientation of common tangents. By the steps 4a, 1 and 2b, we derive a theorem of Laguerre from the 4 curve theorem. Let a and b in 9 be lines; after dualizing they become points. Choose them as I and J. As fixing the tangents to a curve from I and J is equivalent to fixing its foci, and as the product of the ratios of the "distances" of given lines from I and J determines their orientation,[1] we have:

The orientation of the common tangents of two curves is determined by their foci.

It is the same as the orientation of the lines connecting the foci of one curve with those of the other.

We now pass to a single curve. From the 4 curve theorem with a and b as lines we derive a theorem on a, b and a single curve c. Let p be the first polar of ab with regard to c; the meets ap may be determined from ac only, and bp from bc. As the meets cp are the points of contact on the tangents to c from ab, we have:

The meets of a curve c with two given lines a, b determine the ratio product (with regard to a and b) of the tangents from ab.

Conversely, this theorem implies the 4 curve theorem with a and b as lines: take c as a reducible curve $d \cdot e$, and apply the theorem to c, d, e separately, then note that the tangents to c are the same as to d and e plus the lines to de.

Dualizing the preceding theorem, replacing a and b by I and J and remarking that the ratio $c(I):c(J)$ determines the orientation[2] of c, we see that the orientation of a curve is determined by its foci.

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[1] Sum of angles with a given line.

[2] Sum of angles of the asymptotes with a given line.

THE HYPERSURFACE CROSS RATIO

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Introduction. In my note on *A 5 curve theorem generalizing the theorem of Carnot*,[1] I introduced the notion of curve cross ratio. This extension of the ordinary cross ratio is the simplest situationally invariant (3.4) case of the generalized or hypersurface cross ratio of $n+1$ pairs of hypersurfaces in n-space which is the subject of the following lines. The generalized cross ratio is at the same time a generalization of the resultant of $n+1$ quantics; the connection between cross ratios and resultants occurred to me when reading a paper of P. Humbert.[2]

The properties of the generalized cross ratio, including extensions of some of those of the ordinary cross ratio, will be developed, together with the similar and interdependent properties of an analogous generalization of the intersection of n hypersurfaces to pairs of hypersurfaces, in §3. This section, much of the contents of which is known, is parallel to §§1 and 2 on the ordinary resultant and intersection. In each section, after the definition and fundamental properties, the influence of a rational transformation of coordinates and of permutation, variation and linear combination of the hypersurfaces is studied.

1. The resultant. 1.1. Definition. Let $x=(x_0, \cdots, x_n)$ be a point in (complex, or algebraically closed) projective n-space, $n\geqq 0$, and $a=(a_0, \cdots, a_n)$ a system of $n+1$ quantics of positive degrees $\bar{a}_0, \cdots, \bar{a}_n$ in the variables $x_0, \cdots, x_n$. Then the resultant $[a]$ is an irreducible polynomial in the coefficients of a with $[x_k^{\bar{a}_k}]=1$, such that $[a]=0$ if, and only if, an $x\neq 0$ with $a(x)=0$ (that is, $a_0(x)=0, \cdots, a_n(x)=0$ exists. The resultant is unique since the conditions of irreducibility and $[x_k^{\bar{a}_k}]=1$ distinguish it from its powers and multiples respectively.[3]

1.2. *Degree.* $[a]$ is a quantic of degree $\prod_k \bar{a}$ in the coefficients of a_k. Considering $\bar{a}_k$ as degree of the coefficients, we can write $[a]\cdot=\prod \bar{a}$.

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[1] Bull. Amer. Math. Soc. vol. 51 (1945) pp. 972–975.

[2] *Sur l'orientation des systèmes de droites*, Amer. J. Math. vol. 10 (1888) pp. 258–281.

[3] Cf. van der Waerden, *Moderne Algebra* vol. 2, 1931, pp. 18–21. As $|\ |$ is used for the absolute value, $[\ ]$ (already generally used in case of the vector product) is preferable for determinants, resultants and intersections. The product of all degrees shall be denoted by $\prod \bar{a}$, and if $\bar{a}_k$ is omitted, by $\prod_k \bar{a}$ ($=1$ for $n=0$, as all void products including 0 degree in 3.4 and 3.8).

1.3. *Multiplication.* If one of the given quantics a_k is a product $a_k = \prod a_k^{(j)}$, we have $[a] = \prod_j [a_0, \cdots, a_{k-1}, a_k^{(j)}, a_{k+1}, \cdots, a_n]$, in short $[\cdots, \prod, \cdots] = \prod[\ ]$, as both sides vanish simultaneously, are of equal degree, and can be simultaneously equal to 1.

1.4. *Special cases.* If $a_k \equiv 0$, then $[a] = 0$.

For $\prod \bar{a} = 1$, $[a]$ is a determinant.

For $n = 1$, $[a]$ is a product of $\prod \bar{a}$ determinants.

Further note the *monomial* case $[\alpha_k x_k^{\bar{a}_k}] = \prod \alpha_k^{\Pi_k \bar{a}}$.

1.5. *Rational transformation.* Let $f_0, \cdots, f_n$ be quantics of equal degree $\bar{f} > 0$; then $y = f(x)$ defines a rational transformation. If, for $x \neq 0$, also $y \neq 0$, that is, if $[f] \neq 0$, the transformation is called *regular*; in it, to every y, there belong $\bar{f}^n$ points x.[4] Hence, the only regular birational transformations are the projective transformations.

If $a(y) = 0$ for $x \neq 0$, then either $y \neq 0$, $[a] = 0$, or $y = 0$, $[f] = 0$. Hence $[a(f)]$ must be of the form $c[a]^r[f]^s$, $\bar{c} = 0$ (that is, c is constant). In the monomial case $a_k = \alpha_k x_k^{\bar{a}_k}$, $f_k = \beta_k x_k^{\bar{f}}$ we have $a_k(f) = \alpha_k \beta_k^{\bar{a}_k} x_k^{\bar{f}\bar{a}_k}$, $[a(f)] = \prod(\alpha_k \beta_k^{\bar{a}_k})^{\bar{f}^n \Pi_k \bar{a}}$. We obtain $[a(f)] = [a]^{\bar{f}^n} [f]^{\Pi \bar{a}}$.

1.6. *Permutation.* If the quantics a_k are permuted, then $[a]$ and the new resultant $[a']$ are irreducible and of equal degree and vanish together, so $[a'] = c[a]$, $\bar{c} = 0$. Choosing quantics which are products of linear forms, we see by 1.3 and the laws of determinants that $c = (\pm 1)^{\Pi \bar{a}}$ for a permutation of sign ± 1.

1.7. *Variation.* If $\bar{a}_k \geqq \bar{a}_j$, $j \neq k$, and if a_k is replaced by $a_k + \lambda a_j$, where λ is a quantic of degree $\bar{a}_k - \bar{a}_j$, then $[a]$ and the new resultant $[a']$ are of equal degree and vanish together, so $[a'] = c[a]$, $\bar{c} = 0$. Choosing $\lambda = 0$, we see that $c = 1$.

1.8. *Linear combination.* If several quantics a_k of equal degree are replaced by linear combinations $\sum l_{jk} a_k$, $\bar{l}_{jk} = 0$, then by repeated application of 1.7 and 1.2 the new resultant is $[a'] = [l]^{\Pi_k \bar{a}}[a]$.

If all degrees $\bar{a}_k$ are equal, 1.6–1.8 are special cases of 1.5.

2. **The intersection.** 2.1. DEFINITION. If $a_1, \cdots, a_n$, $n \geqq 1$, are fixed quantics of positive degree, so that the intersection $a_1 = \cdots = a_n = 0$ of n hypersurfaces consists only of a finite number of points P_i (for $n = 2$, quantics without a common factor of positive degree), then for $\bar{l} = 1$, $[l, a] = 0$ only if some $l(P_i) = 0$; hence $[l, a] = \prod l(P_i)$ with appropriate multiplicities of sum $\prod \bar{a}$ and with a constant factor according to which the coordinates of the points P_i are determined. The system of these P_i, whose order may be changed and whose coordinates may be multiplied by constants, with product 1, will be de-

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[4] By Bézout's theorem, if the number of points is finite. But for, say, $y_0 \neq 0$, any variety of positive dimension of points x with $y_0 f_k(x) - y_k f_0(x) = 0$ would have a point in common with $f_0(x) = 0$, against the assumption of regularity.

noted by $P=[a]$ (the same symbol as for the resultant, but referring to n quantics only); we consider the intersection P as product of its points and write accordingly $[l, a]=l([a])$.

For a general a_0, $[a_0, a]$ and $a_0([a])$ (that is, $\prod a_0(P_i)$) are of equal degree and vanish together, so they differ only by a constant factor c. Choosing a_0 as product of linear forms, we see by 1.3 that $c=1$, so $[a_0, a]=a_0([a])$ (well known for $n=1$).

2.2. *Degree.* If a_k is replaced by ca_k, $\bar{c}=0$, then $[a]$ becomes $c^{\Pi_k \bar{a}}[a]$; we may again say that $[a]$ is of degree $\prod_k \bar{a}$ in the coefficients of a, and, as in 1.2, write $[a]\cdot=\prod \bar{a}$, so the degree of an intersection is the number of its points. The degree of a resultant in one of the hypersurfaces is the number of meets of the others.

2.3. *Multiplication.* By $[l, a]=l([a])$ and 1.3 we see that again $[\cdots, \prod, \cdots]=\prod[\ ]$, the latter symbol denoting juxtaposition (union) of the points of the respective intersections.

2.4. *Special cases.* If $[a]$ is defined, then no α_k is equal to 0.

For $\prod \bar{a}=1$, $[a]$ consists of n determinants ("vector product").

In the monomial case we have $[\alpha_k x_k^{\bar{a}_k}]=\prod \alpha_k^{\Pi_k \bar{a}}\ (1, 0, \cdots, 0)$.

2.5. *Rational transformation.* Assume $[f]\neq 0$, and $[a(f)]$ definable by 2.1. A point Q_i belongs to $[a(f)]$ if $f(Q_i)$ belongs to $[a]$, so we may write $[a(f)]=[a]:f$. To obtain the exact degree and factor of P in $f(Q)$, put $l=x_k$, $l(f)=f_k$, then, by 2.1, the formula of 1.5 becomes $f_k(Q)=(x_k(P))^{f^n}[f]^{\Pi\bar{a}}=x_k(P^{f^n})[f]^{\Pi\bar{a}}$, or $f([a]:f)=P^{f^n}[f]^{\Pi\bar{a}}$; f^n agrees with 1.5, fourth line.

Rational variety. If $a_{m+1}=\cdots=a_n=0$ is a fixed rational variety $f(t)$, $t=(t_0, \cdots, t_m)$, of degree $\bar{f}$ such that to different t there belong different points $f(t)$, then the cross ratios $[a]$ and $[a(f)]$ as functions of the coefficients of $a_0, \cdots, a_m$ vanish together and are of equal degree, which is the number of meets of the hypersurfaces except $a_k=0$, so they differ only by a constant depending on f and on the form $c^{\Pi a}$ (product up to $\bar{a}_n$). A factor can be given to f so that $c=1$.

For $n=1$, that is, a unicursal curve, $[a(f)]$ is a product of determinants.

If $a_{m+1}=\cdots=a_n=0$ is composed of several rational varieties, $[a]$ equals the product of the corresponding cross ratios. For $m=0$ this is again the final formula of 2.1.

2.6. *Permutation.* For a permutation of sign ±1, 2.1 and 1.6 give $a'=(\pm P_i)$.

2.7. *Variation.* By 2.1 and 1.7, $[\cdots, a_k+\lambda a_s, \cdots]=[a]$.

2.8. *Linear combination.* By 2.7 and 2.2, $[a']=[l]^{\Pi_k \bar{a}}[a]$.

3. **Generalized cross ratio and intersection.** 3.1. DEFINITION. If

$a_0, \cdots, a_n$ are quantics of positive degree, except a_k, which is a quotient $b_k : c_k$ of quantics of positive degree, we may, in agreement with 1.3, define $[a]$ as $[a_0, \cdots, a_{k-1}, b_k, a_{k+1}, \cdots, a_n] : [a_0, \cdots, a_{k-1}, c_k, a_{k+1}, \cdots, a_n]$, and $\lambda b_k : \lambda c_k$, $\bar\lambda > 0$, will give the same value. Likewise, if every a_k is a homogeneous rational function $b_k : c_k$, $[a]$ is defined as quotient of products of 2^n resultants. This "resultant of rational functions" we call also *cross ratio* of the corresponding $n+1$ pairs of hypersurfaces $b_k = 0$ and $c_k = 0$.[5] It may be 0, ∞, or indeterminate. Artificial indetermination, due to the choice of λ, must be avoided.

The intersection may equally be defined for rational functions, in agreement with 2.3, by admitting negative multiplicities of points P_ι, or, in case all multiplicities are 0, a number without point. And the notions of resultant and intersection remain linked by the relation $[a_0, a] = a_0([a])$. Here, for $\bar a_0 = \bar b_0 - \bar c_0 = 0$, every factor $a_0(P_\iota)$ equals $d_i : e_i$, if P_i is on the curve $e_\iota b_0 - d_\iota c_0 = 0$ of the pencil (b_0, c_0), d_i and e_i being constants.

3.2. *Degree.* $[a]$ is of degree $\prod_k \bar a$ in the coefficients of b_k, and of degree $-\prod_k \bar a$ in the coefficients of c_k; so we can again write $[a] = \prod \bar a$. The same statements hold for an intersection $[a]$ of pairs of hypersurfaces.

3.3. *Multiplication.* Again, for cross ratios and intersections, $[\cdots, \prod, \cdots] = \prod[\ ]$; and for both, $[\cdots, 1 : a_k, \cdots] = 1 : [a]$.

3.4. *Special cases.* No b_k or c_k is allowed to be congruent to 0.

In the monomial case, the relations of 1.4 and 2.4 subsist.

If $\bar a_k = 0$, $[a]$ is of degree 0 in the coefficients of all other quantics; if $\bar a_j = \bar a_k = 0$, $[a]$ depends only on the *situation* of the given hypersurfaces.[6] In this case, by 3.1, the cross ratio $[a]$ is a product of "cross ratios $a_k(P_i : Q_i) = a_k(P_i) : a_k(Q_i)$ of a pair of points and a pair of hypersurfaces of equal degree," another situational invariant.[7] As $b_k(P_i : Q_i)$ is the ratio of the first and last coefficient of z in $b_k(z_0 P_\iota + z_1 Q_\iota)$, that is, the product of the ratios $-z_1 : z_0$ corresponding to the meets of $b_k = 0$ with the line $P_\iota Q_i$, we see that $a_k(P_i : Q_i)$ equals a product of ordinary cross ratios on $P_\iota Q_i$, each of which is determined by a meet of $b_k = 0$ and a meet of $c_k = 0$.

For cross ratios and intersections with constant a_k, $[a] = a_k^{\Pi_k \bar a}$, and with constant a_j and a_k, $[a] = 1$. Thus, in contrast to 1.1, the resultant of quantics of non-negative degree may be reducible; and it may be equal to 0 even if an $x\neq 0$ with $a(x)=0$ exists, which happens if there are 2 or more constant a_k, all of them equal to 0 (this case, it is true, has been excluded). Yet, in formal developments, we shall admit either or both $\bar{b}_k=0$ and $\bar{c}_k=0$, so we may assume b_k and c_k without a common factor of positive degree.

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[5] Or of the *virtual* hypersurfaces $a_k = 0$, van der Waerden, *Algebraische Geometrie*, 1939, p. 179.

[6] If $\bar b_k > 0$, $\bar c_k > 0$, we can attain $\bar a_k = 0$ by replacing b_k and c_k by their own powers.

[7] Special cases of which are the ratio in Newton's theorem on algebraic curves, and, passing to the logarithms, the angle, and Laguerre's "orientation."

3.5. *Rational transformation.* The relations of 1.5 and 2.5 subsist. We see that, for $\prod\bar{a}=0$, the cross ratio and intersection are projective invariants. They are not invariant to dualization, common points occurring without common tangents and vice versa.

For a general (not necessarily regular) rational transformation and $\bar{a}_0=0$, $a_0(P_i)$ in 3.1 remains unaltered; thereby the change of the cross ratio and the intersection can be obtained.

Rational variety. The conclusions of 2.5 subsist; hence, for $\prod\bar{a}=0$, $[a]=[a(f)]$. For a unicursal curve $f(t)$ with inhomogeneous t, met by $b_0=0$, $c_0=0$, $b_1=0$, $c_1=0$ respectively in the systems of points $f(\beta_0), f(\gamma_0), f(\beta_1), f(\gamma_1)$ or, admitting negative multiplicities, by $a_0=0$ and $a_1=0$ (of degrees 0) in $f(\alpha_0)$ and $f(\alpha_1)$, we obtain $[a]=\alpha_0-\alpha_1$ $=(\beta_0-\beta_1)(\gamma_0-\gamma_1):(\beta_0-\gamma_1)(\gamma_0-\beta_1)$ where $\beta_0-\beta_1$, and so on, denote products of differences. For a line and $\bar{b}_0\bar{c}_0\bar{b}_1\bar{c}_1=1$, this is the ordinary cross ratio.

It would be interesting to know whether similar laws hold for non-unicursal curves.

3.6. *Permutation.* The relations of 1.6 and 2.6 subsist. Interchanging of b_k and c_k gives, by 3.3, the reciprocal value (or negative multiplicities). All other permutations may change more than the sign of $[a]$, or the sign of the multiplicities. Thus a cross ratio in $(n-1)$-space, or an intersection in n-space, $n>1$, has not $(2n)!$ but (generally) $N=(2n)!:n!2^n=1\cdot 3\cdots(2n-1)$ values, and with the reciprocals $2N$ values. Change of sign occurs only if among the $2n$ hypersurfaces there are n of even and n of odd degree; these must be combined in pairs, which can be done in $n!$ ways, so in this case, together with the opposites, there are respectively $N+n!$ or $2N+2n!$ values.

But the N values are not independent, the number of resultants (or intersections) being only $C_{2n,n}=(2n)!:n!^2$, that is, 6, 20, 70, 252, 924, 3432, 12870, 48620, $\cdots$ instead of 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, $\cdots$. Moreover, a depends only on the $C_{2n,n}-1$ ratios of resultants, and for odd n only on the $C_{2n,n}:2$ ratios of a resultant of certain hypersurfaces and that of all other hypersurfaces.

The true sequence of numbers of independent values begins

2, 5, 14, $\cdots$ as seen below. The quantics will be denoted by 1, 2, $\cdots$, $2n$. Cross ratios and intersections are denoted as in the following example. If the given $[a]$ is $[2:7,\ 3:5,\ 6:1,\ 4:8]$, we bring 1 to the first place: $\pm 1:[a]=[1:6,\ 2:7,\ 3:5,\ 4:8]$. In the second place we see 6, which is the 5th number after 1; so we begin our symbol by 5. The remaining numbers 234578 are now ordered cyclically, beginning after 6, that is, with 7. 7 is brought to the next place, and we write $\pm a=[1:6,\ 7:2,\ 3:5,\ 4:8]$. The figure after 7, that is, 2, is the second in cyclic order; our symbol begins therefore 62. 3 is already in the proper place, 5 is second among 458; hence (622). 8 is not in its place, so $\pm 1:[a]=(622)$.

$n=2$. We have $(1)=[1:2,3:4]=[13][24]:[14][23]$, $(2)=[1:3,4:2]$, $(3)=[1:4,2:3]$. Multiplying, we get A: $(1)(2)(3)=(-1)$ to the power $(\bar{2})(\bar{3})+(\bar{2})(\bar{4})+(\bar{3})(\bar{4})$, hence there are 2 independent values.[8]

$n=3$. Permuting 345 cyclically, we have, as before, $A:(11)(12)(13)=\pm 1$. 234 gives $B:(11)(23)(31)=\pm 1$. 235 gives $(12)(23)(43)=\pm 1$, which follows also from the two preceding types A and B ("types" means that the first figures of the symbols may be replaced by their cyclic successors). So the 5 values (γ_1) determine the others; they are independent, as, for example, (11) contains [145] which occurs in no other (γ_1).

$n=4$. 234 gives $C:(111)(251)(311)=\pm 1$. 235 gives $(123)(221)(451)=\pm 1$, or, by B and C, $D:(221)=\pm(111)(131)(311)(511)$. 236 gives $(131)(213)(511)=\pm 1$, or, by B and C, again D. 237 gives $(143)(212)(651)=\pm 1$, or, by ABC, again D. Finally, 246 gives $(123)(353)(512)=\pm 1$, or, by ABC, a relation E containing (341). By C, D, E respectively, all $(\delta 51)$, $(\delta 21)$, $(\delta 41)$ are expressed by the 14 values $(\delta 11)$, $(\delta 31)$. These are independent, as, for example, (131) contains [1567] which occurs in no other $(\delta 31)$ or $(\delta 11)$, and (111) contains [1457] which occurs in no other $(\delta 11)$.

Every relation for $[a]$ and its permutations holds also if one of the quantics is 1, that is, for $2n-1$ hypersurfaces.[9] The converse is easily seen to be true for relations derived by cyclic permutation of 3 quantics, and 1.6, 2.6 and 3.3. Under the same, perhaps void, restriction the same relations hold for the values $[a]$ formed by $2n-2$

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[8] The relation $[1:2,\ 3:4]+[1:3,\ 2:4]=1$ by which the $2N=6$ values of the ordinary cross ratio are shown to be functions of one of them—and which is equivalent to the additivity of linear measure—ceases to hold, for example, for squares of linear forms. Likewise, for general n, there hold special relations in the linear case, reducing the above sequence to that of squares 1, 4, 9, $\cdots$. A, B, and so on, and the effect of interchanging b_k and c_k, are cases of logarithmic antisymmetry.

[9] A for $n=3$ and the last quantic equal to 1 is the extension of Carnot's theorem given in my paper loc. cit. p. 976.

of $2n-1$ given hypersurfaces, the dimension of space having been diminished by 1.

3.7. *Variation.* $[a]$ remains unchanged if $\bar{b}_k$ is replaced by $b_k+\lambda b_j c_j$, or c_k by $c_k+\lambda b_j c_j$. Hence $[a]$ is determined if, in case $\bar{b}_k \geqq \bar{b}_j+\bar{c}_j$, instead of b_k, only the intersection of $b_k=0$ with $b_j c_j=0$ (and thereby also $\bar{b}_k$) is known. Cf. the end of 3.1, and of 3.5.

In some cases the cross ratio $[a_0, a]$ is quite independent of a_0 (of a given degree).[10] By 3.1, this happens if, and only if, all multiplicities of the points P_i of $[a]$ are equal to 0, that is, if every meet of n hypersurfaces, one of each pair, belongs to a further hypersurface, for example, in the linear combination case 3.8.

Already for $n=2$ this is not the only case of constant $[1, a]$. Let b_1, c_1, b_2, c_2—no two of which shall have a common factor of positive degree—have $p \geqq 0$ common points, and b_1, c_1, b_2 have γ_2 additional common points, and define β_1, γ_1, β_2 similarly. Then, of course, $\bar{b}_1\bar{c}_1 \geqq p+\beta_2+\gamma_2$ and 5 analogous inequalities hold, of which—by the before-said—4 are equations if, and only if, $[1, a]$ is constant. The 4 equations give $\prod \bar{a}=0$ (as stated before); let $\bar{a}_1=0$, so that $\bar{b}_1=\bar{c}_1$, $\beta_1=\gamma_1$, and $\gamma_2-\beta_2=\bar{b}_1\bar{a}_2$, which may be assumed not less than 0. $\bar{b}_1$, $\bar{c}_2$, $\bar{a}_2$, p, β_1, β_2 are connected by (1) $\bar{b}_1^2 \geqq p+2\beta_2+\bar{b}_1\bar{a}_2$, (2) $\bar{c}_2^2+\bar{c}_2\bar{a}_2 \geqq p+2\beta_1$, (3) $\bar{b}_1\bar{c}_2=p+\beta_1+\beta_2$. Eliminating p, there remains (4) $0,\ \bar{b}_1\bar{c}_2-\bar{a}_1\bar{c}_2-\bar{c}_2^2+\beta_1 \leqq \beta_2 \leqq \bar{b}_1\bar{c}_2-\beta_1,\ b_1^2-\bar{b}_1\bar{a}_2-\bar{b}_1\bar{c}_2+\beta_1$. Eliminating β_2, we have (5) $0,\ \bar{b}_1\bar{a}_2+\bar{b}_1\bar{c}_2-\bar{b}_1^2 \leqq \beta_1 \leqq \bar{b}_1\bar{c}_2,\ (\bar{a}_2+\bar{c}_2)\bar{c}_2/2$ and $(\bar{b}_1-\bar{c}_2)\bar{a}_2 \leqq (\bar{b}_1-\bar{c}_2)^2$,[11] that is, either $S1$: $\bar{a}_2 \leqq \bar{b}_1-\bar{c}_2$, which may be written $\bar{c}_2 \leqq \bar{b}_2 \leqq \bar{b}_1$, or $S2$: $\bar{a}_2 \leqq \bar{b}_1 \leqq \bar{c}_2$, which may be written $\bar{b}_2-\bar{c}_2 \leqq \bar{b}_1 \leqq \bar{c}_2 \leqq \bar{b}_2$ ($\bar{a}_2 \leqq \bar{b}_1$ follows by eliminating β_1, or from (1), except for $\bar{b}_1=\bar{c}_1=0$, a linear combination case). In both cases, the inequality (6) $\bar{b}_1\bar{a}_2+\bar{b}_1\bar{c}_2-\bar{b}_1^2 \leqq (\bar{a}_1\bar{c}_2+\bar{c}_2^2)/2$ is automatically fulfilled: for $S1$ this is obvious, for $S2$ we write (6) $(2\bar{b}_1-\bar{c}_2)\bar{a}_2 \leqq \bar{b}_1^2+(\bar{b}_1-\bar{c}_2)^2$, the left side, if greater than 0, being not greater than $(2\bar{b}_1-\bar{c}_2)\bar{b}_1$ not greater than the right side.

For given $\bar{b}_1$, $\bar{c}_2$, we now may choose $\bar{a}_2>0$ according to $S1$ or $S2$, or $S3$: $\bar{a}_2=0$, with $\bar{b}_1>\bar{c}_2$, or $S4$: $\bar{a}_2=0$, $\bar{b}_1=\bar{c}_2$; then β_1 according to (5), which in cases $S1$, $S3$, $S4$ reduces to $0 \leqq \beta_1 \leqq (\bar{a}_1+\bar{c}_2)\bar{c}_2/2$; then β_2 according to (4), which in case $S4$ reduces to $\beta_2=\beta_1>0$ (for $\beta_1=0$ we have a linear combination case); then p by (3), $\bar{b}_2$, γ_2. $S3$ and $S4$ are the situationally invariant cases. Arranging by ascending $\max(\bar{b}_1, \bar{c}_2)$, the table of "nonlinear-combination-cases" begins

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[10] By 3.3 it is necessary and sufficient for this, that it be so for a linear form $a_0=1$. By 3.2 there must be $\prod \bar{a}=0$. If the cross ratio is 1, it is also independent of $\bar{a}_0$.

[11] Also immediately as $(1)+(2)-2(3)$.

S	1	2	2	1	1	3	3	2	2	4	4	2	2	2
$\bar{b}_1$	1	1	1	2	2	2	2	2	2	2	2	2	2	2
$\bar{c}_2$	0	1	2	0	0	1	1	1	1	2	2	2	2	2
$\bar{a}_2$	1	1	1	1	2	0	0	1	1	0	0	1	1	2
β_1	0	1	2	0	0	0	0	0	1	1	2	2	3	4
β_2	0	0	0	0	0	1	2	0	1	1	2	0	1	0
p	0	0	0	0	0	1	0	2	0	2	0	2	0	0
$\bar{b}_2$	1	2	3	1	2	1	1	2	2	2	2	3	3	4
γ_2	1	1	1	2	4	1	2	2	3	1	2	2	3	4
	A	B		C	D	E	F	G	H	I	J			

All 10 cases $A, \cdots, J$, containing only lines and conics, are realizable. Of the 4 situationally invariant cases E, F, I, J, the latter concerns 4 conics in the position described by the theorem:

Of 4 *given pairs of points every* 3 *pairs are on a conic if, and only if, either* (1) *all* 4 *pairs are on a conic, or* (2) *a point and a line harmonic to the* 4 *pairs exist.*

Indeed, let B_1, C_1, B_2, C_2 be pairs of points, and $B_1C_1B_2$ on a conic c_2, and so on. Choose $C_2=(I, J)$, then b_1, c_1, b_2 are circles. If they coincide, we have case (1); otherwise B_1, C_1, B_2 are pairs of inverse points for a certain circle O. If O is a line, we have case (2); otherwise, in the inversion at O, c_2 becomes a quartic c_2', meeting c_2 at B_1, C_1, B_2 and in 4 points of O, which is impossible.[12]

F (on a pair of conics intersecting in 4 points $ABCD$ and the pair of lines AB, CD) is a case of cross ratio 1, for the cross ratio is the same for all pairs of conics, as it cannot become 0 (or ∞), and equals 1 for coinciding conics; for two circles, F becomes the theorem of the radical axis. E concerns two conics as in F and the lines AB, AC; by a limiting case of F, either conic may be replaced by BC and its tangent at A, so the fixed cross ratio equals that of AB, AC and the two tangents. I yields a theorem on 4 circles, which by inversion becomes the statement that a variable circle is met by a fixed triangle and its circumcircle in a constant cross ratio.

3.8. *Linear combination.* 1.8 and 2.8 extend to the case where the unaltered a_k are rational functions.

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[12] The non-existence of c_2, in this case, is also easily proved algebraically, or by remarking that the directions of its axes would have to be harmonic to any 2 of the 3 lines through O and B_1, C_1, B_2.

If $m+1>0$ quantics b_k are linear forms β_k of $m+1$ quantics d_k, and if the corresponding c_k are also linear forms γ_k of d_k, then by repeated application of 1.8, or 2.8, we have $[a]=[\alpha]^{\Pi_k d}$, where $\alpha_k=\beta_k:\gamma_k$. In particular we mention that (a) two pairs of hypersurfaces belonging to a pencil are met by every line in the same cross ratio (and by any other curve in a power of that cross ratio)—a generalization of a fundamental theorem of projective geometry—and this cross ratio is a projective invariant; (b) n pairs of hyperplanes through a point are cut by every hyperplane in the same cross ratio.

The cross ratio of $n+1$ pairs of hyperplanes, or points, considered as function of one of them, is a quotient of two quantics of degree 2^{n-1}, and constant on a hypersurface of degree 2^{n-1}. For $n=2$, the cross ratio of 3 pairs of points, or lines, is 1 if, and only if, they are on a conic, which is the fact underlying the projective generation of conics. For general n, the cross ratio of $n+1$ points a and of the fundamental points is 1 if, and only if, the product of the coaxial minors of even degree of a equals that of those of odd degree. For $n=1$, this leads to $m=0$.

UNIVERSITY OF JERUSALEM

THE PENTAGON IN THE PROJECTIVE PLANE, WITH A COMMENT ON NAPIER'S RULE

TH. MOTZKIN

1. **Introduction.** The relations between the 6 values of the cross ratio of 4 points obtained by permuting the points are well known (§2). The additional relations for the cross ratios of any 4 of 5 collinear points are given in §3. 5 points (or lines) in the plane determine cross ratios with the same relations (4.1). The vertices of a pentagon are on a conic, the sides on another, the pentagon is polar to itself for a third conic. The last may be without real points; in this case the relations between the cross ratios turn out to be equivalent to the famous rule of Napier on the 5 parts of a rectangular spherical triangle, the cross ratios becoming the squares of the sines of the 5 parts. Thus the said relations are a very simple projective formulation of Napier's rule.[1] These considerations form the contents of 4.2. The meets of the diagonals of a pentagon form another pentagon, projective to the first. Repeating the procedure, we obtain a bothways infinite sequence of pentagons, converging to the vertices and sides of the common polar triangle of the 3 above-mentioned conics (4.3). In §5 the relation between the generalized cross ratios of 6 lines (or points) in the plane is determined. Finally we remark that the study of the pentagon in the projective plane is connected, in addition to that of the rectangular spherical triangle, to the study of the triangle in the metrical plane, that is, in elementary Euclidean geometry, and may be important because of its applications to the latter subject.

2. **Cross ratios of 4 elements in 1 dimension.** A point $a=(a_0, a_1)$ on a line is given by a binary linear form $ax=a_0x_0+a_1x_1$ vanishing at the point. Denoting by $[a, b]$ the determinant of the coefficients a, b of two linear forms, the cross ratio of the pairs a, b and c, d is $s=[a:b, c:d]=([a, c]:[a, d]):([b, c]:[b, d])$. Hence $[c:d, a:b]=s$, $[a:b, d:c]=1:s$. By the relation of Ptolemy-Euler-Plücker $[a, b][c, d]+[a, c][d, b]+[a, d][b, c]=0$, there is $[a:c, b:d]=1-s$. Hence, $[b:c, d:a]=[a:d, c:b]=t=s:(s-1)$, $1:s+1:t=1$. $1:s$, $1-s$, and t may be called respectively the *reciprocal*, *complementary*, and *opposite* cross ratios to s. The 6 values s, $1:s$, $1-s$, t, $1:t$, $1-t$ lie, if real,

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[1] And of its dual equivalent, cf, for example, R. E. Moritz, *Napier's theorem for quadrantal triangles*, Congrès International des Mathématiciens, Oslo, 1936, II, pp. 170–171.

in the 6 intervals between $-\infty, -1, 0, 1/2, 1, 2, \infty$, one in each interval. We shall denote the intervals between $-\infty, 0, 1, \infty$ by I, II, III; II and III, I and III, I and II are interchanged respectively by passing to $1:s$, $1-s$, t. 4 points in cyclic order determine two opposite cross ratios. 4 points may be ordered cyclically in 3 ways; for real points there is a *natural* cycle, with cross ratios in III, and two *interlaced* cycles.[2] If each point is given by its coordinates, or coordinate, A, the cross ratio shall be denoted by $A, B; C, D$, for example, $[\infty, 0; 1, s]=s$.[3]

3. Cross ratio of 5 elements in 1 dimension. Of 5 elements, 12 cycles and 30 cross ratios may be formed. In addition to the former relations, we have 10 identities of the type $[a:b, d:e][b:c, d:e][c:a, d:e]=1$. Denoting $1:[b:c, d:e]$ by $\bar{a}$, and cyclically, we obtain $\bar{a}\cdot\bar{c}=1-\bar{b}$. 5 points are projectively determined by 2 cross ratios, say $\bar{b}$ and $\bar{d}$; $\bar{a}, \cdots, \bar{e}$ are then $1-\bar{b}/1-\bar{b}\bar{d}$, $\bar{b}$, $1-\bar{b}\bar{d}$, $\bar{d}$, $1-\bar{d}/1-\bar{b}\bar{d}$. Inverting the cyclic order does not change the cross ratio quintuple. To the *opposite cycle acebd* belong the opposite cross ratios $1:[c:e, b:d]=\bar{a}:(\bar{a}-1)$, and so on. Of the other 10 cycles, 5, such as *ebcda*, are *bordering* cycles to *abcde*, and 5 are bordering to *acebd*; the latter are also bordering or opposite to the former. The 12-graph giving the relation of bordering, having no closed 3-chains, is not the icosahedric graph, but has 30 closed 4-chains. To *ebcda* belong the cross ratios $1-\bar{e}$, $1:\bar{b}$, $\bar{c}:(\bar{c}-1)$, $1:\bar{d}$, $1-\bar{a}$. For real points the natural cycle has its whole cross ratio quintuple $\bar{a}, \cdots, \bar{e}$ in II, the opposite in I. The bordering cycles have quintuples with cross ratios in II, III, I, III, II, their opposites in III, I, II, I, III.

4. Cross ratios of 5 elements in 2 dimensions. 4.1. A line $a=(a_0, a_1, a_2)$ in the plane is given by its equation, that is, by a ternary linear form $ax=a_0x_0+a_1x_1+a_2x_2$ vanishing at its points. If we denote by $[a, b, c]$ the determinant of the coefficients a, b, c of 3 linear forms, and by $[a, b]$ the meet of a and b, given by the coefficients of $x=(x_0, x_1, x_2)$ in $[x, a, b]$, the cross ratio of the meets $[a, b]$, $[a, c]$, $[a, d]$, $[a, e]$ equals $[a, b:c, d:e]=([a, b, d]:[a, b, e]):([a, c, d]$

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[2] The same holds for complex points, the corresponding regions being bounded by circular arcs through $1/2\pm(-3)^{1/2}/2$. For example, the natural cyclic order (of $\infty, 0, 1, s$) is characterized by $|s|>1$, $|s|>|s-1|$.

[3] The fractional notation a/b would lead to the typographically less convenient symbol

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix},$$

which reflects the group properties of the cross ratio and has been proposed about 50 years ago.

ways considered as ratio of ordinary cross ratios; hence for each of e given pairs of lines the ratio of the cross ratios determined on its lines by the two other pairs is the same. If the third pair (of points) s I, J, then $iln/2$ of the (generalized) cross ratio is the difference of the angles at which one pair is seen from the points of the other. The cross ratio is 1 if, and only if, the 6 points (lines) are on the same conic—in particular, if two pairs are collinear (concurrent). There are again 30 cross ratios. The multiplicative relations of §§2–3 subsist and allow us to obtain the other values from $\bar{a}=[a:b,\ c:d,\ e:f]$, $\bar{b}=[b:c,\ d:e,\ a:f]$, $\cdots$, $\bar{e}=[e:a,\ b:c,\ d:f]$. As 6 lines are projectively determined by 4 constants, there must be a relation between $\bar{a}, \cdots, \bar{e}$. Indeed, if $a'=1:[f,\ a:b,\ c:d]$, $a''=1:[e,\ a:b,\ c:d]$, and so on, then $a', \cdots, e'$ and $a'', \cdots, e''$ are cross ratio quintuples as in §§3–4, and $\bar{a}, \cdots, \bar{e}$ equal $((1-b'):(1-b'd')):((1-b''):(1-b''d''))$, $b':b''$, $(1-b'd'):(1-b''d'')$, $d':d''$, $((1-d'):(1-b'd')):((1-d'') :(1-b''d''))$. By $b'=\bar{b}b''$, $d'=\bar{d}d''$ we have $1-\bar{b}\bar{d}b''d''=\bar{c}(1-b''d'')$, $b''d''=(1-\bar{c}):(\bar{b}\bar{d}-\bar{c})$. By $(1-\bar{b}b''):(1-b'')=\bar{a}\bar{c}$, $b''=(1-\bar{a}\bar{c}):(\bar{b}-\bar{a}\bar{c})$, and similarly $d''=(1-\bar{c}\bar{e}):(\bar{d}-\bar{c}\bar{e})$. Multiplying and equating, we get $(1-\bar{a}\bar{c})(1-\bar{c}\bar{e})(\bar{c}-\bar{b}\bar{d})=(\bar{c}-1)(\bar{b}-\bar{a}\bar{c})(\bar{d}-\bar{c}\bar{e})$. $\bar{c}$ divides the difference of both sides, so finally $\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}-\bar{a}\bar{c}\bar{e}-\bar{c}\bar{e}\bar{b}-\bar{e}\bar{b}\bar{d}-\bar{b}\bar{d}\bar{a}-\bar{d}\bar{a}\bar{c}+\bar{a}\bar{c}+\bar{c}\bar{e}+\bar{e}\bar{b}+\bar{b}\bar{d}+\bar{d}\bar{a}-1=0$, an identity between the determinants of a 6, 3-matrix. Conversely, to any $\bar{a}, \cdots, \bar{e}=0, 1$ with $\bar{b}=\bar{a}\bar{c}$, and so on, and fulfilling the last equation there belong certain b', d', b'', d'', and 6 lines not on a conic, no 3 of which are concurrent.

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RECTILINEAR CONGRUENCES WHOSE DEVELOPABLES INTERSECT A SURFACE IN ITS LINES OF CURVATURE

C. E. SPRINGER

Introduction. It is well known that the developable surfaces of the congruence of normals to a surface intersect the surface in its lines of curvature. One may inquire if there exist congruences other than the congruence of normals to a surface the developables of which intersect the surface in its lines of curvature. It is the chief aim of this paper to give an affirmative answer to this query. The exhibition of a congruence of the required variety depends upon the solution of a partial differential equation of Laplace—a circumstance which occurs frequently in problems of differential geometry.

The notation employed here is that of Eisenhart,[1] with the exception that $\Gamma^{\alpha}_{\beta\gamma}$ will be used for the Christoffel symbol of the second kind. Greek letters will take the range 1, 2, and Latin letters the range 1, 2, 3. The convention of the tensor analysis as to summation on repeated indices will be observed.

1. **Analytical development.** Consider a surface S represented by $x^i = x^i(u^1, u^2)$ $(i = 1, 2, 3)$ referred to a rectangular cartesian system of coordinates. The functions $x^i(u^1, u^2)$, together with their partial derivatives to the second order, are understood to be continuous at any point P of the surface. A unique line λ of a congruence Λ is determined at each point P of the surface S by the direction cosines

$$\lambda^i = \lambda^i(u^1, u^2), \qquad \lambda^i\lambda^i = 1, \tag{1}$$

where the functions λ^i and their first partial derivatives are continuous at points of the surface under consideration.

The functions λ^i may be expressed in terms of the direction numbers $x^i_{,\alpha}$ $(\alpha = 1, 2)$ of the tangents to the coordinate curves on the surface through P, and the direction cosines X^i of the normal to the surface at P. Thus,

$$\lambda^i = p^{\alpha} x^i_{,\alpha} + qX^i, \tag{2}$$

where p^{α} are the contravariant components of a vector in the surface at P, q is a positive scalar function, and $x^i_{,\alpha}$ denotes covariant

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[1] Eisenhart, *Differential geometry*, Princeton University Press, 1940.

differentiation of x^i with respect to u^α, based on the first fundamental tensor $g_{\alpha\beta} \equiv x^i_{,\alpha} x^i_{,\beta}$ of the surface S.

On using (2) and the second of equations (1), there results

$$\lambda^i\lambda^i \equiv (p^\alpha x^i_{,\alpha} + qX^i)(p^\beta x^i_{,\beta} + qX^i) \equiv g_{\alpha\beta}p^\alpha p^\beta + q^2 = 1. \tag{3}$$

If θ is the angle between the normal to S at P and the line λ of the congruence Λ at P, it follows from (3) that

$$\cos\theta \equiv \lambda^i X^i = q.$$

Thus, from equation (3), it is seen that the length of the vector with contravariant components p^α is $\sin\theta$. That is, the last of equations (3) assumes the form

$$p_\alpha p^\alpha = \sin^2\theta.$$

It may be noticed here that the angle ϕ between the line λ of the congruence Λ at P and the direction dx^i/ds of any curve C: $u^\alpha = u^\alpha(s)$ through P on S is given by

$$\cos\phi \equiv \lambda^i dx^i/ds = (p^\alpha x^i_{,\alpha} + qX^i)x^i_{,\beta}du^\beta/ds = g_{\alpha\beta}p^\alpha du^\beta/ds,$$

where s denotes arc length along C.

Covariant differentiation of λ^i in equation (2) gives

$$\lambda^i_{,\beta} = p^\alpha x^i_{,\alpha\beta} + qX^i_{,\beta} + x^i_{,\alpha}p^\alpha_{,\beta} + X^i q_{,\beta},$$

which, by means of the Gauss and Weingarten equations[2]

$$x^i_{,\alpha\beta} = d_{\alpha\beta}X^i, \qquad X^i_{,\beta} = -\,d_{\beta\gamma}g^{\gamma\delta}x^i_{,\delta},$$

can be written in the form

$$\lambda^i_{,\alpha} = \mu^\gamma{}_\alpha x^i_{,\gamma} + \nu_\alpha X^i, \tag{4}$$

where, in turn, $\mu^\gamma{}_\alpha$ and ν_α are defined by

$$\mu^\gamma{}_\alpha \equiv p^\gamma{}_{,\alpha} - q d_{\alpha\sigma}g^{\sigma\gamma}, \qquad \nu_\alpha \equiv q_{,\alpha} + p^\beta d_{\alpha\beta}. \tag{5}$$

Differentiation of $\lambda^i\lambda^i = 1$ yields $\lambda^i\lambda^i_{,\alpha} = 0$, which can be written, by use of (2) and (4), in the form

$$(p^\sigma x^i_{,\sigma} + qX^i)(\mu^\gamma{}_\alpha x^i_{,\gamma} + \nu_\alpha X^i) = 0,$$

or

$$p_\gamma \mu^\gamma{}_\alpha + q\nu_\alpha = 0 \qquad (\alpha = 1, 2). \tag{6}$$

Because q does not vanish, h_γ may be written for p_γ/q, so that equation (6) takes the form

$$\nu_\alpha = -\,h_\gamma \mu^\gamma{}_\alpha \qquad (\alpha = 1, 2). \tag{7}$$

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[2] Eisenhart, loc. cit. pp. 216, 217.

2. **Developables of the congruence and their intersections with the surface.** In order to arrive at the differential equation of the two one-parameter families of curves on the surface S in which the developables of the congruence Λ intersect the surface, take a point with coordinates $x^i+t\lambda^i$ on the line λ through P on S, and let P move along a curve $C: u^\alpha=u^\alpha(s)$ on S for which the point with coordinates $x^i+t\lambda^i$ will describe a curve tangent to the line λ. This requires that

$$d(x^i + t\lambda^i) \equiv dx^i + t d\lambda^i + \lambda^i dt = m\lambda^i, \tag{8}$$

where m is to be determined. Multiplication of equation (8) by λ^i and use of $\lambda^i\lambda^i=1$ show that $m=dt+\lambda^i dx^i$, by means of which equation (8) becomes

$$dx^i + t d\lambda^i = \lambda^i(\lambda^k dx^k),$$

or

$$[x^i{}_{,\alpha} + t\lambda^i{}_{,\alpha} - \lambda^i(\lambda^k x^k{}_{,\alpha})]du^\alpha = 0. \tag{9}$$

From equations (2), (4), (7), it is observed that

$$\lambda^i x^i{}_{,\alpha} = p^\sigma g_{\sigma\alpha} = p_\alpha, \tag{10}$$

$$\lambda^i{}_{,\alpha}\lambda^i{}_{,\beta} = \mu^\sigma{}_\alpha\mu^\tau{}_\beta g_{\sigma\tau} + \nu_\alpha\nu_\beta = \mu^\sigma{}_\alpha\mu^\tau{}_\beta(g_{\sigma\tau} + h_\sigma h_\tau), \tag{11}$$

$$\lambda^i{}_{,\alpha}x^i{}_{,\beta} = \mu^\gamma{}_\alpha x^i{}_{,\gamma}x^i{}_{,\beta} = \mu^\gamma{}_\alpha g_{\gamma\beta} = \mu_{\beta\alpha}. \tag{12}$$

Multiplication of equation (9) by $\lambda^i{}_{,\beta}$, together with use of equations (10), (11), (12), yields

$$(\mu_{\alpha\beta} + tG_{\alpha\beta})du^\alpha = 0 \qquad (\beta = 1, 2), \tag{13}$$

where $G_{\alpha\beta}$ is an abbreviation for

$$\lambda^i{}_{,\alpha}\lambda^i{}_{,\beta} \equiv \mu^\sigma{}_\alpha\mu^\tau{}_\beta(g_{\sigma\tau} + h_\sigma h_\tau). \tag{14}$$

Elimination of the parameter t from equations (13) gives the differential equation

$$e^{\gamma\delta}\mu_{\alpha\gamma}G_{\beta\delta}du^\alpha du^\beta = 0, \tag{15}$$

in which $e^{\gamma\delta}$ is defined as follows: $e^{12}=1$, $e^{21}=-1$, $e^{11}=e^{22}=0$. *The curves on the surface which satisfy equation* (15) *constitute the net of curves in which the developables of the congruence intersect the surface.* This net will be referred to as the *intersector net.*

3. **Particular congruences.** It will be shown next that equation (15) can be written in the form

$$e_{\alpha\gamma}(\mu^\gamma{}_\beta - h^\gamma\nu_\beta)du^\alpha du^\beta = 0. \tag{16}$$

From the definition of $G_{\alpha\beta}$ in (14) it follows that

$$
\begin{aligned}
e^{\gamma\delta}\mu_{\alpha\gamma}G_{\beta\delta} &\equiv e^{\gamma\delta}\mu_{\alpha\gamma}\mu^{\sigma}{}_{\beta}\mu^{\tau}{}_{\delta}(g_{\sigma\tau} + h_{\sigma}h_{\tau}) \\
&\equiv e^{\gamma\delta}\mu^{\epsilon}{}_{\gamma}g_{\alpha\epsilon}\mu^{\sigma}{}_{\beta}\mu^{\tau}{}_{\delta}(g_{\sigma\tau} + h_{\sigma}h^{\rho}g_{\tau\rho}).
\end{aligned} \tag{17}
$$

By use of the relations $e^{\gamma\delta}\mu^{\epsilon}{}_{\gamma}\mu^{\tau}{}_{\delta} \equiv \mu e^{\epsilon\tau}$, $e^{\epsilon\tau}g_{\epsilon\alpha}g_{\sigma\tau} \equiv g e_{\alpha\sigma}$, where $\mu \equiv |\mu^{\alpha}{}_{\beta}|$, $g \equiv |g_{\alpha\beta}|$, and $e_{\alpha\sigma}$ is defined by $e_{12}=1$, $e_{21}=-1$, $e_{11}=e_{22}=0$, the last member of (17) takes the form

$$
\mu g e_{\alpha\sigma}(\mu^{\sigma}{}_{\beta} - h^{\sigma}\nu_{\beta}). \tag{18}
$$

Thus, on rejecting the nonvanishing factors μ, g, equation (16) follows.

A necessary and sufficient condition[3] that the intersector net given by (16) be a conjugate net is that

$$
e_{\alpha\sigma}(\mu^{\sigma}{}_{\beta} - h^{\sigma}\nu_{\beta})d^{\alpha\beta} = 0,
$$

which, by use of (5) and (7), can be written in the form

$$
e_{\sigma\beta}(p^{\sigma}/q)_{,\alpha}d^{\alpha\beta} = 0.
$$

Thus, *a necessary and sufficient condition that the congruence have the property that its intersector net on S be a conjugate net is that the parameters h^{σ} of the congruence satisfy the partial differential equation of first order*

$$
e_{\sigma\beta}h^{\sigma}{}_{,\alpha}d^{\alpha\beta} = 0. \tag{19}
$$

The intersector net on S is orthogonal, if, and only if,

$$
e_{\alpha\sigma}(\mu^{\sigma}{}_{\beta} - h^{\sigma}\nu_{\beta})g^{\alpha\beta} = 0.
$$

Again, by use of (5) and (7), and the identical vanishing of the term $e_{\gamma\beta}d_{\alpha\sigma}g^{\sigma\gamma}g^{\alpha\beta}$, it is found that *a necessary and sufficient condition for the orthogonality of the intersector net on S is that the parameters h^{σ} of the congruence Λ satisfy*

$$
e_{\gamma\beta}(h^{\gamma}{}_{,\alpha} - d_{\alpha\sigma}h^{\gamma}h^{\sigma})g^{\alpha\beta} = 0. \tag{20}
$$

It is to be observed that if Λ is the congruence of normals to S ($h^{\sigma} \equiv 0$), then equations (19), (20) are satisfied identically, and equation (16) reduces to the lines of curvature net represented by

$$
e_{\beta\sigma}d_{\alpha\gamma}g^{\gamma\sigma}du^{\alpha}du^{\beta} = 0.
$$

The congruence Λ is a normal congruence if, and only if, $\mu_{\alpha\beta} = \mu_{\beta\alpha}$. This can be shown as follows. If every line of Λ is normal to a surface $\overline{S}$ given by $y^i = y^i(u^1, u^2)$, then $y^i = x^i + t\lambda^i$ and $\lambda^i d(x^i + t\lambda^i) = 0$ identically in u^1, u^2. Because $\lambda^i\lambda^i_{,\alpha}du^\alpha = 0$ and $\lambda^i(x^i_{,\alpha}+\lambda^i t_{,\alpha})du^\alpha = 0$ for arbitrary direction du^α on $\bar{S}$, it follows that

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[3] Eisenhart, loc. cit. p. 231.

$$\lambda^i x^i_{,1} + t_{,1} = 0, \qquad \lambda^i x^i_{,2} + t_{,2} = 0.$$

On differentiating the last two equations with respect to u^2, u^1 respectively, and eliminating $\partial^2 t/\partial u^1 \partial u^2$, there results

$$x^i_{,\alpha}\lambda^i_{,\beta} = x^i_{,\beta}\lambda^i_{,\alpha},$$

which, by (4), can be written as

$$g_{\alpha\gamma}\mu^\gamma_\beta = g_{\beta\gamma}\mu^\gamma_\alpha,$$

or

$$\mu_{\alpha\beta} = \mu_{\beta\alpha}. \tag{21}$$

On multiplying equation (5) by $g_{\beta\gamma}$, it follows from equation (21) that *a necessary and sufficient condition for the congruence* Λ *to be a normal congruence is that*

$$p_{\alpha,\beta} = p_{\beta,\alpha}.$$

4. Congruences for which the intersector net coincides with the lines of curvature net. It follows from equation (16) that the intersector net is the parametric net on S if, and only if,

$$\mu^\gamma_\beta = h^\gamma \nu_\beta \qquad (\gamma \neq \beta), \tag{22}$$

which, by means of equations (5) and the fact that $h^\gamma = p^\gamma/q$, can be expressed in the form

$$h^\gamma_{,\alpha} = (g^{\gamma\sigma} + h^\gamma h^\sigma)d_{\alpha\sigma} \qquad (\gamma \neq \alpha). \tag{23}$$

The first order partial differential equations (23) appear as a two-dimensional analogue of an ordinary differential equation of Riccati.

It will now be shown that the two first order partial differential equations (23) in h^γ become two partial differential equations of the second order by the substitution

$$h^\gamma = -d^{\gamma\sigma}\psi_{,\sigma} \qquad (\gamma = 1, 2), \tag{24}$$

where $\psi \equiv \log \omega(u^1, u^2)$. Covariant differentiation of equations (24) yields

$$h^\gamma_{,\alpha} = -d^{\gamma\sigma}\psi_{,\sigma\alpha} - d^{\gamma\sigma}_{,\alpha}\psi_{,\sigma},$$

by use of which equations (23) become

$$-d^{\gamma\sigma}\psi_{,\sigma\alpha} - d^{\gamma\sigma}_{,\alpha}\psi_{,\sigma} = (g^{\sigma\gamma} + d^{\sigma\tau}d^{\gamma\rho}\psi_{,\tau}\psi_{,\rho})d_{\alpha\sigma}.$$

On setting $\psi = \log \omega$, and using the fact that $d^{\sigma\tau}d_{\alpha\sigma} = \delta^{\tau}_{\alpha}$, there results

$$d^{\gamma\sigma}\omega_{,\sigma\alpha} + d^{\gamma\sigma}{}_{,\alpha}\omega_{,\sigma} + g^{\sigma\gamma}d_{\alpha\sigma}\omega = 0 \qquad (\gamma \neq \alpha)$$

or

$$(d^{\gamma\sigma}\omega_{,\sigma})_{,\alpha} + g^{\sigma\gamma}d_{\alpha\sigma}\omega = 0 \qquad (\gamma \neq \alpha). \tag{25}$$

A solution $\omega(u^1, u^2)$ *of the two differential equations* (25) *determines the parameters* h^σ *in* (24) *of a congruence whose intersector net on* S *is parametric.*

It is to be required next that the parametric intersector net be the lines of curvature net on the surface S. For this, it is necessary and sufficient that $g_{12} = d_{12} = 0$. In this case, both of equations (25) reduce to the single equation of Laplace

$$(d^{\gamma\sigma}\omega_{,\sigma})_{,\alpha} = 0 \qquad (\gamma \neq \alpha). \tag{26}$$

Equation (26) may be expanded into the form

$$d^{\gamma\sigma}\partial^2\omega/\partial u^\alpha\partial u^\sigma + (d^{\gamma\sigma}{}_{,\alpha} - d^{\gamma\rho}\Gamma^{\sigma}_{\alpha\rho})\partial\omega/\partial u^\sigma = 0,$$

nd this, in turn, on using

$$d^{\gamma\sigma}{}_{,\alpha} = \partial d^{\gamma\sigma}/\partial u^\alpha + d^{\rho\sigma}\Gamma^{\gamma}_{\rho\alpha} + d^{\gamma\rho}\Gamma^{\sigma}_{\rho\alpha},$$

akes the form

$$d^{\gamma\sigma}\partial^2\omega/\partial u^\alpha\partial u^\sigma + (\partial d^{\gamma\sigma}/\partial u^\alpha + d^{\rho\sigma}\Gamma^{\gamma}_{\rho\alpha})\partial\omega/\partial u^\sigma = 0 \qquad (\gamma \neq \alpha). \tag{7)}$$

Use of the Codazzi equations,[4] together with $g_{12} = d_{12} = 0$, allows equaion (27) to be written out in the form

$$2\frac{\partial^2\omega}{\partial u^1\partial u^2} = \frac{d_{22}}{g_{22}d_{11}}\cdot\frac{\partial g_{11}}{\partial u^2}\cdot\frac{\partial\omega}{\partial u^1} + \frac{d_{11}}{g_{11}d_{22}}\cdot\frac{\partial g_{22}}{\partial u^1}\cdot\frac{\partial\omega}{\partial u^2}\,. \tag{28}$$

Thus, it can be concluded that *the exhibition of a congruence whose developables intersect a given surface in its lines of curvature depends upon finding a solution of equation* (28). After h^σ are found from equations (24), p^γ and q are determined by the relations $h^\sigma \equiv p^\sigma/q$ and equation (3).

To the solution $\omega(u^1, u^2) \equiv$ constant, of equation (28), there corresponds the congruence of normals to the surface S.

It should be oberved that if g_{11} is a function of u^1 alone, or if g_{22} is a function of u^2 alone, then a solution of equation (28) can be obtained by quadratures.

5. Coincidence of the intersector net with the asymptotic net on

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[4] Eisenhart, loc. cit. p. 230.

the surface. The intersector net may be required to coincide with a net other than the lines of curvature. For example, in case the asymptotic net on S is parametric, $d_{11}=d_{22}=0$, so that if the intersector net is required to coincide with the asymptotic net, equations (25) become

$$(d^{\gamma\alpha}\omega_{,\alpha})_{,\alpha}+g^{\gamma\gamma}d_{\alpha\gamma}\omega=0 \qquad (\gamma\neq\alpha;\ \alpha,\gamma \text{ not summed}). \tag{29}$$

On putting $\gamma=2$, $\alpha=1$, and making use of the Codazzi relation

$$\partial d_{12}/\partial u^1=d_{12}(\Gamma^1{}_{11}-\Gamma^2{}_{12}),$$

there results from equations (29)

$$\partial^2\omega/\partial u^1\partial u^1+\Gamma^1{}_{11}\partial\omega/\partial u^1-\Gamma^2{}_{11}\partial\omega/\partial u^2+g^{22}(d_{12})^2\omega=0,$$

and similarly, with $\gamma=1$, $\alpha=2$,

$$\partial^2\omega/\partial u^2\partial u^2-\Gamma^1{}_{22}\partial\omega/\partial u^1+\Gamma^2{}_{22}\partial\omega/\partial u^2+g^{11}(d_{12})^2\omega=0.$$

To a solution $\omega(u^1, u^2)$ of the last two equations there corresponds a congruence Λ for which the developables intersect the surface S in its asymptotic curves.

University of Oklahoma

INTEGRAL DISTANCES

PAUL ERDÖS

In a note under the same title (Bull. Amer. Math. Soc. vol. 51 (1945) pp. 598–600) it was shown that there does not exist in the plane an infinite set of noncollinear points with all mutual distances integral.

It is possible to give a shorter proof of the following generalization: if A, B, C are three points not in line and $k=[\max(AB, BC)]$, then there are at most $4(k+1)^2$ points P such that $PA-PB$ and $PB-PC$ are integral. For $|PA-PB|$ is at most AB and therefore assumes one of the values $0, 1, \cdots, k$, that is, P lies on one of $k+1$ hyperbolas. Similarly P lies on one of the $k+1$ hyperbolas determined by B and C. These (distinct) hyperbolas intersect in at most $4(k+1)^2$ points. An analogous theorem clearly holds for higher dimensions.

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A NOTE ON THE FIRST NORMAL SPACE OF A V_m IN AN R_n

YUNG-CHOW WONG[1]

Let N be the normal plane at a point p of a surface V_2 in a Euclidean 4-space R_4. Calapso[2] proved that the hypersphere S in R_4 passing through p and with center c in N cuts V_2 in a curve with a double point at p, at which the two tangents to the curve coincide if and only if c lies on the Kommerell conic. The Kommerell conic is the locus of the point in which N (at p) is cut by the neighboring normal planes of V_2.

The purpose of this note is to generalize this result to the case of a subspace V_m in a Euclidean n-space R_n. To do this we shall first state some definitions and known results concerning the first (or principal) normal space of V_m in R_n.[3]

Let X^k $(k=1, \cdots, n)$ be the rectangular cartesian coordinates in R_n and let

$$X^k = x^k(u^a) \qquad (a, b, c = 1, \cdots, m) \tag{1}$$

be the equations of a V_m. Put

$$\overset{k}{B_a} = \partial_a \overset{k}{x} \equiv \partial x^k / \partial u^a. \tag{2}$$

Then the fundamental tensor and curvature tensor of V_m in R_n are, respectively,

$$'g_{cb} = \sum_k \overset{k}{B_c} \overset{k}{B_b}, \tag{3}$$

$$H_{cb}^{\cdot\cdot k} = \partial_c \overset{k}{B_b} - '\Gamma^a_{cb} \overset{k}{B_a}, \tag{4}$$

where $'\Gamma^a_{cb}$ is the Christoffel symbol of the second kind for V_m.

Let us consider the figure surrounding a certain point p of V_m. We have at p a tangent m-plane and a normal $(n-m)$-plane N. Let i^a be the unit tangent vector at p of an arbitrary curve in V_m passing through p. Then the component in N of the first curvature vector of the curve with respect to R_n is

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[1] Harrison research fellow at the University of Pennsylvania.

[2] R. Calapso, *Sulle reti di Voss di uno spazio lineare quadri dimensionale*, Rendiconti Seminario matematico Roma (4) vol. 2 (1938) pp. 276–311.

[3] See J. A. Schouten and D. J. Struik, *Einführung in der neueren Methoden der Differentialgeometrie* II, Groningen, 1938, chap. 3; D. Perepelkine, *Sur la courbure et les espaces normaux d'une V_m dans R_n*, Rec. Math. (Mat. Sbornik) N.S. vol. 42 (1935) pp. 81–100.

$$u^{k} = H_{cb}^{\cdot\cdot k} i^{c} i^{b}. \tag{5}$$

The vector u^k spans the *first* normal m'-plane N' (in N) of V_m in R_n.

The arithmetic mean of the vector u^k for m mutually orthogonal directions i^a is the *mean* normal curvature vector

$$M^{k} = {}'g^{cb} H_{cb}^{\cdot\cdot k}/m. \tag{6}$$

Any vector n_k $(=n^k)$ in N orthogonal to N' is such that

$$n_k H_{cb}^{\cdot\cdot k} = 0. \tag{7}$$

p is called a *semi-umbilical* point if there exists a vector v_k such that

$$v_k H_{cb}^{\cdot\cdot k} = {}'g_{cb} \tag{8}$$

is satisfied. Because of (7) we may suppose that v_k is a vector in N'.

The normal $(n-m)$-plane at the neighboring point $p+dp$ may or may not intersect N' (at p) at points other than p depending on the direction of dp. But we call the locus of the intersection of N' at p (the point p being excluded) by the normal $(n-m)$-planes of all the neighboring points $p+dp$ the *K-variety* at p of V_m in R_n. The equation of the K-variety is

$$\operatorname{Det}\,(Y_k H_{cb}^{\cdot\cdot k} - {}'g_{cb}) = 0, \tag{9}$$

where Y_k is a variable vector in N'. The K-variety is an algebraic hypersurface of order m in N'. At a semi-umbilical point, it is a hypercone in N' with vertex at the end point $v(x^k+v^k)$ (cf. (8)).

We are now in a position to prove the following theorems.

THEOREM 1. *The hypersphere S in R_n passing through p and with center at a point c in N intersects V_m in a V_{m-1} with a singular point at p. The tangent lines to V_{m-1} at p form a hypercone C (in the tangent m-plane to V_m) of generally the second degree.*

THEOREM 2. *p is semi-umbilical if and only if there exists a hypercone C at p which is of at least the third degree.*

THEOREM 3. *All the points in N with the same projection in N' have the same hypercone C. There exist two points in N' having the same hypercone C if and only if p is semi-umbilical. At a semi-umbilical point all the points (with the exception of the point v (x^k+v^k)) on each line in N' passing through v have the same hypercone C. No two points in N' noncollinear with v have the same hypercone C.*

THEOREM 4. *The locus of the point in N' whose hypercone C sustains* *an orthogonal ennuple of generators is the polar hyperplane of the end point of the mean normal curvature vector with respect to the unit hypersphere in N' (with center at p).*

THEOREM 5. *The K-variety is the locus of the point in N' whose hypercone C has a line of vertices.*

Theorem 5 for $m=2$, $n=4$, $N'=N$ reduces to the theorem of Calapso quoted at the beginning of this paper.

PROOF. The expansion of $x^k(u^a)$ in the neighborhood of $p: x_0^k = x^k(u_0^a)$ is

$$x^k = x_0^k + (\partial_a x^k)_0 du^a + 2^{-1}(\partial_c \partial_b x^k)_0 du^c du^b + \cdots .$$

But by (2) and (4),

$$\partial_a x^k = B_a^k, \qquad \partial_c \partial_b x^k = \partial_c B_b^k = H_{cb}^{\cdot\cdot k} + {}'\Gamma_{cb}^a B_a^k.$$

Therefore

$$x^k = x_0^k + (B_a^k)_0 du^a + 2^{-1}(H_{cb}^{\cdot\cdot k} + {}'\Gamma_{cb}^a B_a^k)_0 du^c du^b + \cdots . \tag{10}$$

The equation of the hypersphere S in R_n passing through p and with center at a point $c(x_0^k + c^k)$ in N is

$$\sum_k (X^k - x_0^k - c^k)^2 = \sum_k (c^k)^2.$$

Using (1) for X^k we see that S intersects V_m at the points $(u_0^a + du^a)$ at which

$$\begin{aligned} \sum_k [- c^k + (B_a^k)_0 du^a + 2^{-1}(H_{cb}^{\cdot\cdot k} + {}'\Gamma_{cb}^a B_a^k)_0 du^c du^b + \cdots]^2 \\ = \sum_k (c^k)^2. \end{aligned} \tag{11}$$

Let us arrange this in powers of du^a. Then the constant term disappears. The first term vanishes because c^k is orthogonal to the tangent m-plane spanned by $(B_a^k)_0$:

$$\sum_k c^k (B_a^k)_0 = 0. \tag{12}$$

The second degree term is

$$\begin{aligned} \left[\sum_k (B_c^k B_b^k)_0 - \sum_k c^k (H_{cb}^{\cdot\cdot k} + {}'\Gamma_{cb}^a B_a^k)_0 \right] du^c du^b \\ = ({}'g_{cb} - c_k H_{cb}^{\cdot\cdot k})_0 du^c du^b \end{aligned}$$

by (3) and (12). This proves Theorem 1 and gives us the equation of the hypercone C as

$$(c_k H_{cb}^{\cdot\cdot k} - {}'g_{cb}) Z^c Z^b = 0, \tag{13}$$

where Z^a is a variable direction in the tangent m-plane at p.

Theorem 2 follows at once from (8), (13) and (11).

The hypercones at p for the two distinct points c $(x_0^k + c^k)$ and d $(x_0^k + d^k)$ are the same if and only if a constant ρ exists such that

$$(d_k - \rho c_k) H_{cb}^{\cdot\cdot k} = (1 - \rho)\, {}'g_{cb} \tag{14}$$

is satisfied. If d and c have the same projection in N' we have by (7)

$$(d_k - c_k) H_{cb}^{\cdot\cdot k} = 0. \tag{15}$$

Therefore (14) will be satisfied by $\rho = 1$. Hence all the points in N with the same projection in N' have the same hypercone C.

This shows that we may confine our attention to the points in N' for the consideration of the hypercone C. Let this be done. Then (15) can no longer hold, and condition (14) cannot be satisfied by $\rho = 1$. Consequently, (14) may be put into the form (8) with

$$v_k = (d_k - \rho c_k)/(1 - \rho). \tag{16}$$

Therefore if there exist two points in N' with the same hypercone C then p must be a semi-umbilical point. Conversely, at a semi-umbilical point the locus of the point in N' whose hypercone C is the same as that of the point c (distinct from v) is the straight line cv minus the point v.

The hypercone (13) sustains an orthogonal ennuple, that is, contains m mutually orthogonal generators if and only if

$$'g^{cb}(c_k H_{cb}^{\cdot\cdot k} - {}'g_{cb}) = 0,$$

that is, by (6) if

$$c_k M^k = 1. \tag{17}$$

Theorem 4 is an immediate consequence of this.

The hypercone (13) has a line of vertices if and only if

$$\operatorname{Det}\,(c_k H_{cb}^{\cdot\cdot k} - {}'g_{cb}) = 0, \tag{18}$$

that is, if c lies on the K-variety (9). This proves Theorem 5.

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ERRATA, VOLUME 50

J. Hadamard, *A known problem of geometry and its cases of indetermination.*

p. 520. Footnote 1 should read "Mathematicae Notae, Boletin del Instituto de Matematica vol. 3 (1945) p. 155, Rosario (Argentine Republic)."

ERRATA, VOLUME 51

D. H. Hyers, *Linear topological spaces.*

p. 9, line 1. For "neighborhood system U" read "neighborhood system $\mathcal{V}$."

p. 10, line 17. For "point $y \neq x$" read "point y of S, with $y \neq x$."

p. 11, line 28. For "set D" read "set E."

p. 14, line 20. To conditions (2), (a) and (b), of Definition 6.1 add the condition:

"(c) $\epsilon(x_0, x_1, x_2)$ is continuous in the pair (x_1, x_2) at the point (θ, x_2) for each x_2 in L_1;".

p. 14, line 36. For "$\epsilon(\theta, y, z)$" read "$\epsilon_r(\theta, y, z)$."

p. 15, last line. For "m-differential" read "M-differential.".

H. S. Wall, *Note on the expansion of a power series into a continued fraction.*

p. 98, line 5. Instead of

$$“\frac{A_{n+1}(z)}{B_{n+1}(z)} - \frac{A_n(z)}{B_n(z)} = \frac{a_0 a_1 \cdots a_n}{z^{2n+1}} + \frac{h_n}{z^{2n+2}} + \cdots”$$

read

$$“\frac{A_{n+1}(z)}{B_{n+1}(z)} B_n(z) - A_n(z) = \frac{a_0 a_1 \cdots a_n}{z^{n+1}} + \frac{h_n}{z^{n+2}} + \cdots .”$$

H. E. Salzer, *Formulas for direct and inverse interpolation of a complex function tabulated along equidistant circular arcs*, abstract 51-7-127.

p. 522, line 12 of abstract. For "$\sum C_k^{(n)}(p_m + iq_m)$" read "$\sum C_k^{(m)}(p_m + iq_m)$."

H. Bateman, *The control of an elastic fluid.*

p. 629, line 4. The footnote symbol should be 79 instead of 69.

p. 645, lines 4 and 5. For "equation we form (A) the function" read "equation (A) we form the function."

Isaac Opatowski, *Direct and reverse transitions in Markoff chains*, abstract 51-9-184.

p. 665, line 6 of abstract. Instead of "k_1" read "k_i." Instead of "g_1" read "g_i."

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$[a, c, e]$), as is immediately verified by assuming $a=(1, 0, 0)$. In ddition to the relations of §1, we have the Menelaos-Möbius identity $[a, b:c, d:e][b, c:a, d:e][c, a:b, d:e]=1$, with the same consequences s in §2. This is also clear by considering the lines as elements on their common conic, say $(1, u, u^2)$, the cross ratio of 4 lines on the fifth being equal to that determined on any fixed tangent $(1, 0, 0)$, that is, to that of the meets $(0, u, -1)$.[4]

4.2. A pentagon consists of 5 sides a, b, c, d, e and 5 vertices $[a, b]$, $[b, c]$, $[c, d]$, $[d, e]$, $[e, a]$ in cyclic order. Let C_0 be the *mean conic*, that is, the one for which a and c are conjugate, and so on cyclically; then the polarity at C_0 interchanges every side a and its opposite vertex $[c, d]$, and also the conics C_1 and C_{-1} on which the sides and the vertices are. Let A and A' be the meets of C_0 and a, and $[A, A'; [e, a], [a, b]]=a'$. Give to A, A', and $[e, a]$ the inhomogeneous coordinates ∞, 0, and 1, then the coordinates of $[a, b]$, $[a, c]$, $[a, d]$ will be a', -1, $-a'$, so $\bar{a}=1:[a, b:c, d:e]=[a', -1; 1, -a']$ $=-(a'-1)^2:4a'$. For $\bar{a}$ in I or III, a' is real; for $\bar{a}$ in II, a' is purely imaginary. Hence C_0 has real points except for natural cycles. For the latter, throw one of the lines to infinity; an ellipse round the opposite vertex would cut its sides, so that C_0 cannot have real points. If the pentagon is on the plane at infinity of 3-space, C_0 may be the isotropic conic; then the corresponding spherical pentagon will have quadrantal diagonals, that is of arc $\pi:2$. The sides, or diagonal angles, are given by $[i, -i; 0, \tan\alpha]=a'$, whence $(i-\tan\alpha):(i+\tan\alpha)=a'$, $\bar{a}=\tan^2\alpha:(1+\tan^2\alpha)=\sin^2\alpha$; they are also the parts of any of the rectangular spherical triangles *abc*, *bcd*, and so on (for example, *abc* has the sides $\pi:2-\alpha$, β, $\pi:2-\gamma$ and the angles δ, ϵ), the Napierian cycle becoming $\alpha\gamma\epsilon\beta\delta$. From $\bar{a}\bar{c}=1-\bar{b}$ we obtain $\sin\alpha\sin\gamma=\cos\beta$ (the sign, for the usual orientation of angles, being verified for small angles). The opposite of $\sin^2$ being $-\tan^2$, we have $\cot\alpha\cot\beta=\cos\delta$, the second equation of Napier.[5]

4.3. To the opposite line cycle *acebd* of a pentagon belongs the same C_1, but other vertices on a conic C_3, and a new mean conic C_2. The opposite vertex cycle on C_3 leads to C_5 and C_4, and so on. We obtain a bothways infinite sequence of cycles, determined by any of i cycles, and conics C_m for every integer m. As the sides and vertic, of a pentagon have equal cross ratios, the same cross ratios belo to all cycles of the sequence; consecutive pentagons have opposi cross ratios, and every pentagon is projective to the next but on The projectivity P between both pentagons transforms every lin cycle or point cycle into the next, and C_m into C_{m+4}. The polarity at C_0 inverts the whole sequence and P; the fixed triangle $\overline{P}$ of P is, therefore, a polar triangle of C_0 (and of every C_{2m}). We may choose $\overline{P}$ as fundamental triangle and C_0 as $xx=0$; then $ad=0$, $bd=0$, so that d is proportional to $[a, b]$, and P transforms $a=(a_k)$ to $a'=(\lambda_k a_k)$ which connects $[b, d]$ and $[c, e]$. Hence $[a', b, d]=0$, $[a', b][a, b]=0$, whence by a formula of Lagrange $a'a\cdot b^2-a'b\cdot ab=0$; likewise $b'b\cdot a^2-b'a\cdot ab=0$, so $a'a\cdot b^2=b'b\cdot a^2$; We see that $a'a:aa=q$ is the same for $a, \cdots, e$, so that C_1, given by $a'a-qaa=0$, and hence or similarly every C_{2m+1}, has also $\overline{P}$ as polar triangle. λ and the cross ratio quintuple determine each other. By Poncelet's closure theorem, to a sequence of conics there belong one-dimensionally many (∞^1) sequences of pentagons. For a sequence of pentagons consisting of natural cycles and their opposites (as in case of the quadrantal pentagon), there are mean conics without real points, so the vertices and sides of $\overline{P}$ are real; if $|\lambda_0|<|\lambda_1|<|\lambda_2|$, the sides of the pentagons converge for $m\rightarrow+\infty$ to (0, 0, 1), the vertices to (1, 0, 0), and vice versa for $m\rightarrow-\infty$. Otherwise, that is, for a sequence of cycles bordering to natural and their opposites, only one vertex v and side s of $\overline{P}$ are real, for C_{-1} and C_3 have exactly 2 real meets, as is easily seen for particular cases and therefore generally true, a change in the reality of $\overline{P}$ being only possible for touching C_{-1} and C_1, whereas touching conics have no closure polygons of Poncelet. P being projective to a transformation of similitude, the vertices of the pentagons converge to v, while the sides approach v without converging to any line through v; vice versa for s. The limit points (1, 0, 0), (0, 0, 1), v are roots of cubic equations, which might lead to an approximative construction of the roots of cubic equations. Among the projectivities belonging to the 12 point cycles of a given line cycle it is easy to find pairs whose product has a line of fixed points.

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[4] The relations between the cross ratios of $n+3$ elements in $n>2$ dimensions obtained by joining any 4 of them to the linear variety of the others, that is, between their mutual projective coordinates, can also be derived from identities of the types Menelaos and Ptolemy, and are therefore the same as for $n+3$ elements in 1 dimension. This can, as above, be otherwise obtained, together with the natural cyclic order, by use of the norm curve through the elements.

[5] Taking the mean conic as the conic of the correspondence of Hesse, we see that the cross ratio quintuples are also connected with the mutual cross ratios of 5 pairs of points on a line, which are harmonic to each other in cyclic order.

5. **Cross ratios of 6 elements in 2 dimensions.** The generalized cross ratio $[a:b, c:d, e:f]=[a, c:d, e:f]:[b, c:d, e:f]$[6] can be in 3

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[6] For general quantics cf. my papers *The hypersurface cross ratio*, and *A 5-curve theorem generalizing the theorem of Carnot*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 976–984, 972–975.